COMPUTABILITY

EXERCISE : $A = \{ x \in \mathbb{N} \mid W_x \subseteq E_x \}$

* A is saturated

$A = \{ x \mid \varphi_x \in \mathcal{A} \}$ $\qquad \mathcal{A} = \{ f \mid dom(f) \subseteq cod(f) \}$

* A is not r.e.

observe $\mathbb{1} \notin \mathcal{A}$ $\qquad \underbrace{dom(\mathbb{1})}_{= \mathbb{N}} \not\subseteq \underbrace{cod(\mathbb{1})}_{= \{1\}}$

but $\vartheta = \emptyset \subseteq \mathbb{1}$ and $\vartheta \in \mathcal{A}$

hence A is not r.e. by Rice - Shapiro (hence not recursive)

* $\bar{A}$ is not r.e.

take $pred(x) = x \dot{-} 1$ $\qquad dom(pred) = cod(pred) = \mathbb{N}$

hence $pred \in \mathcal{A} \rightsquigarrow pred \notin \bar{\mathcal{A}}$

but if you take $\vartheta \subseteq pred$

$$\vartheta(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \uparrow & \text{otherwise} \end{cases}$$

$dom(\vartheta) = \{0, 1\} \not\subseteq cod(\vartheta) = \{0\}$

hence $\vartheta \notin \mathcal{A} \rightsquigarrow \vartheta \in \bar{\mathcal{A}}$

hence, by Rice - Shapiro $\bar{A}$ is not r.e. (hence $\bar{A}$ not recursive)

EXERCISE

Call $f : \mathbb{N} \to \mathbb{N}$ injective if $\forall x, y \in dom(f)$ $\quad f(x) = f(y)$ then $x = y$.

$$A = \{ x \mid \varphi_x \text{ is injective} \}$$

conjecture : $\bar{A}$ r.e. ($SC_{\bar{A}}$), not recursive (Rice)

$\rightsquigarrow$ A not r.e. (hence not recursive)

* $\bar{A}$ is r.e.

$SC_{\bar{A}}(x)$ = look for y, z s.t. $\varphi_x(y) = \varphi_x(z)$

$$= \mathbb{1}\left(\mu (\underset{(\omega)_1}{y}, \underset{(\omega)_2}{z}, \underset{(\omega)_3}{t}, \underset{(\omega)_4}{v}) . \quad S(x, y, v, t) \land S(x, z, v, t) \atop y \neq z \right)$$

$$= \mathbb{1}\left(\mu \omega . \quad S(x, (\omega)_1, (\omega)_4, (\omega)_3) \land S(x, (\omega)_2, (\omega)_4, (\omega)_3) \land (\omega)_1 \neq (\omega)_2 \right)$$

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$$= \mathbb{1}\left(\mu \omega . \quad S(x, (\omega)_1, (\omega)_4, (\omega)_3) \land S(x, (\omega)_1 + 1 + (\omega)_2, (\omega)_4, (\omega)_3)\right)$$

computable

$\rightsquigarrow$ $\bar{A}$ r.e.

* $\bar{A}$ not recursive

(1[st] possibility) reduction $K \leq_m \bar{A}$

define

$$g(x, y) = \begin{cases} \text{not injective in } y \quad 1 & x \in K \\ \text{injective} \quad \uparrow & x \notin K \end{cases}$$

$$= SC_K(x) \qquad \text{computable}$$

By smn theorem there is $s: \mathbb{N} \to \mathbb{N}$ total computable such that $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} 1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

Now s is the reduction function for $K \leq_m \bar{A}$

not injective

* if $x \in K$ then $\varphi_{s(x)}(y) = 1 \quad \forall y$ hence $\varphi_{s(x)} = \mathbb{1} \in \bar{A}$

  and thus $s(x) \in \bar{A}$

injective

* if $x \notin K$ then $\varphi_{s(x)}(y) \uparrow \quad \forall y$ hence $\varphi_{s(x)} = \emptyset \notin \bar{A}$

  and thus $s(x) \notin \bar{A}$

Since $K \leq \bar{A}$ and K not recursive then $\bar{A}$ not recursive.

(2^{nd} possibility) Observe that $\bar{A}$ is saturated and not trivial

- if e_1 is s.t. $\varphi_{e_1} = \mathbb{1}$ then $e_1 \in \bar{A} \neq \emptyset$
- if e_0 is s.t. $\varphi_{e_0} = \emptyset$ then $e_0 \notin \bar{A} \neq \mathbb{N}$

by Rice's theorem $\bar{A}$ not recursive.

<u>EXERCISE</u> :

Say $f: \mathbb{N} \to \mathbb{N}$ is <u>monotone</u> if f is total and

$\forall x, y \in \mathbb{N}$ if $x \leq y$ then $f(x) \leq f(y)$

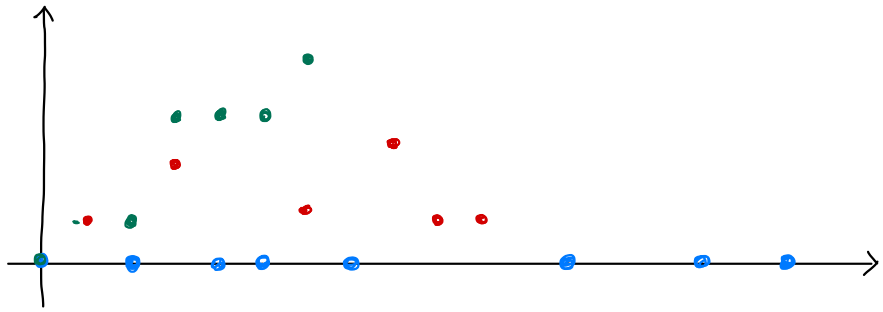
<u>Question</u> : is there a monotone non computable function ?

Consider

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x) \downarrow \\ 0 & \text{otherwise} \end{cases}$$

we know that it is total and not computable

$(\chi_K(x) = sg(f(x)))$



define $\quad g(x) = \sum_{y \leq x} f(y)$

→ total

→ not computable $\quad \forall x \quad g(x) \neq \varphi_x(x)$

→ $\varphi_x(x)\downarrow \qquad g(x) = \sum_{y \leq x} f(y) \geq f(x) = \varphi_x(x) + 1$

$\rightsquigarrow g(x) > \varphi_x(x)$

→ $\varphi_x(x)\uparrow \qquad g(x)\downarrow \neq \varphi_x(x)$

→ g is monotone, $\quad \forall x, y \quad x \leq y$

$$g(x) = \sum_{z \leq x} f(z) \leq \sum_{z \leq x} f(z) + \sum_{x < z \leq y} f(z)$$

$$= \sum_{z \leq y} f(z) = g(y)$$

* Alternative solution

$g: \mathbb{N} \to \mathbb{N}$

$$g(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \text{ and } \varphi_x(x) \neq x+1 \\ x & \text{otherwise } (\text{if } \varphi_x(x) = x+1 \text{ or } \varphi_x(x)\uparrow) \end{cases}$$

→ g total

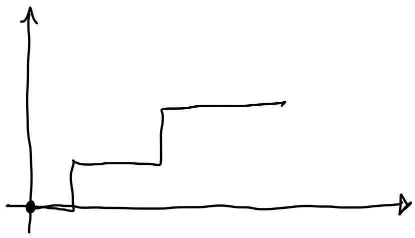
→ g not computable $\quad \forall x \quad \varphi_x(x) \neq g(x)$

- if $\varphi_x(x)\downarrow \quad \begin{cases} \varphi_x(x) \neq x+1 & \text{then } g(x) = x+1 \neq \varphi_x(x) \\ \varphi_x(x) = x+1 & \text{then } g(x) = x \neq \varphi_x(x) \end{cases}$

- if $\varphi_x(x)\uparrow$ then $g(x)\downarrow \neq \varphi_x(x)$

→ g is monotone $\forall x, y \quad x < y$

$$g(x) \leq x+1 \leq y \leq g(y)$$



Even simpler

$$g(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{otherwise} \end{cases}$$

→ total & monotone

→ not computable

$$\chi_K(x) = g(x) \dot{-} x = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

if g were computable then χ_K, composition of computable functions would be computable. ⇝ g not computable.

<u>EXERCISE</u>

Is there a total non computable function $f: \mathbb{N} \to \mathbb{N}$ s.t.

$$g(x) = \sum_{y<x} f(x) \quad \text{is computable ?}$$

NO :

$$f(x) = g(x+1) \dot{-} g(x)$$
$$= \sum_{y<x+1} f(y) \dot{-} \sum_{y<x} f(y)$$

hence if g were computable also f would be computable, by composition.

- What about the case in which f is <u>non-total</u> ?

$$f(x) = \begin{cases} \uparrow & \text{if } x=0 \\ \chi_K(x) & \text{if } x>0 \end{cases} \qquad \text{not computable}$$

in fact, observe that, if $\chi_K(0) = b$ then

$$\chi_K(x) = \begin{cases} b & \text{if } x = 0 \\ f(x) & \text{if } x > 0 \end{cases}$$

if f were computable and $e \in \mathbb{N}$ be such that $f = \varphi_e$ then

$$\chi_K(x) = \Big(\mu w \,\big(S(e, x, (w)_1, (w)_2) \wedge x > 0 \big) \vee \big((w)_1 = b \wedge x = 0 \big) \Big)_1$$

would be computable

Hence f is not computable.

Moreover

$$g(x) = \sum_{y<x} f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mu z.\, x \qquad \text{computable}$$

<u>EXERCISE</u> : show that there is $x \in \mathbb{N}$ s.t. $\varphi_x(y) = x \dot{-} y$

Define $\quad g(x, y) = x \dot{-} y \quad$ computable

By smn $\quad g(x,y) = \varphi_{s(x)}(y) \quad$ for $s : \mathbb{N} \to \mathbb{N}$ total computable

using the 2nd recursion theorem there is x_0 s.t. $\varphi_{x_0} = \varphi_{s(x_0)}$

$$\varphi_{x_0}(y) = \varphi_{s(x_0)}(y) = g(x_0, y) = x_0 \dot{-} y$$

$\square$