COMPUTABILITY (13/01/2024)

EXERCISE: Given $f : \mathbb{N} \to \mathbb{N}$ fixed function and let

$$B_f = \{ e \in \mathbb{N} \mid \varphi_e = f \}$$

The set is saturated

$$B_f = \{ e \in \mathbb{N} \mid \varphi_e \in \mathcal{B}_f \} \qquad \text{with} \qquad \mathcal{B}_f = \{ f \}$$

2 cases:

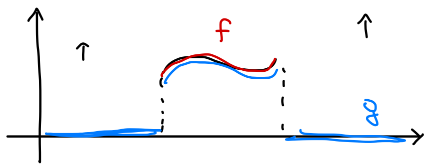
(1) f is not computable

$B_f = \emptyset$ $\qquad \overline{B_f} = \mathbb{N}$ $\qquad$ recursive (and thus r.e.)

(2) f is computable

* B_f is not r.e.

(2.a) <u>if f is finite</u> $\qquad f = \vartheta$



let g be a total function s.t.

$f \subseteq g$

(for instance $\qquad g(x) = \begin{cases} f(x) & \text{if } x \in dom(f) \\ 0 & \text{otherwise} \end{cases}$)

Then $g \notin \mathcal{B}_f$ but $f \subseteq g$ finite subfunction and $f \in \mathcal{B}_f$

hence by Rice - Shapiro B_f is not r.e.

(2.b) <u>f is not finite</u>

we have $f \in \mathcal{B}_f$ and for any $\vartheta \subseteq f$, ϑ finite $\vartheta \neq f$

hence $\vartheta \notin \mathcal{B}_f$. Hence, by Rice - Shapiro, B_f is not r.e.

* $\overline{B_f}$

* $f = \emptyset$ ($f(x)\uparrow \ \forall x$) $\rightsquigarrow$ $\overline{B_f}$ is r.e.

$e \in \overline{B_f}$ iff there is some x such that $\varphi_e(x)\downarrow$

some x, t such that $H(e, x, t)$

hence $sc_{\overline{B_f}}(e) = \mathbb{1}(\mu(x,t).\ H(e,x,t))$

$$= \mathbb{1}(\mu\omega.\ H(e, (\omega)_1, (\omega)_2))$$

$$= \mathbb{1}(\mu\omega.\ |\chi_H(e, (\omega)_1, (\omega)_2) - 1|)$$

computable.

Thus $\overline{B_f}$ r.e.. Not recursive since B_f not r.e. (hence not recursive)

* $f \neq \emptyset$

$\overline{B_f}$ is not r.e.

$f \notin \overline{B_f}$ and $\vartheta = \emptyset \subseteq f$ finite and $\emptyset \in \overline{B_f}$

hence by Rice-Shapiro, $\overline{B_f}$ is not r.e. (hence not recursive).

<u>EXERCISE</u>: show that $gcd : \mathbb{N}^2 \to \mathbb{N}$ is PR

$gcd(x,y)$ = greatest common divisor of x and y

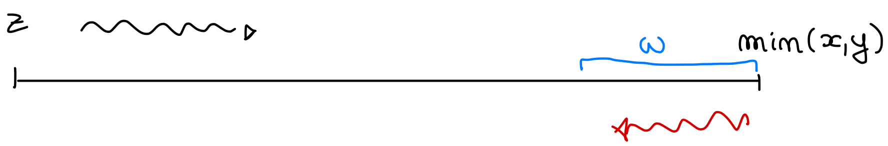
$$gcd(x,y) = \max \underset{\leq \, min(x,y)}{z}.\ \underbrace{z \text{ divisor of } x}_{rm(z,x) = 0} \text{ and } \underbrace{z \text{ divisor of } y}_{rm(z,y)=0}$$

$$= \max z \leq min(x,y).\ (rm(z,x) + rm(z,y) = 0)$$

$\max z \ldots P(z)$ not working

$\shortparallel$ $gcd(15, 30)$

$\min z \ldots \neg P(z+1)$ .... 3 $\cancel{4}$



$z = \min(x,y) - \omega$

$$\gcd(x,y) = \min(x,y) - \left(\mu \omega \leq \min(x,y).\ \left(z = \min(x,y) - \omega \text{ and } rm(z,x) + rm(z,y) = 0\right)\right)$$

$$= \underbrace{\min(x,y) \dot{-} \left(\mu \omega \leq \min(x,y).\ \underbrace{rm(\min(x,y) \dot{-} \omega, x) + \underbrace{rm\big(\underbrace{\underbrace{\min(x,y)}_{PR} \dot{-} \omega}_{PR}, y\big)}_{PR}}_{PR}\right)}_{PR}$$

hence gcd is in PR.

<u>EXERCISE</u> show there are $m, n \in \mathbb{N}$ s.t.

(i) $\varphi_n = \varphi_{n+1}$

(ii) $\varphi_m \neq \varphi_{m+1}$

(i) observe that $succ : \mathbb{N} \to \mathbb{N}$ $succ(n) = n+1$ is total and computable. Hence, by the II recursion theorem, there is n s.t. $\varphi_n = \varphi_{succ(n)} = \varphi_{n+1}$

(ii) if it were that $\forall m$ $\varphi_m = \varphi_{m+1}$, inductively you would have

$\varphi_0 = \varphi_1 = \varphi_2 \ldots$ $\forall n\ m$ $\varphi_n = \varphi_m$

which is false (e.g. $succ \neq id$).

EXERCISE : Define PR. Using only the definition show that

$max_2 : \mathbb{N} \to \mathbb{N}$

$max_2(x) = max\{2, x\}$ is in PR

(1) Rebuild max

sum $x+y$

$x+0 = x$
$x+(y+1) = (x+y)+1$

predecessor $y \dot{-} 1$

$0 \dot{-} 1 = 0$
$(y+1) \dot{-} 1 = y$

difference $x \dot{-} y$

$x \dot{-} 0 = x$
$x \dot{-} (y+1) = (x \dot{-} y) \dot{-} 1$

max $max(x,y) = x + (y \dot{-} x)$

$max_2(x) = max(2, x) = max(succ(succ(0)), x)$

(2) Proceed "on demand"

$max_2(0) = 2$

$max_2(y+1) = \begin{cases} 2 \ (1)+1 & \text{if } y=0 \\ y+1 & \text{if } y>0 \end{cases} = y+1 + \overline{sg}(y)$

$max(1,y)$

sum as above

$\overline{sg}(0) = 1$
$\overline{sg}(y+1) = 0$

even shorter ....

$$\begin{cases} max_2(0) = 2 \\ max_2(y+1) = \begin{cases} \overbrace{2}^{(1)+1} & \text{if } y=0 \\ \underbrace{y}_{max(1,y)}+1 & \text{if } y>0 \end{cases} = \underbrace{max(1,y)}_{max_1(y)} + 1 \end{cases}$$

$$\begin{cases} max_1(0) = 1 \\ max_1(y+1) = y+1 \end{cases}$$

EXERCISE :

Say $f: \mathbb{N} \to \mathbb{N}$ is monotone if f is total

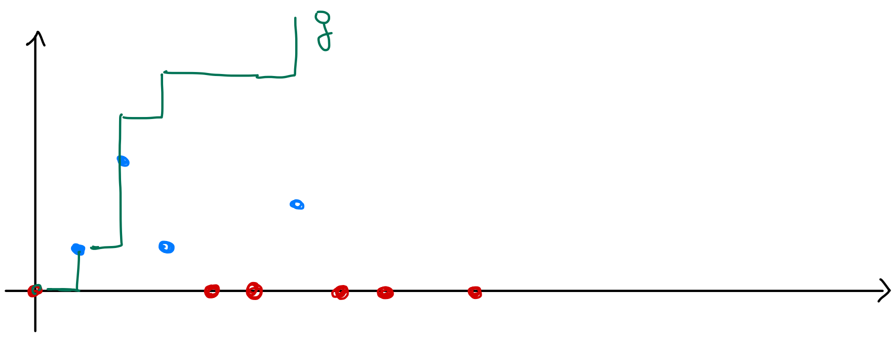
$\forall x,y \quad x \leq y$ then $f(x) \leq f(y)$

Question : is there a monotone, non computable function ?

Consider

$$f(x) = \begin{cases} \varphi_x(x)+1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

total, not computable



$$g(x) = \sum_{y \leq x} f(x)$$

→ g is total

→ g is not computable

$\forall x.$ if φ_x is total then

$$g(x) = \sum_{y \leq x} f(y) \geq f(x) = \varphi_x(x) + 1 \neq \varphi_x(x)$$

hence g total and different from all total computable functions, hence not computable.

- g is monotone

  if $x \leq y$ then

$$g(x) = \sum_{z \leq x} f(z) \leq \sum_{z \leq x} f(z) + \sum_{x < z \leq y} f(z)$$

$$= \sum_{z \leq y} f(z) = g(y)$$

* Alternative solution

$$g(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \text{ and } \varphi_x(x) \neq x+1 \\ x & \text{if } (\varphi_x(x)\downarrow \text{ and } \varphi_x(x) = x+1 \\ & \quad \text{or } \varphi_x(x)\uparrow) \end{cases}$$

- g is total
- g is not computable

  (total and different from all total computable function

  in fact, if $x \in \mathbb{N}$ s.t. φ_x is total

  - if $\varphi_x(x) = x+1$ then $g(x) = x \neq \varphi_x(x)$
  - if $\varphi_x(x) \neq x+1$ then $g(x) = x+1 \neq \varphi_x(x)$)

- g is monotone

  $\forall x, y$ s.t. $x < y$ then

$$g(x) \leq x+1 \leq y \leq g(y)$$

Even simpler

$$g(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{otherwise} \end{cases}$$

- g total
- g monotone
- g not computable

in fact $\quad \chi_K(x) = g(x) \dot{-} x = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$

hence g is not computable otherwise χ_K would be computable.

EXERCISE : show that there is $x \in \mathbb{N}$ s.t. $\varphi_x(y) = x \dot{-} y$

Define

$$g(x,y) = x \dot{-} y \qquad \text{computable}$$

hence by smn there is $s: \mathbb{N} \to \mathbb{N}$ total computable s.t. $\forall x,y$

$$\varphi_{s(x)}(y) = g(x,y) = x \dot{-} y$$

By second recursion theorem there is $x_0 \in \mathbb{N}$ s.t. $\varphi_{x_0} = \varphi_{s(x_0)}$

hence

$$\varphi_{x_0}(y) = \varphi_{s(x_0)}(y) = g(x_0, y) = x_0 \dot{-} y \quad .$$