COMPUTABILITY (18/11/2024)

UNIVERSAL FUNCTION

Def: Given $k \geq 1$ the universal function (for functions of arity k) is

$$\psi_U^{(k)} : \mathbb{N}^{k+1} \to \mathbb{N}$$

$$\psi_U^{(k)}(e, \vec{x}) = \varphi_e^{(k)}(\vec{x}) \qquad \text{(well-defined)}$$

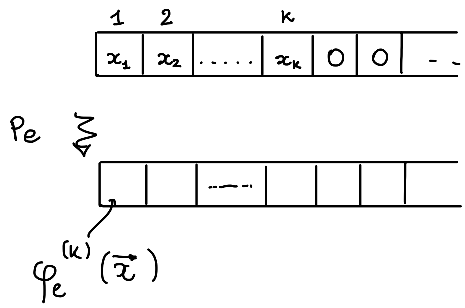
Theorem: $\psi_U^{(k)} : \mathbb{N}^{k+1} \to \mathbb{N}$ is computable

proof

given $e \in \mathbb{N}$ $\quad \vec{x} \in \mathbb{N}^k$

we want $\psi_U^{(k)}(e, \vec{x}) = \varphi_e^{(k)}(\vec{x})$

intuitive idea
→ get the program $P_e = \gamma^{-1}(e)$
→ execute P_e



→ configuration of the memory

r_1	r_2	r_3	. . . - - -	0	0

$$c = \prod_{i \geq 1} p_i^{r_i} \qquad r_i = (c)_i$$

$$c_k : \mathbb{N}^{k+2} \to \mathbb{N}$$

$$c_k(e, \vec{x}, t) = \begin{cases} \text{configuration of the memory after } t \text{ steps of } P_e(\vec{x}) \\ (\text{if } P_e(\vec{x})\downarrow \text{ in } t \text{ steps or fewer} \rightsquigarrow \text{final configuration}) \end{cases}$$

$$j_k : \mathbb{N}^{k+2} \to \mathbb{N}$$

$$j_k(e, \vec{x}, t) = \begin{cases} \text{index of the instruction to be executed after } t \text{ steps of } P_e(\vec{x}) \\ \text{if } P_e(\vec{x}) \text{ does not halt in } t \text{ steps or fewer} \\ 0 \qquad \text{otherwise} \end{cases}$$

Observe

→ if $P_e(\vec{x})\downarrow$ then it halts in $t_0 = \mu t.\, J_K(e,\vec{x},t)$ steps

hence $\varphi_e^{(K)}(\vec{x}) = \left(c_K(e,\vec{x},\mu t.\, J_K(e,\vec{x},t))\right)_1$

→ if $P_e(\vec{x})\uparrow$ then $\mu t.\, J_K(e,\vec{x},t)\uparrow$

$$\varphi_e^{(K)}(\vec{x}) = \left(c_K(e,\vec{x},\mu t.\, J_K(e,\vec{x},t))\right)_1 \uparrow$$

Therefore

$$\psi_U^{(K)}(e,\vec{x}) = \varphi_e^{(K)}(\vec{x}) = \left(c_K(e,\vec{x},\mu t.\, J_K(e,\vec{x},t))\right)_1$$

and thus if c_K, J_K computable ⇝ $\psi_U^{(K)}$ computable

$\boxed{\text{AIM: } c_K, J_K \text{ computable}}$

* given $i \in \mathbb{N}$ instruction code i.e. $i = \beta(\text{Instr})$

$zarg(i) = qt(4,i)+1$ $\qquad i = \beta(Z(n)) = 4*(n-1)$

$sarg(i) = qt(4,i)+1$ $\qquad i = \beta(S(n)) = 4*(n-1)+1$

$Targ_1(i) = \pi_1(qt(4,i))+1$ $\qquad i = \beta(T(m,n)) = 4*\pi(m-1,n-1)+2$

$Targ_2(i) = \pi_2(qt(4,i))+1$

$Jarg_i \quad i=1,2,3$

computable

* effect of executing algebraic instruction on a configuration

$zero(c,n) = qt(p_n^{(c)_n}, c)$

r_1	r_2	r_3	...	r_n

$succ(c,n) = p_n \cdot c$

$c = p_1^{r_1}\cdot p_2^{r_2} \cdots\cdots p_n^{r_n} \cdots$

$(c)_n$

$transf(c,m,n) = zero(c,n)\cdot p_n^{(c)_m}$

computable

* effect of executing instruction with code i in configuration c

$$change(c,i) = \begin{cases} zero(c, zarg(i)) & \text{if } rm(4,i) = 0 \\ succ(c, sarg(i)) & \text{if } rm(4,i) = 1 \\ transf(c, Targ_1(i), Targ_2(i)) & \text{if } rm(4,i) = 2 \\ c & \text{if } rm(4,i) = 3 \end{cases}$$

computable

* configuration of registers from configuration c and executing instruction at line t in program P_e

$$\text{nextconf}(e, c, t) = \begin{cases} \text{change}(c, a(e,t)) & \text{if } 1 \le t \le \ell(e) \\ c & \text{otherwise} \end{cases}$$

computable

* number of the next instruction to be executed after $i = \beta(\text{Instr})$ and Instr is at line t in the program

$$\text{ni}(c, i, t) = \begin{cases} t+1 & \text{if } rm(4,i) \neq 3 \text{ or } (rm(4,i) = 3 \text{ and } (c)_{\text{Jarg}_1(i)} \neq (c)_{\text{Jarg}_2(i)}) \\ \text{Jarg}_3(i) & \text{otherwise} \end{cases}$$

computable

* next instruction to be executed, if we execute instruction at line t of P_e in configuration c

$$\text{next instr}(e, c, t) = \begin{cases} \text{ni}(c, a(e,t), t) & \text{if } 1 \le t \le \ell(e) \text{ and } 1 \le \text{ni}(c, a(e,t), t) \le \ell(e) \\ 0 & \text{otherwise} \end{cases}$$

computable

Now

$$c_k(e, \vec{x}, 0) = \prod_{i=1}^{k} p_i^{x_i}$$

1		k		
x_1	...	x_k	0	...

$$j_k(e, \vec{x}, 0) = 1$$

$$c_k(e, \vec{x}, t+1) = \text{nextconf}(e, c_k(e, \vec{x}, t), j_k(e, \vec{x}, t))$$

$$j_k(e, \vec{x}, t+1) = \text{next inst}(e, c_k(e, \vec{x}, t), j_k(e, \vec{x}, t))$$

c_k, j_k defined by primitive recursion from computable functions (actually primitive recursive) and so they are computable (actually primitive recursive)

Now

$$\varphi_U^{(k)}(e,\vec{x}) = \left(c_k(e,\vec{x}, \mu t.\, J_P(e,\vec{x},t)) \right)_1$$

computable

$\square$

<u>Corollary</u>: The following predicates are <u>decidable</u>

(a) $H_k(e,\vec{x},t) \equiv$ " $P_e(\vec{x})\downarrow$ in t steps or fewer "

(b) $S_k(e,\vec{x},y,t) \equiv$ " $P_e(\vec{x})\downarrow y$ in t steps or fewer "

<u>proof</u>

(a) $\chi_{H_k} : \mathbb{N}^{k+2} \to \mathbb{N}$

$$\chi_{H_k}(e,\vec{x},t) = \begin{cases} 1 & \text{if } H_k(e,\vec{x},t) \\ 0 & \text{otherwise} \end{cases}$$

$$= \overline{sg}\left(J_k(e,\vec{x},t) \right)$$

$\downarrow$

$\neq 0$ if $P_e(\vec{x})\not\downarrow$ in t steps

$= 0$ if $P_e(x)\downarrow$ in t steps

computable by composition.

(b)

$\chi_{S_k}(e,\vec{x},y,t)$

$$= \underbrace{\chi_{H_k}(e,\vec{x},t)}_{\substack{1 \text{ if } P_e(\vec{x})\downarrow \\ 0 \text{ otherwise}}} \cdot \underbrace{\overline{sg}\left(\underbrace{|y - (c_k(e,\vec{x},t))_1|}_{\substack{=0 \text{ if } y = \text{result} \\ \neq 0 \text{ otherwise}}} \right)}_{\substack{1 \text{ if } y \text{ is } = \text{result} \\ 0 \text{ otherwise}}}$$

computable by composition

$\varphi_e^{(k)}, W_e^{(k)}, E_e^{(k)} \rightsquigarrow \varphi_e, W_e, E_e$ if $k=1$

$H_k, S_k \rightsquigarrow H, S$ if $k=1$

EXERCISE : Computability of the inverse (reprise)

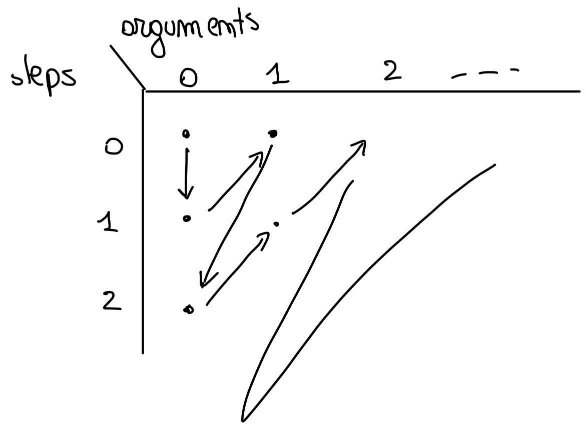
Let $f : \mathbb{N} \to \mathbb{N}$ total computable injective

then
$$f^{-1}(y) = \begin{cases} x & x \text{ s.t. } f(x) = y \text{ , if it exists} \\ \uparrow & \text{otherwise} \end{cases}$$

is computable

$$f^{-1}(y) = \mu x. \, |f(x) - y|$$

without totality?



try t steps
over argument x
for all possible t, x

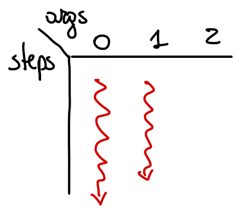
if $y = f(x)$

f is computable $\iff$ there is $e \in \mathbb{N}$ s.t. $f = \varphi_e$

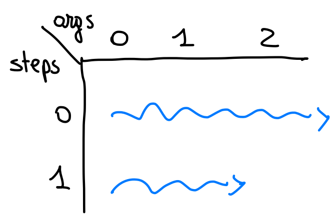
look for input x s.t. $\underbrace{\varphi_e(x)\downarrow y \text{ in } t \text{ steps}}_{S(e,x,y,t)}$
number of steps t

$\uparrow$
minimalisation

$$f^{-1}(y) = \mu x. \cancel{\mu t. S(e, x, y, t)}$$



$$= \mu t. \cancel{\mu x. S(e, x, y, t)}$$



$$f^{-1}(y) = \pi_1\left(\mu w.\ \ S(e, \pi_1(w), y, \pi_2(w))\right)$$

$$\pi^{-1}(w) = (\underbrace{\pi_1(w)}_{x}, \underbrace{\pi_2(w)}_{t})$$

ok

but we do it differently

$$f^{-1}(y) = \text{"}\left(\mu w.\ \ S(e, (w)_1, y, (w)_2)\right)_1\text{"}$$

$$w \to (w)_1, (w)_2$$

$w = 3 = 2^0 \cdot 3^1 \ldots \to (0, 1)$

$w = 6 = 2^1 \cdot 3^1 \ldots \to (1, 1)$

$w = 30 = 2^1 \cdot 3^1 \cdot 5^1 \ldots \to (1, 1)$

$$= \left(\mu w.\ \ |\chi_S(e, (w)_1, y, (w)_2) - 1|\right)_1$$

computable

$\square$

OBSERVATION: function which is total and not computable

$$f(x) = \begin{cases} \boxed{\varphi_x(x) + 1} & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{if } \varphi_x(x)\uparrow \end{cases}$$

ok

PROBLEM

$$= \begin{cases} \Psi_U(x, x) + 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{if } \varphi_x(x)\uparrow \end{cases}$$

OBSERVATION: (Halting problem) The following predicate

$$Halt(x) = \begin{cases} true & \text{if } \varphi_x(x)\downarrow \\ false & \text{if } \varphi_x(x)\uparrow \end{cases}$$

is UNDECIDABLE

[EXERCISE]