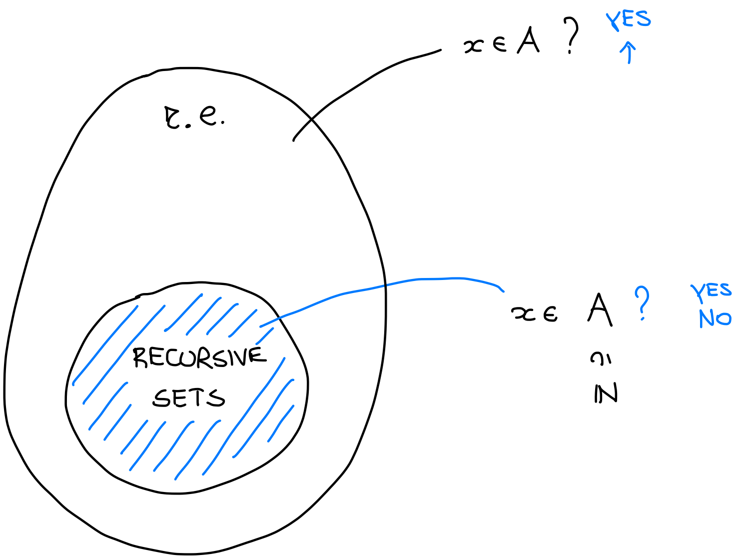
COMPUTABILITY (28/11/2023)



RECURSIVELY ENUMERABLE SETS

Def (r.e. set) : A set $A \subseteq \mathbb{N}$ is recursively enumerable (r.e.) if the semi-characteristic function $sc_A : \mathbb{N} \to \mathbb{N}$

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

is computable

A property $Q(\vec{x}) \subseteq \mathbb{N}^k$ is semidecidable if

$sc_Q : \mathbb{N}^k \to \mathbb{N}$

$$sc_Q(\vec{x}) = \begin{cases} 1 & \text{if } Q(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases}$$

computable

Note : if $Q(x) \subseteq \mathbb{N}$

$Q(x)$ semidecidable iff $\{x \mid Q(x)\}$ r.e.

(we could define also recursive / r.e. sets $A \subseteq \mathbb{N}^k$)

OBSERVATION : Let $A \subseteq \mathbb{N}$ be a set

A recursive $\iff$ $A, \bar{A}$ r.e.

proof

($\Longrightarrow$) let $A \subseteq \mathbb{N}$ be recursive, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \quad \text{is computable}$$

we want to show A r.e., i.e.

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases} \quad \text{is computable}$$

intuitively:

you have P_{χ_A} for "$x \in A$" $\nearrow 1$, $\searrow 0$ $\leadsto$

def $sc_A(x)$:
    if $P_{\chi_A}(x) = 1$:
        return 1
    else
        loop

formally:

$$sc_A(x) = \mathbb{1}\Big(\mu w. \underbrace{\underbrace{|\chi_A(x) - 1|}_{\substack{0 \text{ if } x \in A \\ 1 \text{ if } x \notin A}}}_{\substack{0 \text{ if } x \in A \\ \uparrow \text{ otherwise}}}\Big)$$

computable since it is composition, minimalisation of computable functions.

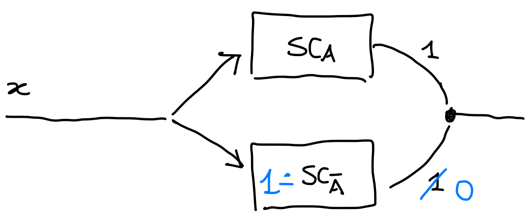
hence A r.e.

Concerning $\bar{A}$, note that since A recursive also $\bar{A}$ recursive

Hence by the argument above $\bar{A}$ is r.e..

($\Longleftarrow$) let $A, \bar{A}$ be r.e., i.e. the semi-characteristic functions are computable

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases} \qquad 1 \dot{-} sc_{\bar{A}}(x) = \begin{cases} \cancel{1}\ 0 & \text{if } x \in \bar{A} \\ \uparrow & \text{otherwise} \end{cases}$$



let $e_1, e_0 \in \mathbb{N}$ s.t. $sc_A = \varphi_{e_1}$ and $1 \dot{-} sc_{\bar{A}} = \varphi_{e_0}$

idea "$\big(\mu(y,t).\ S(e_1, x, y, t) \vee S(e_0, x, y, t)\big)_{\downarrow y}$"

formally

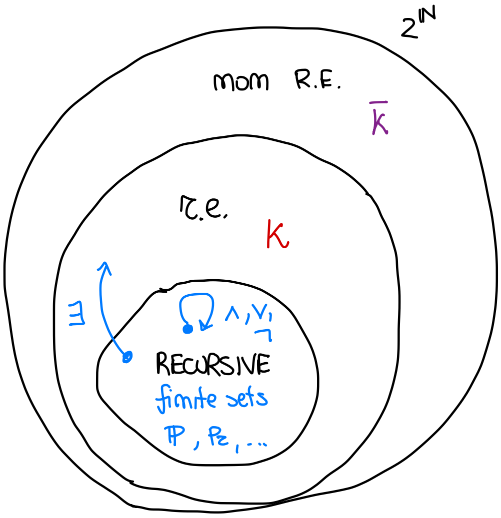
$$\chi_A(x) = \big(\underset{\substack{\uparrow \\ (w)_1 = y \quad (w)_2 = t}}{\mu w}.\ S(e_1, x, (w)_1, (w)_2) \vee S(e_0, x, (w)_1, (w)_2)\big)_1$$

$$= \Big(\mu w\ \overline{sg}\big(\max\big(\chi_S(e_1, x, (w)_1, (w)_2),\ \chi_S(e_0, x, (w)_1, (w)_2)\big)\big)\Big)_1$$

computable.

Hence A is recursive

□



* K not recursive, it is r.e.

$$sc_K(x) = \begin{cases} 1 & \text{if } x \in K \quad (\varphi_x(x)\downarrow) \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}(\varphi_x(x))$$

$$= \mathbb{1}(\Psi_U(x, x))$$

* $\bar{K}$ is not r.e.

otherwise if $\bar{K}$ r.e., since K r.e. we would have K recursive

* Existential quantification

$Q(t, \vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable

$P(\vec{x}) \equiv \exists t.\ Q(t, \vec{x})$ semi-decidable

STRUCTURE THEOREM

Let $P(\vec{x}) \subseteq \mathbb{N}^k$ a predicate

$P(\vec{x})$ semi-decidable $\iff$ there is $Q(t,\vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable s.t. $P(\vec{x}) = \exists t. Q(t,\vec{x})$

proof

($\Rightarrow$) let $P(\vec{x}) \subseteq \mathbb{N}^k$ be semi-decidable

$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases} \quad \text{is computable}$$

i.e. there is $e \in \mathbb{N}$ s.t. $sc_P = \varphi_e^{(k)}$

Observe $\boxed{P(\vec{x})}$ iff $sc_P(\vec{x}) = 1$

iff $sc_P(\vec{x})\downarrow$

iff $\varphi_e(\vec{x})\downarrow$

iff $\boxed{\exists t. \ H^{(k)}(e,\vec{x},t)}$

If we let $Q(t,\vec{x}) = H^{(k)}(e,\vec{x},t)$ decidable and

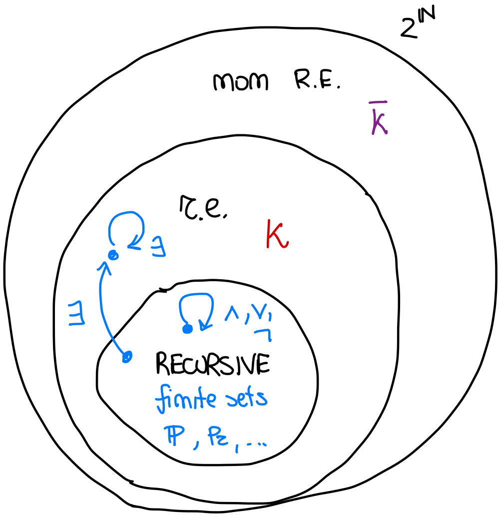
$$P(\vec{x}) \equiv \exists t. Q(t,\vec{x})$$

($\Leftarrow$) We assume $P(\vec{x}) \equiv \exists t. Q(t,\vec{x})$ with $Q(t,\vec{x})$ decidable

$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \iff \exists t. Q(t,\vec{x}) \iff \exists t. \ \chi_Q(t,\vec{x}) = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}\big(\underbrace{\mu t. \ |\chi_Q(t,\vec{x}) - 1|}_{\substack{t \text{ s.t. } Q(t,\vec{x}) \text{ if it exists} \\ \uparrow \text{ otherwise}}}\big)$$

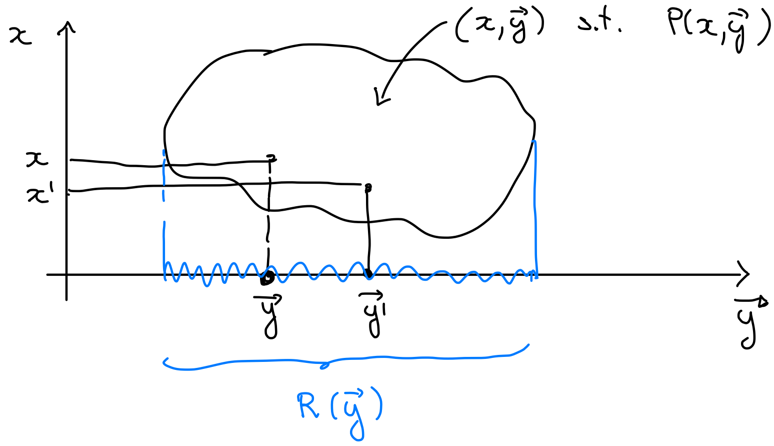
$\square$



Projection Theorem

Let $P(x,\vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable

Then $R(\vec{y}) \equiv \exists x.\ P(x,\vec{y})$ is semi-decidable



proof

Let $P(x,\vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable. Hence, by structure th., there is $Q(t,x,\vec{y}) \subseteq \mathbb{N}^{k+2}$ decidable s.t.

$$P(x,\vec{y}) \equiv \exists t.\ Q(t,x,\vec{y})$$

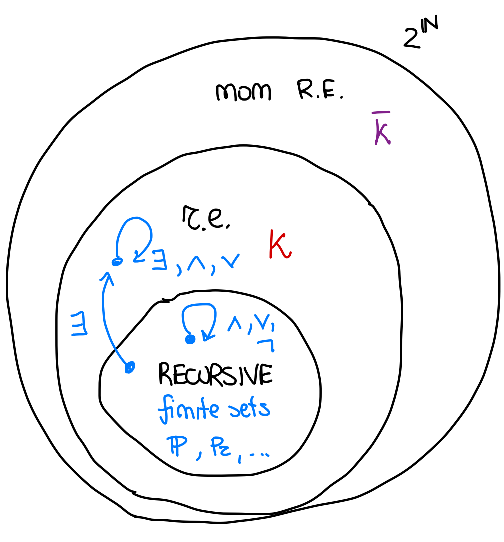
Now

$$\boxed{R(\vec{y}) \equiv} \ \exists x.\ P(x,\vec{y}) \equiv \exists x.\ \exists t.\ Q(t,x,\vec{y})$$

$$\boxed{\equiv \exists \omega.\ \underbrace{Q((\omega)_1, (\omega)_2, \vec{y})}_{\text{decidable}}}$$

Hence R is the existential quantification of a decidable predicate

$\Rightarrow$ by structure th. it is semi-decidable $\square$



Conjunction / Disjunction

Let $P(\vec{x}), Q(\vec{x}) \subseteq \mathbb{N}^k$ semi-decidable predicates. Then

① $P(\vec{x}) \wedge Q(\vec{x})$

② $P(\vec{x}) \vee Q(\vec{x})$

semi-decidable

proof

Since $P(\vec{x}), Q(\vec{x})$ are semi-decidable, by structure theorem

$$P(\vec{x}) \equiv \exists t.\ P'(t,\vec{x})$$
$$Q(\vec{x}) \equiv \exists t.\ Q'(t,\vec{x})$$

with $P'(t,\vec{x}), Q'(t,\vec{x})$ decidable

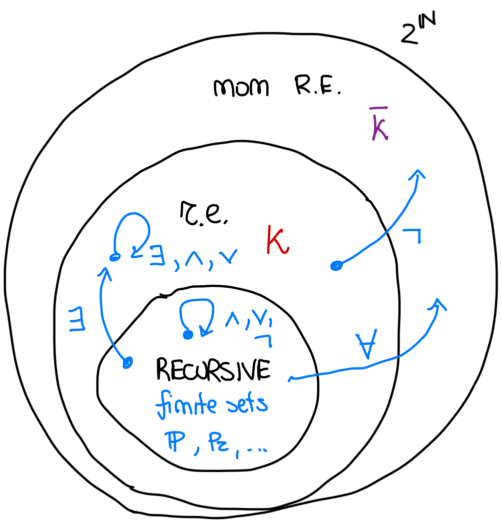
Then

① $$P(\vec{x}) \wedge Q(\vec{x}) \equiv \exists t.\ P'(t,\vec{x}) \wedge \exists t.\ Q'(t,\vec{x})$$
$$\equiv \exists \omega.\ (\underbrace{P'((\omega)_1,\vec{x}) \wedge Q'((\omega)_2,\vec{x})}_{\text{decidable}})$$

hence, by the structure theorem, $P(\vec{x}) \wedge Q(\vec{x})$ is semi-decidable

② $$P(\vec{x}) \vee Q(\vec{x}) \equiv \exists t.\ P'(t,\vec{x}) \vee \exists t.\ Q'(t,\vec{x})$$
$$\equiv \exists t.\ \underbrace{\left(P'(t,\vec{x}) \vee Q'(t,\vec{x})\right)}_{\text{decidable}}$$

hence, by the structure theorem, $P(\vec{x}) \vee Q(\vec{x})$ is semi-decidable. □



* **Negation** ?

$Q(x) \equiv$ "$x \in K$" $\equiv$ "$\varphi_x(x)\downarrow$"

semi-decidable

$\neg Q(x) \equiv$ "$x \notin K$" $\equiv$ "$\varphi_x(x)\uparrow$"

not semi-decidable

* <u>Universal quantification</u>

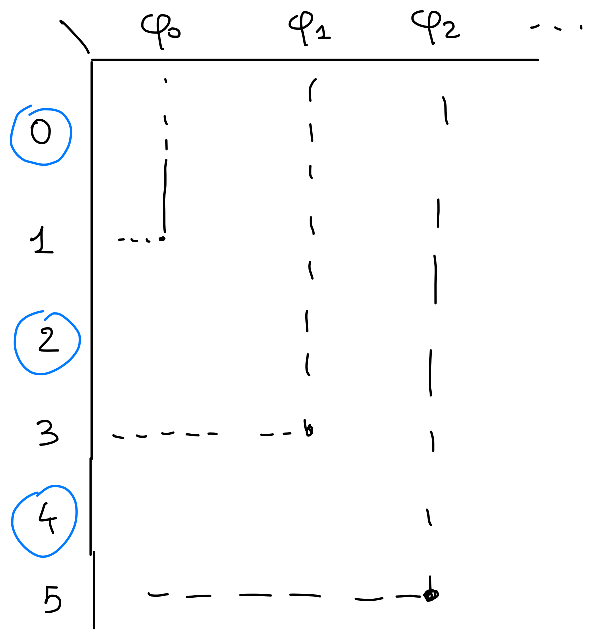
$R(t, x) \equiv \neg H(x, x, t)$ decidable

"$x \in \overline{K}$" $\equiv \forall t.\ R(t, x) \equiv \forall t.\ \neg H(x, x, t)$ non semi-decidable.

<u>EXERCISE</u>: Define a function total and non-computable $f: \mathbb{N} \to \mathbb{N}$

s.t. $f(x) = x$ on infinitely many $x \in \mathbb{N}$

<u>1st idea</u>



$$f(x) = \begin{cases} x & \text{if } x \text{ is even} \\ \varphi_{\frac{x-1}{2}}(x) + 1 & \text{if } x \text{ is odd and } \varphi_{\frac{x-1}{2}}(x)\downarrow \\ 0 & \text{if } x \text{ is odd and } \varphi_{\frac{x-1}{2}}(x)\uparrow \end{cases}$$

- f total
- $f(x) = x \quad \forall x$ even (infinite set)
- f not computable (total and $\neq$ from all total computable functions)
  ($\forall x$ if φ_x is total $\quad f(2x+1) = \varphi_x(2x+1) + 1 \neq \varphi_x(2x+1)$)

<u>2nd idea</u>

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \varphi_x(x)\downarrow \\ x & \varphi_x(x)\uparrow \end{cases}$$

- total

- not computable ($\forall x$ if φ_x is total $f(x) = \varphi_x(x)+1 \neq \varphi_x(x)$)

  hence f is different from all total computable functions

- $f(x) = x$ $\forall x \in \bar{K}$ ($\bar{K}$ is infinite, otherwise it would be recursive)

<u>3rd idea</u>

$$f(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{otherwise} \end{cases}$$

- f total

- $f(x) = x$ $\forall$ $x \in \bar{K}$

- f not computable [EXERCISE]

<u>EXERCISE</u>: If f is computable

and g coincides with f almost everywhere (except for a finite set of inputs)

then g is computable.