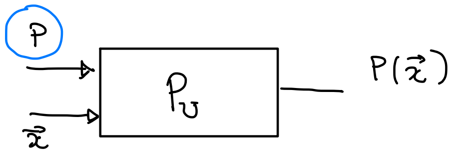
COMPUTABILITY (31/10/2023)



* Enumeration of URM programs

set X countable if $|X| \leq |\mathbb{N}|$

i.e. there is $f : \mathbb{N} \to X$ surjective (enumeration)

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad \dots\dots}_{X}$$

if f is also injective then it is called bijective enumeration

f effective

Lemma: there are bijective enumerations of
effective

1. $\mathbb{N}^2$
2. $\mathbb{N}^3$
3. $\bigcup_{k \geq 1} \mathbb{N}^k$

(1) $\pi : \mathbb{N}^2 \to \mathbb{N}$

$$\pi(x,y) = \underbrace{2^x (2y+1) - 1}_{n} \qquad \text{[computable]}$$

$\pi^{-1} : \mathbb{N} \to \mathbb{N}^2$ [effective]

$$\pi^{-1}(n) = (\pi_1(n), \pi_2(n))$$

$\pi_1, \pi_2 : \mathbb{N} \to \mathbb{N}$ $\qquad \pi_1(n) = (n+1)_1$

computable

$$\pi_2(n) = \left(\frac{(n+1)}{2^{\pi_1(n)}} / 2\right) \dot{-} 1$$

$$= qt(2, qt(2^{\pi_1}, n+1)) \dot{-} 1$$

(2) $\nu : \mathbb{N}^3 \to \mathbb{N}$

$$\nu(x,y,z) = \pi(x, \pi(y,z))$$

$\nu^{-1} : \mathbb{N} \to \mathbb{N}^3$

$\nu^{-1}(n) = (\nu_1(n), \nu_2(n), \nu_3(n))$

$\nu_1(n) = \pi_1(n)$

$\nu_2(n) = \pi_1(\pi_2(n))$

$\nu_3(n) = \pi_2(\pi_2(n))$

(3) $\tau : \bigcup_{k \geq 1} \mathbb{N}^k \to \mathbb{N}$

$$\tau(x_1, \ldots, x_k) = \prod_{i=1}^{k} p_i^{x_i} \dot{-} 1$$

not injective

$(1,0) \to p_1^1 \cdot p_2^0 \dot{-} 1 = 2^1 \cdot 3^0 \dot{-} 1 = 1$

$(1,0,0) \to p_1^1 \cdot p_2^0 \cdot p_3^0 = 2^1 \cdot 3^0 \cdot 5^0 \dot{-} 1 = 1$

$$\tau(x_1, \ldots, x_k) = \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) \cdot p_k^{x_k+1} \dot{-} 2$$

$\tau^{-1} : \mathbb{N} \to \bigcup_{k \geq 1} \mathbb{N}^k$

$a : \mathbb{N}^2 \to \mathbb{N}$

$$\tau^{-1}(n) = \underbrace{(a(n,1) \quad a(n,2) \quad \ldots \quad a(n, \ell(n)))}_{\ell(n)} \qquad \ell : \mathbb{N} \to \mathbb{N}$$

$$n = \tau(\ldots) = \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) \cdot p_k^{x_k+1} \dot{-} 2$$

$k = \ell(n)$

$\ell(n) =$ largest k such that p_k divides $n+2$

(computable, exercise)

$$a(n,i) = \begin{cases} (n+2)_i & i < \ell(n) \\ (n+2)_i \dot{-} 1 & i = \ell(n) \end{cases}$$

$\ell : \mathbb{N} \to \mathbb{N}$, $a : \mathbb{N}^2 \to \mathbb{N}$ computable

<u>OBSERVATION</u> : Let $\mathcal{P}$ the set of URM programs.

There is an "effective" enumeration which is bijective

$$\gamma : \mathcal{P} \to \mathbb{N}$$

Let $\mathcal{J} = \{ Z(n), S(n), T(m,n), J(m,n,t) \ : \ m, n, t \geq 1 \}$

we consider

$$\beta : \mathcal{J} \to \mathbb{N}$$

Z(0)	S(0)	T	J	Z(1)	S(1)	T	J	Z(2)	S(2)	T	...
0	1	2	3	4	5	6	7	8	9	10	...

$$\beta(Z(n)) = 4 * (n \dot{-} 1)$$

$$\beta(S(n)) = 4 * (n \dot{-} 1) + 1$$

$$\beta(T(m,n)) = 4 * \pi(m-1, n-1) + 2$$

$$\beta(J(m,n,t)) = 4 * \nu(m-1, n-1, t-1) + 3$$

$\beta^{-1} : \mathbb{N} \to \mathcal{J}$ $\qquad\qquad x \rightsquigarrow$ $r = rm(4, x)$
$q = qt(4, x)$

$$\beta^{-1}(x) = \begin{cases} Z(q+1) & r = 0 \\ S(q+1) & r = 1 \\ T(\pi_1(q)+1, \pi_2(q)+1) & r = 2 \\ J(\nu_1(q)+1, \nu_2(q)+1, \nu_3(q)+1) & r = 4 \end{cases}$$

Given program $P \in \mathcal{P}$ URM program

$$P = \begin{cases} I_1 \\ I_2 \\ \vdots \\ I_s \end{cases} \qquad \gamma(P) = \tau(\beta(I_1) \ \ldots \ \beta(I_s))$$

inverse : $\gamma^{-1} : \mathbb{N} \to \mathcal{P}$

$$\gamma^{-1}(n) = P = \begin{cases} I_1 \\ \vdots \\ I_{\ell(n)} \end{cases} \qquad I_i = \beta^{-1}(a(n,i))$$

* γ fixed enumeration of URM programs

$\gamma(P)$ (Gödel) number of P

given n $\qquad P_n = \gamma^{-1}(n)$

<u>Example</u>

* P

$$\begin{cases} T(1,2) \rightsquigarrow 4 * \pi(1-1,2-1) + 2 = 4 * \underbrace{\pi(0,1)}_{\underbrace{2^0(2\cdot 1+1)-1}_{2}} + 2 = 10 \\ S(2) \rightsquigarrow 4 * (2-1) + 1 = 5 \\ T(2,1) \rightsquigarrow \text{———} \quad \text{——— — —} = 6 \end{cases}$$

$$\gamma(P) = \tau(10 \quad 5 \quad 6)$$
$$= p_1^{10} \cdot p_2^{5} \cdot p_3^{6+1} - 2 = 2^{10} \cdot 3^5 \cdot 5^7 - 2$$
$$= 19\,439\,999\,998$$

* P' $\qquad S(1)$

$$\gamma(P') = \tau(\beta(S(1))) = \tau(4*(1-1)+1)$$
$$= \tau(1) = p_1^{1+1} - 2 = 2^2 - 2 = 2$$

* given $n = 100$

what is P_{100} ?
$\qquad\quad \| $
$\qquad \gamma^{-1}(100)$

$$\underbrace{\left(\prod_{i=1}^{k-1} p_i^{x_i}\right) \cdot p_k^{x_k+1} \dot{-} 2}_{n}$$

$$n + 2 = 100 + 2 = 2^1 \cdot 3^1 \cdot 17^1$$
$$= p_1^1 \cdot p_2^1 \cdot p_3^0 \cdot p_4^0 \cdot p_5^0 \cdot p_6^0 \cdot p_7^1$$

$\ell(100) = 7$

I_1	$\beta^{-1}(1)$	$S(1)$
I_2	$\beta^{-1}(1)$	$S(1)$
I_3	$\beta^{-1}(0)$	$Z(1)$
I_4	$\beta^{-1}(0)$	$Z(1)$
I_5	$\beta^{-1}(0)$	$Z(1)$
I_6	$\beta^{-1}(0)$	$Z(1)$
I_7	$\beta^{-1}(1-1)$	$Z(1)$

* Fixed $\gamma : \mathcal{P} \to \mathbb{N}$

this induces an enumeration of the computable functions

$\varphi_n^{(k)} : \mathbb{N}^k \to \mathbb{N}$ function of k arguments computed by $\gamma^{-1}(n) = P_n$

$f_{P_n}^{(k)}$

$$W_n^{(k)} = dom(\varphi_n^{(k)}) = \{ \vec{x} \in \mathbb{N}^k \mid \varphi_n^{(k)}(\vec{x}) \downarrow \} \subseteq \mathbb{N}^k$$

$$E_n^{(k)} = cod(\varphi_n^{(k)}) = \{ \varphi_n^{(k)}(\vec{x}) \mid \vec{x} \in W_n^{(k)} \} \subseteq \mathbb{N}$$

When $k = 1$ we omit it

φ_n for $\varphi_n^{(1)}$

<u>Example</u> : $\varphi_{100} : \mathbb{N} \to \mathbb{N}$

$\varphi_{100}(x) = 0 \quad \forall x \in \mathbb{N}$

$W_{100} = \mathbb{N} \qquad E_{100} = \{0\}$

$$\underbrace{\varphi_0 \quad \varphi_1 \quad \underset{\text{successor}}{\overset{\text{successor}}{\varphi_2}} \quad \varphi_3 \quad \varphi_4 \quad \dots \quad \varphi_{19\,439\,999\,998}}_{\text{enumeration of all unary computable functions}}$$

↑ there are repetitions (infinite) (not injective)

$|\mathcal{C}^{(1)}| \leq |\mathbb{N}|$

$|\mathcal{C}^{(k)}| \leq |\mathbb{N}| \quad \forall k$

$\mathcal{C} = \bigcup_{k \geq 1} \mathcal{C}^{(k)}$ denumerable $|\mathcal{C}| \leq |\mathbb{N}|$

Exercise : $\mathcal{R}$ partial recursive function

least rich class , i.e.

→ includes basic functions

→ closed under
- composition
- primitive recursion
- minimalisation

Originally defined by Gödel - Kleene $\mathcal{R}_0$

least class

→ includes basic functions

→ closed under
- composition
- primitive recursion
- minimalisation used only when result is total

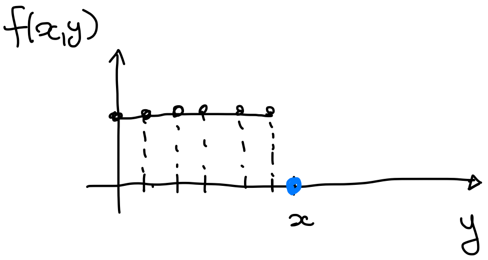
$$\mathcal{R}_0 \subseteq \underbrace{\mathcal{R} \cap Tot}$$

$$? \supseteq$$

not obvious since one can obtain total functions from partial ones

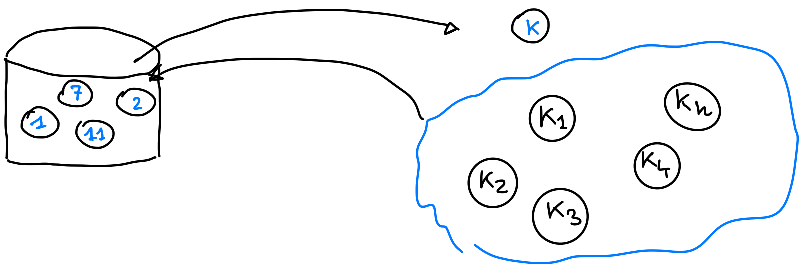
$f : \mathbb{N}^2 \to \mathbb{N}$

$$f(x,y) = \begin{cases} 1 & y < x \\ 0 & y = x \\ \uparrow & \text{otherwise} \end{cases}$$



$$h(x) = \mu y.\ f(x,y) = x$$

Exercise :



$K_1, \dots, K_n < K$

→ Does this process terminate ? Why ?