COMPUTABILITY (17/10/2023)

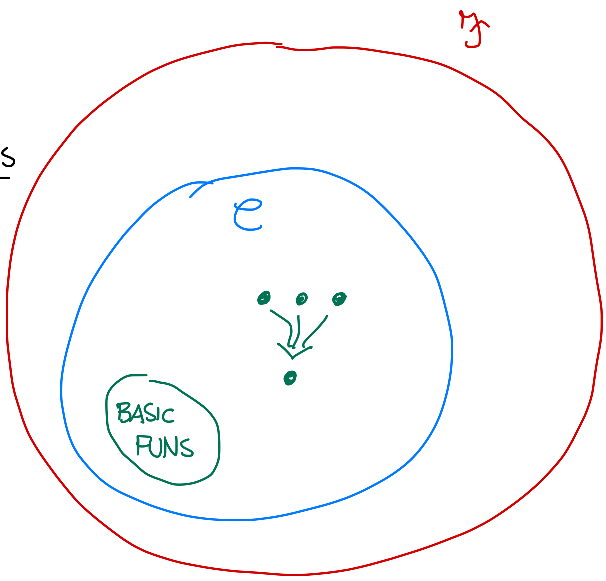
* Generation of computable functions

$\mathcal{C}$ closed under

→ composition

→ primitive recursion

→ unbounded minimalisation



$\vec{x} = (x_1, \dots, x_k)$

* BASIC FUNCTIONS

① constant zero $\quad z : \mathbb{N}^k \to \mathbb{N} \quad z(\vec{x}) = 0 \quad \forall \vec{x} \in \mathbb{N}^k$

② successor $\quad S : \mathbb{N} \to \mathbb{N} \quad S(x) = x+1 \quad \forall x \in \mathbb{N}$

③ projection $\quad U_j^k : \mathbb{N}^k \to \mathbb{N} \quad U_j^k(\vec{x}) = x_j \quad \forall x \in \mathbb{N}^k$

They are in $\mathcal{C}$ as they are computed by

① $Z(1)$

② $S(1)$

③ $T(j, 1)$

* Notation :

given a program P

- $\rho(P) = \max\{ n \mid \text{register } R_n \text{ is referred in } P \}$
- $\ell(P) =$ length of P
- P is in standard form if whenever it terminates it does at instruction $\ell(P) + 1$

- concatenation : given P, Q programs

$$\begin{matrix} P \\ Q \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} P \\ Q' \end{matrix} \quad \leftarrow \text{update } J(m,n,t) \text{ with } J(m,n,t+\ell(P))$$

- given P a program we write

$$P[i_1, \dots, i_k \to i] \qquad (*)$$

program taking the input from $R_{i_1}, \dots, R_{i_k}$ and outputs in R_i without assuming registers different from the input are set to 0

$$\left.\begin{array}{l} T(i_1, 1) \\ \vdots \\ T(i_k, k) \\ Z(k+1) \\ \vdots \\ Z(\rho(P)) \\ P \\ T(1, i) \end{array}\right\} (*)$$

problem $P[2, 1 \to 1]$

1	2
x	y

you want $\rightsquigarrow$

y	x

$T(2, 1)$
$T(1, 2)$
P
$\vdots$

you get

y	y

EXERCISE : write $(*)$ properly

* COMPOSITION

Given $f: \mathbb{N}^k \to \mathbb{N}$, $g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$

you define $h: \mathbb{N}^m \to \mathbb{N}$ for $\vec{x} \in \mathbb{N}^m$

$$h(\vec{x}) = \begin{cases} f(g_1(\vec{x}), \dots, g_k(\vec{x})) & \text{if } g_1(\vec{x})\downarrow, \dots, g_k(\vec{x})\downarrow \text{ and } f(g_1(\vec{x}), \dots, g_k(\vec{x}))\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

E.g. $z(x) = 0 \quad \forall x$

$\phi(x)\uparrow \quad \forall x \qquad\qquad z(\phi(x))\uparrow \quad \forall x$

$U_1^2(x, y) = x \qquad\qquad U_1^2(x, \phi(y))\uparrow \quad \forall x, y$

Proposition: $\mathcal{C}$ is closed under (generalised) composition

proof

Given $f : \mathbb{N}^k \to \mathbb{N}$, $g_1, \dots, g_k : \mathbb{N}^n \to \mathbb{N}$ in $\mathcal{C}$

then $h : \mathbb{N}^n \to \mathbb{N}$

$$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x})) \quad \text{is in } \mathcal{C}$$

Let $F, G_1, \dots, G_k$ be programs (in std form) for $f, g_1, \dots, g_k$

The program for h can be

1		n		m	m+1		m+n	m+n+1		m+n+k
x_1	...	x_n			x_1	..	x_n	$g_1(\vec{x})$	...	$g_k(\vec{x})$

$$m = \max\{\rho(F), \rho(G_1), \dots, \rho(G_k), k, n\}$$

$T(1, m+1)$

$\vdots$

$T(n, m+n)$

$G_1[m+1, \dots, m+n \to m+n+1]$

$\vdots$

$G_k[m+1, \dots, m+n \to m+n+k]$

$F[m+n+1, \dots, m+n+k \to 1]$

$\square$

Example: $f(x_1, x_n) = x_1 + x_2$ known to be in $\mathcal{C}$

$g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$= f(f(x_1, x_2), x_3) \qquad \vec{x} = (x_1, x_2, x_3)$$

$$= \underbrace{f\Big(\underbrace{f(\underbrace{U^3_1(\vec{x})}_{\mathbb{N}^3 \to \mathbb{N}}\ \underbrace{U^3_2(x)}_{\mathbb{N}^3 \to \mathbb{N}})}_{\mathbb{N}^3 \to \mathbb{N}}, \underbrace{U^3_3(\vec{x})}_{\mathbb{N}^3 \to \mathbb{N}}\Big)}_{\mathbb{N}^3 \to \mathbb{N}}$$

* Example : Let $f : \mathbb{N} \to \mathbb{N}$ computable and total

$$Q_f(x,y) = \text{" } f(x) = y \text{ "} \quad \text{decidable ?}$$

$$\chi_{Q_f}(x,y) = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise} \end{cases} \quad \text{computable ?}$$

We know that

$$\chi_{Eq} : \mathbb{N}^2 \to \mathbb{N}$$

$$\chi_{Eq}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \quad \text{computable}$$

Then

$$\chi_{Q_f}(x,y) = \chi_{Eq}(f(x), y) \quad \text{computable by composition}$$

* Primitive Recursion

$$\begin{cases} 0! = 1 \\ (n+1)! = \underbrace{n!}_{} * (n+1) \end{cases}$$

$$\begin{cases} fib(0) = 1 \\ fib(1) = 1 \\ fib(n+2) = \underbrace{fib(n)}_{} + \underbrace{fin(n+1)}_{} \end{cases}$$

Def : Given $f : \mathbb{N}^k \to \mathbb{N}$

$g : \mathbb{N}^{k+2} \to \mathbb{N}$

define $h : \mathbb{N}^{k+1} \to \mathbb{N}$

$$\begin{cases} h(\vec{x}, 0) = f(\vec{x}) \\ h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y)) \end{cases}$$

take x $\boxed{\dfrac{\sqrt{x}}{e^x} = \log x}$

→ is there a solution ?

→ is it unique

($(\mathcal{F}, \subseteq)$ complete partial order, continuous operation } → existence solution (least fix point))

uniqueness - - - - - - - - → induction

Examples:

→ $h: \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x+y$

$$\begin{cases} x+0 = x \\ x+(y+1) = (x+y)+1 \end{cases}$$

$f(x) = x = U_1^1(x)$
$g(x,y,z) = z+1$

→ $h': \mathbb{N}^2 \to \mathbb{N}$

$h'(x,y) = x * y$

$x * 0 = 0$
$x*(y+1) = (x*y) + x$

$f(x) = 0$
$g(x,y,z) = z + x$

Proposition: $\mathcal{C}$ is closed by primitive recursion

proof Let $f: \mathbb{N}^k \to \mathbb{N}$, $g: \mathbb{N}^{k+2} \to \mathbb{N}$ be in $\mathcal{C}$

and let F, G programs in std form for f, g

Define $h: \mathbb{N}^{k+1} \to \mathbb{N}$

$$\begin{cases} h(\vec{x},0) = f(\vec{x}) \\ h(\vec{x},y+1) = g(\vec{x},y,h(\vec{x},y)) \end{cases}$$

idea:

$h(\vec{x},0) = f(\vec{x})$ (use F)

$h(\vec{x},1) = g(\vec{x},0,h(\vec{x},0))$ (use G)

$\vdots$

$h(\vec{x},i) = g(\vec{x},i-1, h(\vec{x},i-1))$ (use G)

$i = y$? if so output

no continue with i++

1		K	K+1		m	m+1		m+K	m+K+1	m+K+2	m+K+3
x_1	..	x_K	y			x_1	..	x_K	i	$h(\vec{x}, i)$	y
									initially 0		$y = i$?

$$m = \max\{\rho(F), \rho(G), K+2\}$$

$T(1, m+1)$

$\vdots$

$T(K, m+K)$

$T(K+1, m+K+3)$

$F[m+1, \dots, m+K \to m+K+2]$ // $h(\vec{x}, 0) = f(\vec{x})$

LOOP: $J(m+K+1, m+K+3, \text{RES})$ // $i = y$?

$G[m+1, \dots, m+K+2 \to m+K+2]$ // $h(\vec{x}, i+1) = g(\vec{x}, i, h(\vec{x}, i))$

$S(m+K+1)$ // $i{+}{+}$

$J(1, 1, \text{LOOP})$

RES: $T(m+K+2, 1)$

$\square$

Examples:

→ $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x, y) = x + y$

$$\begin{cases} x + 0 = x \\ x + (y+1) = (x+y) + 1 \end{cases}$$

$f(x) = x = U_1^1(x)$

$g(x, y, z) = z + 1$

→ $h' : \mathbb{N}^2 \to \mathbb{N}$

$h'(x, y) = x * y$

$x * 0 = 0$

$x * (y+1) = (x * y) + x$

$f(x) = 0$

$g(x, y, z) = z + x$

→ exponential x^y

$$\begin{cases} x^0 = 1 \\ x^{y+1} = (x^y) * x \end{cases}$$

→ predecessor $y \dot{-} 1$

$0 \dot{-} 1 = 0$

$(y+1) \dot{-} 1 = y$

→ difference $x \dot{-} y = \begin{cases} 0 & x \leq y \\ x - y & x > y \end{cases}$

$x \dot{-} 0 = x$

$x \dot{-} (y+1) = (x \dot{-} y) \dot{-} 1$

→ sign $sg(y) = \begin{cases} 0 & \text{if } y = 0 \\ 1 & \text{if } y > 0 \end{cases}$

$$\begin{cases} sg(0) = 0 \\ sg(y+1) = 1 \end{cases}$$

→ $\overline{sg}(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$ exercise (SOLUTION $\overline{sg}(x) = 1 \dot{-} sg(x)$)

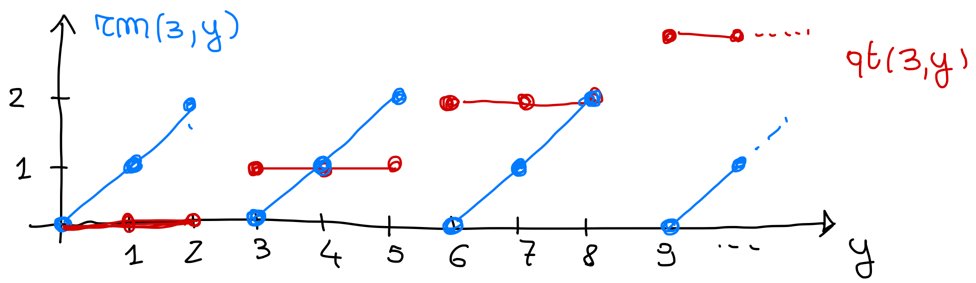
→ $\min(x, y) = \overbrace{x \dot{-} (\underbrace{x \dot{-} y}_{0})}^{x}$ $x \leq y$

$x - (\underbrace{x - y}) = y$ $x > y$

→ $\max(x, y)$ exercise (SOLUTION $\max(x,y) = x + y \dot{-} x$)

→ $rm(x, y)$ = remainder of y divided by x

$= \begin{cases} y \bmod x & x > 0 \\ y & x = 0 \end{cases}$



$$
\begin{cases}
rm(x,0) = 0 \\
rm(x, y+1) = \begin{cases} rm(x,y)+1 & \text{if } rm(x,y)+1 < x \\ 0 & \text{otherwise} \end{cases}
\end{cases}
$$

$$
= (rm(x,y)+1) * \underbrace{sg(x \dot{-} (rm(x,y)+1))}_{\text{something} \begin{cases} 1 & \text{if } rm(x,y)+1<x \\ 0 & \text{otherwise} \end{cases}}
$$

$*$ $qt(x,y) = y \text{ div } x$ (convention $qt(0,y) = 0$)

(exercise) SOLUTION:

$$
qt(x,0) = 0
$$

$$
qt(x,y+1) = \begin{cases} qt(x,y)+1 & \text{if } rm(x,y+1) = 0 \\ qt(x,y) & \text{otherwise} \end{cases}
$$

$$
= qt(x,y) + \overline{sg}(rm(x,y+1))
$$