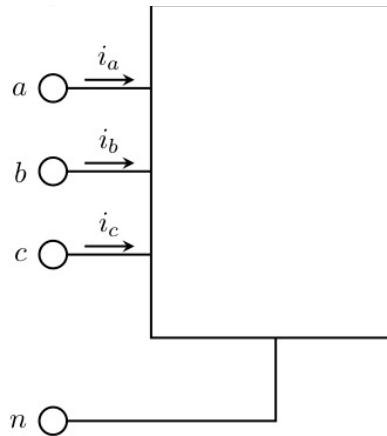


Space vectors

Any three phase system can be represented as in the following picture:



We can identify three electrical ports considering the n terminal, the neutral point, as common reference for the voltages.

We can then identify the three currents i_a , i_b and i_c and the voltages v_a , v_b and v_c of each phase respect to the neutral n .

We can also define the phase-to-phase voltages as:

$$v_{ab} = v_a - v_b$$

$$v_{bc} = v_b - v_c$$

$$v_{ca} = v_c - v_a$$

Considering three-phase systems other set of three quantities arise during the analysis of the system, such as the flux-linkage of each phase λ_a , λ_b and λ_c .

The **space vectors** are useful mathematical tools to manage more efficiently the electrical quantities in three-phase systems allowing a simpler management of equations.

Some examples of three phase quantities

```
In [4]: import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

f      = 50  # the frequency of the current
Im     = 100 # the peak value
t      = np.linspace(0,1/f,101) # we take 1 periods

# Find the three-phase voltages
case = 7
if case == 1:
    ia = Im*np.cos(2*np.pi*f*t)
    ib = Im*np.cos(2*np.pi*f*t - 2/3*np.pi)
    ic = Im*np.cos(2*np.pi*f*t - 4/3*np.pi)

if case == 2:
```

```

ia = Im*np.cos(2*np.pi*f*t) + 0.2*Im*np.sin(5*2*np.pi*f*t)
ib = Im*np.cos(2*np.pi*f*t - 2/3*np.pi) + 0.2*Im*np.sin(5*2*np.pi*f*t)
ic = Im*np.cos(2*np.pi*f*t - 4/3*np.pi) + 0.2*Im*np.sin(5*2*np.pi*f*t)

if case == 3:
    ia = Im*np.cos(2*np.pi*f*t) + 0.2*Im*np.sin(7*2*np.pi*f*t)
    ib = Im*np.cos(2*np.pi*f*t - 2/3*np.pi) + 0.2*Im*np.sin(7*2*np.pi*f*t)
    ic = Im*np.cos(2*np.pi*f*t - 4/3*np.pi) + 0.2*Im*np.sin(7*2*np.pi*f*t)

if case == 4:
    ia = Im*np.sign(np.cos(2*np.pi*f*t))
    ib = Im*np.sign(np.cos(2*np.pi*f*t - 2/3*np.pi))
    ic = Im*np.sign(np.cos(2*np.pi*f*t - 4/3*np.pi))

if case == 5:
    ia = Im*(np.cos(2*np.pi*f*t) + 0.2*Im*np.sin(3*2*np.pi*f*t))
    ib = Im*(np.cos(2*np.pi*f*t - 2/3*np.pi) + 0.2*Im*np.sin(3*2*np.pi*f*t))
    ic = Im*(np.cos(2*np.pi*f*t - 4/3*np.pi) + 0.2*Im*np.sin(3*2*np.pi*f*t))

if case == 6:
    ia = Im*np.cos(2*np.pi*f*t)
    ib = 0.2*Im*np.cos(2*np.pi*f*t - 2/3*np.pi)
    ic = -ia-ib #just to force ia+ib+ic=0

if case == 7:
    t = np.linspace(0,1/f,121)
    ia = np.zeros(121)
    ib = np.zeros(121)
    ic = np.zeros(121)

    ia[0:20] = Im/3
    ia[20:40] = 2*Im/3
    ia[40:60] = Im/3
    ia[60:80] = -Im/3
    ia[80:100] = -2*Im/3
    ia[100:121] = -Im/3

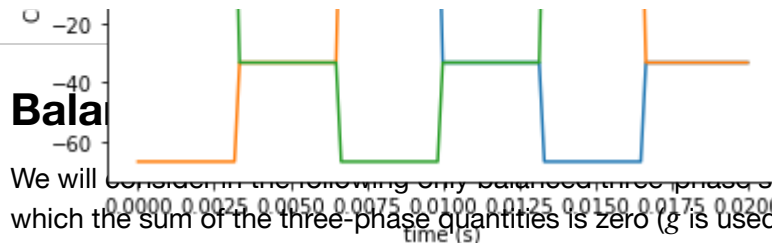
    ib[40:60] = Im/3
    ib[60:80] = 2*Im/3
    ib[80:100] = Im/3
    ib[100:121] = -Im/3
    ib[0:20] = -2*Im/3
    ib[20:40] = -Im/3

    ic[80:100] = Im/3
    ic[100:121] = 2*Im/3
    ic[0:20] = Im/3
    ic[20:40] = -Im/3
    ic[40:60] = -2*Im/3
    ic[60:80] = -Im/3

plt.plot(t, ia, label="$i_a$")
plt.plot(t, ib, label="$i_b$")
plt.plot(t, ic, label="$i_c$")
plt.plot(t, ia+ib+ic, label="$i_n$")

plt.xlabel('time (s)')
plt.ylabel('Current (A)')
plt.legend(ncol=1);

```



We will consider in the following only balanced three phase systems, i.e. systems in which the sum of the three-phase quantities is zero (g is used to indicate general quantities that could be voltage, current or other in different situations):

$$g_0(t) = \frac{g_a(t) + g_b(t) + g_c(t)}{3}$$

Every quantity is a time dependent value as explicitly reported. g_0 is called homopolar component of the three-phase system, here we are assuming that the homopolar component is null.

We define a new complex function in the following way:

$$\vec{g}(t) = \frac{2}{3} \left(g_a(t) + g_b(t) \cdot e^{i\frac{2\pi}{3}} + g_c(t) \cdot e^{i\frac{4\pi}{3}} \right) \quad (1)$$

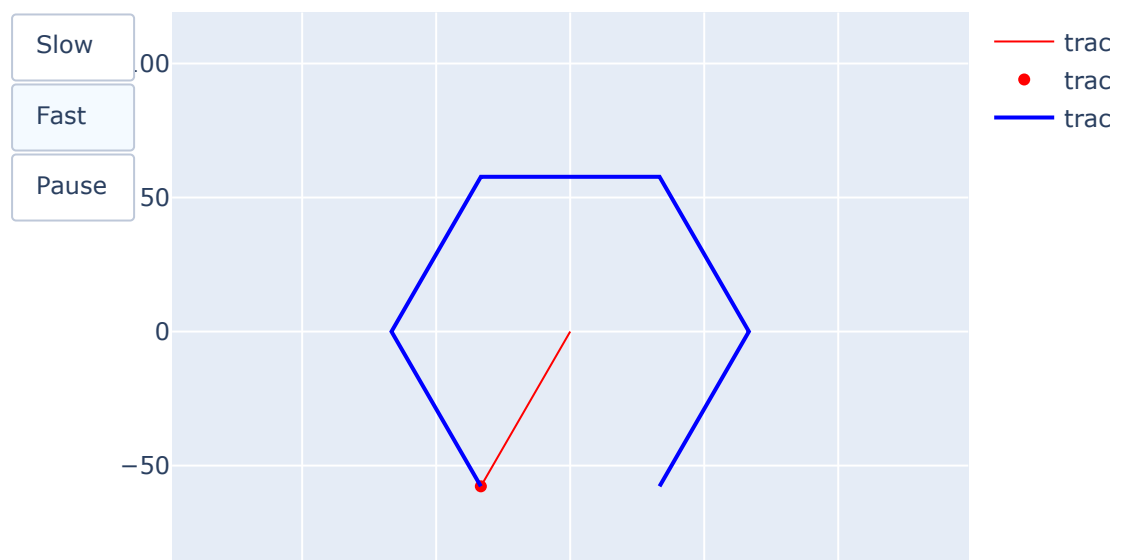
The **space vector** $\vec{g}(t)$ is a complex function of the three real variables $g_a(t)$, $g_b(t)$ and $g_c(t)$. As the time change, $\vec{g}(t)$ move in the complex plane describing a trajectory.

Considering the currents just defined in the examples above, we can plot the locus described by $\vec{i}(t)$:

```
In [5]: from math import e, pi
from cmath import *

i_sv = 2/3*(ia + ib*e**(2j*pi/3) + ic*e**(4j*pi/3))
```

In [6]:

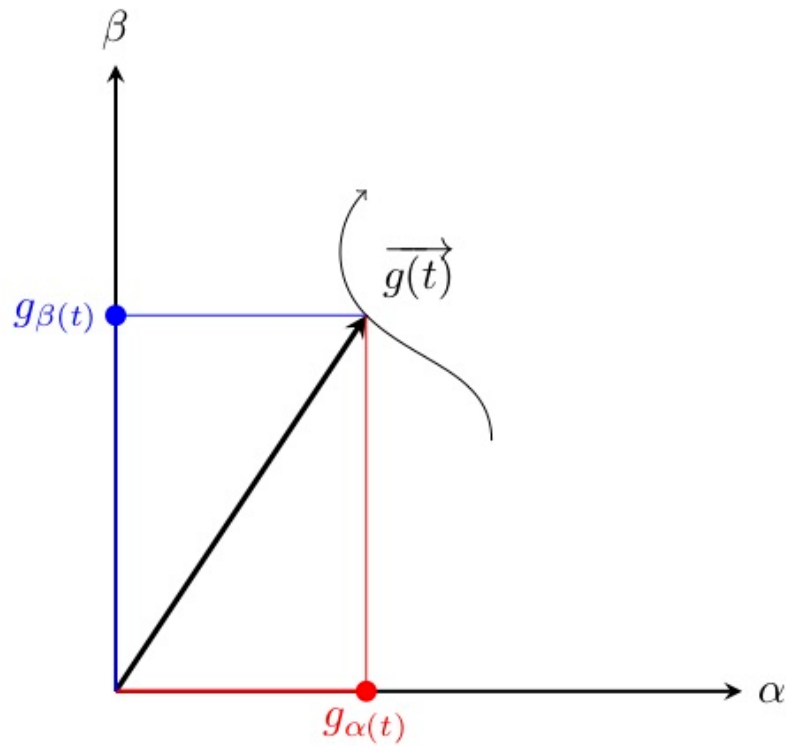


The space vector \vec{g} is drawn in the complex plane whose axes are denoted as α -axis (the real axis) and β -axis (the imaginary axis).

The space vector can be expressed in cartesian form as:

$$\vec{g}(t) = g_\alpha(t) + j \cdot g_\beta(t)$$

This representation is also referred as *space vector in $\alpha\beta$ reference frame* and is typically reported as in the following picture:



We can express explicitly the real and imaginary part of the space vector as:

$$\begin{aligned} g_\alpha(t) &= \Re [\vec{g}(t)] = \frac{2}{3} \left(g_a(t) - \frac{g_b(t)}{2} - \frac{g_c(t)}{2} \right) = g_a(t) \\ g_\beta(t) &= \Im [\vec{g}(t)] = \frac{2}{3} \left(\frac{\sqrt{3}}{2} g_b(t) - \frac{\sqrt{3}}{2} g_c(t) \right) = \frac{1}{\sqrt{3}} (g_b(t) - g_c(t)) \end{aligned} \quad (2)$$

Eq. (1) defines how to transform the three time function into a *space vector*.

Eq. (2) expresses the two space vector components, i.e. its real and imaginary part, separately.

We can also perform the inverse transformation, i.e. to compute the time function $g_a(t)$, $g_b(t)$ and $g_c(t)$ starting from the space vector $\vec{g}(t)$.

This inverse transformation is possible thanks to the assumption $g_a(t) + g_b(t) + g_c(t) = 0$ which is, actually, a constraint on the three functions.

In particular, from the first of (2) we have that

$$g_a(t) = \Re [\vec{g}(t)] = g_\alpha(t)$$

$g_b(t)$ and $g_c(t)$ can be achieved considering the following two new space vectors defined as:

$$\begin{aligned}\vec{g}_b(t) &= \vec{g}(t) \cdot e^{-i\frac{2\pi}{3}} = \frac{2}{3} \left(g_a(t) \cdot e^{-i\frac{2\pi}{3}} + g_b(t) + g_c(t) \cdot e^{i\frac{2\pi}{3}} \right) \\ \vec{g}_c(t) &= \vec{g}(t) \cdot e^{-i\frac{4\pi}{3}} = \frac{2}{3} \left(g_a(t) \cdot e^{-i\frac{4\pi}{3}} + g_b(t) \cdot e^{i\frac{2\pi}{3}} + g_c(t) \right)\end{aligned}$$

and hence their projections on the real axis, i.e.:

$$\begin{aligned}g_b(t) &= \Re [\vec{g}_b(t)] = -\frac{g_a(t)}{2} + \frac{\sqrt{3}}{2} g_\beta(t) \\ g_c(t) &= \Re [\vec{g}_c(t)] = -\frac{g_a(t)}{2} - \frac{\sqrt{3}}{2} g_\beta(t)\end{aligned}$$

Grafically this correspond to project the space vectors on the "phase axes".

We have found how to obtain the space vector from the phase values and viceversa. Such transformation process can be profitably described in matrix notation.

Matrix expression of the transformation

The space vector definition can be expressed in matrix form as:

$$\begin{bmatrix} g_\alpha(t) \\ g_\beta(t) \\ g_0(t) \end{bmatrix} = \frac{2}{3} \cdot \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} g_a(t) \\ g_b(t) \\ g_c(t) \end{bmatrix}$$

where the matrix is intended to *transform* the vector $g_{abc} = [g_a(t) \ g_b(t) \ g_c(t)]^T$ in the vector $g_{\alpha\beta 0} = [g_\alpha(t) \ g_\beta(t) \ g_0(t)]^T$. In the transformation it has been considered also the homopolar component $g_0(t)$ (which has been supposed null in the previous development).

The transformation can then be expressed in compact form as:

$$g_{\alpha\beta 0} = T_{abc \rightarrow \alpha\beta 0} \cdot g_{abc}$$

We can also define the inverse transformation matrix which is:

$$\begin{bmatrix} 1 & 0 & 1 \\ \sqrt{3} & 1 & 0 \end{bmatrix}$$

Power balance

If we consider the phase voltages and current of a three-phase system, we can express the instantaneous power as:

$$p(t) = v_a(t)i_a(t) + v_b(t)i_b(t) + v_c(t)i_c(t)$$

that can be expressed in compact form as

$$p(t) = v_{abc}^T \cdot i_{abc}$$

Using the α - β reference frame, i.e. computing the space vectors, we can rewrite the power as:

$$\begin{aligned} p(t) &= (T_{\alpha\beta 0 \rightarrow abc} v_{\alpha\beta 0})^T \cdot T_{\alpha\beta 0 \rightarrow abc} i_{\alpha\beta 0} \\ &= v_{\alpha\beta 0}^T T_{\alpha\beta 0 \rightarrow abc}^T \cdot T_{\alpha\beta 0 \rightarrow abc} i_{\alpha\beta 0} \end{aligned}$$

Computing the matrix product in the previous expression yields:

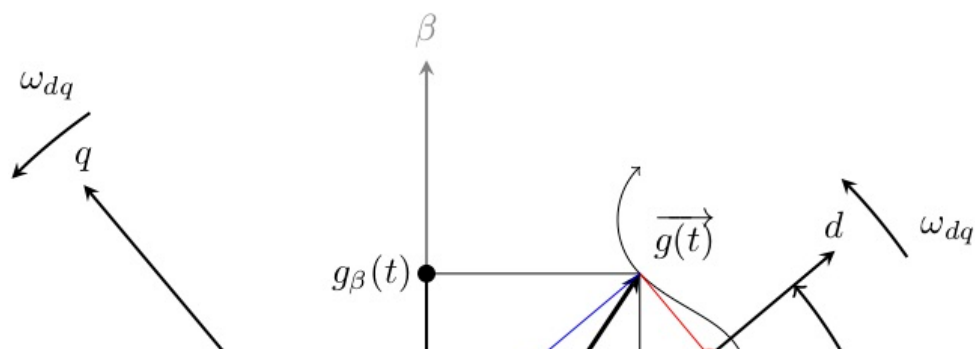
$$\begin{aligned} p(t) &= \begin{bmatrix} v_\alpha(t) & v_\beta(t) & v_0(t) \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} i_\alpha(t) \\ i_\beta(t) \\ i_0(t) \end{bmatrix} \\ &= \frac{3}{2} (v_\alpha(t) i_\alpha(t) + v_\beta(t) i_\beta(t)) + 3v_0(t)i_0(t) \end{aligned}$$

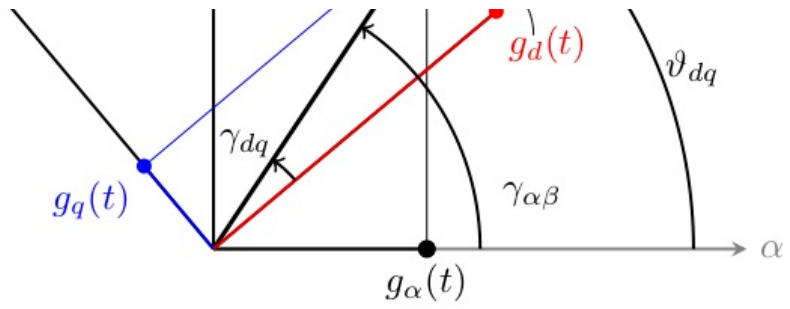
This last relation, shows that the considered transformation does not preserve the power, i.e. we have to consider the factor $\frac{3}{2}$ during the power computations.

Rotating reference frame

In many electric drives applications, it is useful to consider a rotating reference frame to express the space vectors. This allows to simplify the equations describing the dynamic of the system.

The rotating reference frame is defined with two rotating axes, d - and q -axis, sharing the origin with the α - β reference frame which is stationary. Both reference systems are reported in the following picture:





The space vector $\vec{g}(t)$ is the same as before. Now we are just describing its coordinates in a new reference frame rotating at speed ω_{dq} . The new coordinates will be $g_d(t)$ and $g_q(t)$, i.e. its projections on the d and q axes.

The rotation angle of the d - q reference frame is ϑ_{dq} that can be computed from the rotational speed ω_{dq} as:

$$\vartheta_{dq} = \int_0^t \omega_{dq}(t) dt$$

The space vector can be expressed in both reference frame, the following equations are valid:

$$\vec{g}(t)_{\alpha\beta} = g_\alpha(t) + i \cdot g_\beta(t) = |\vec{g}(t)| \cdot e^{i\gamma_{\alpha\beta}}$$

$$\vec{g}(t)_{dq} = g_d(t) + i \cdot g_q(t) = |\vec{g}(t)| \cdot e^{i\gamma_{dq}}$$

$$\gamma_{\alpha\beta} = \gamma_{dq} + \vartheta_{dq}$$

Using this relations, the representation in the two reference frames can be written as:

$$\begin{aligned} \vec{g}(t)_{\alpha\beta} &= |\vec{g}(t)| \cdot e^{i\gamma_{\alpha\beta}} \\ &= |\vec{g}(t)| \cdot e^{i(\gamma_{dq} + \vartheta_{dq})} \\ &= |\vec{g}(t)| \cdot e^{i\gamma_{dq}} \cdot e^{i\vartheta_{dq}} \\ &= \vec{g}(t)_{dq} \cdot e^{i\vartheta_{dq}} \end{aligned}$$

or we can express the coordinates in the d - q reference frame as:

$$\vec{g}(t)_{dq} = \vec{g}(t)_{\alpha\beta} \cdot e^{-i\vartheta_{dq}}$$

Expressing the components we obtain:

$$\begin{bmatrix} g_d(t) \\ g_q(t) \end{bmatrix} = \begin{bmatrix} \cos(\vartheta_{dq}) & \sin(\vartheta_{dq}) \\ -\sin(\vartheta_{dq}) & \cos(\vartheta_{dq}) \end{bmatrix} \begin{bmatrix} g_\alpha(t) \\ g_\beta(t) \end{bmatrix} = T_{\alpha\beta \rightarrow dq} \begin{bmatrix} g_\alpha(t) \\ g_\beta(t) \end{bmatrix}$$

The new transformation matrix $T_{\alpha\beta \rightarrow dq}$ is orthonormal so that the inverse is its transpose, i.e.

$$T_{dq \rightarrow \alpha\beta} = \begin{bmatrix} \cos(\vartheta_{dq}) & -\sin(\vartheta_{dq}) \\ \sin(\vartheta_{dq}) & \cos(\vartheta_{dq}) \end{bmatrix}$$

It is then possible to combine both the transformation realizing the conversion from the phase variables (abc reference frame) to the rotating d - q reference frame.

In fact:

$$g_{dq} = T_{\alpha\beta \rightarrow dq} g_{\alpha\beta} = T_{\alpha\beta \rightarrow dq} T_{abc \rightarrow \alpha\beta} g_{abc} = T_{abc \rightarrow dq} g_{abc}$$

Computing the complete transformation (and including also the homopolar component) we obtain:

$$T_{abc \rightarrow dq0} = \frac{2}{3} \begin{bmatrix} \cos(\vartheta_{dq}) & \cos\left(\vartheta_{dq} - \frac{2\pi}{3}\right) & \cos\left(\vartheta_{dq} - \frac{4\pi}{3}\right) \\ -\sin(\vartheta_{dq}) & -\sin\left(\vartheta_{dq} - \frac{2\pi}{3}\right) & -\sin\left(\vartheta_{dq} - \frac{4\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$T_{dq0 \rightarrow abc} = \begin{bmatrix} \cos(\vartheta_{dq}) & -\sin(\vartheta_{dq}) & 1 \\ \cos\left(\vartheta_{dq} - \frac{2\pi}{3}\right) & -\sin\left(\vartheta_{dq} - \frac{2\pi}{3}\right) & 1 \\ \cos\left(\vartheta_{dq} - \frac{4\pi}{3}\right) & -\sin\left(\vartheta_{dq} - \frac{4\pi}{3}\right) & 1 \end{bmatrix}$$

Example and applications

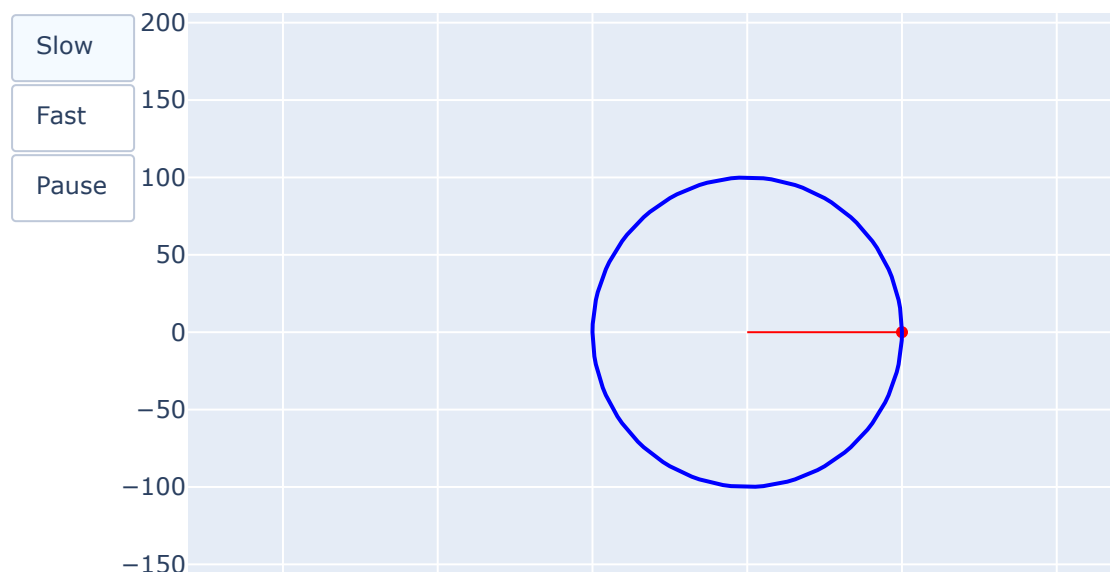
Some additional examples of the use of space vectors are given in the following

In [29]: *# Direct three-phase system in alpha-beta reference frame*

```
ia = Im*np.cos(2*np.pi*f*t)
ib = Im*np.cos(2*np.pi*f*t - 2/3*np.pi)
ic = Im*np.cos(2*np.pi*f*t - 4/3*np.pi)

i_alpha_beta = 2/3*(ia + ib*e**(2j*pi/3) + ic*e**(4j*pi/3))

plot(i_alpha_beta)
```




```
In [122]: # we can also consider the quantites
# in the d-q rotating reference frame

# we define the speed of the rotating reference frame
wdq = 2*np.pi*f

# we transform the space vector in the d-q reference frame
i_dq = i_alpha_beta * e**(-1j*wdq*t)

# we can inspect i_dq to check that it is constant
print(i_dq)

# its plot will be stationary in the (rotating) complex plane
plot(i_dq,110,-110,110,-110)
```

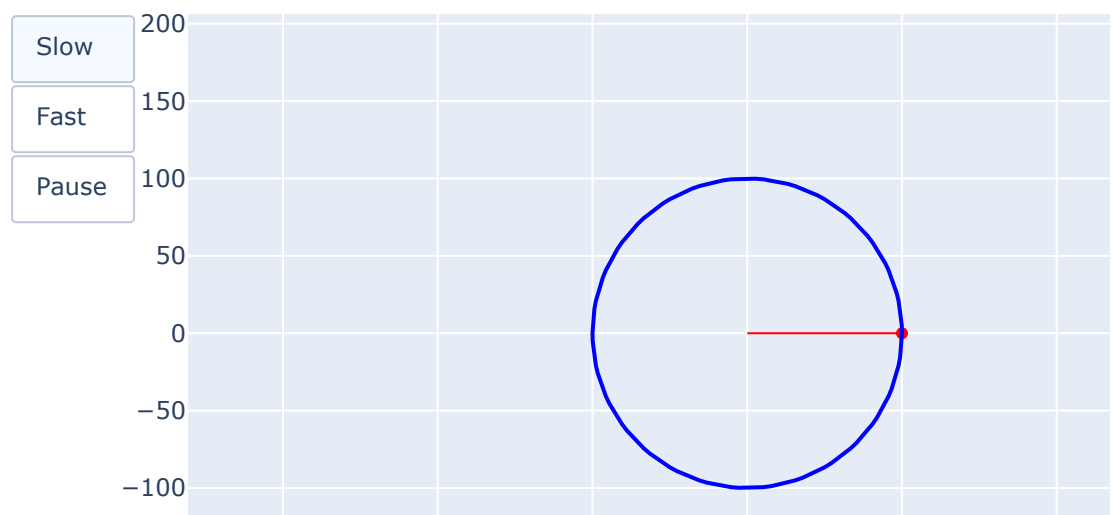
```
[100.+1.89478063e-14j 100.+7.10542736e-15j 100.+1.77635684e-14j
 100.+1.77635684e-14j 100.-3.55271368e-15j 100.+0.00000000e+00j
 100.+0.00000000e+00j 100.+7.10542736e-15j 100.-2.13162821e-14j
 100.+7.10542736e-15j 100.-7.10542736e-15j 100.-1.42108547e-14j
 100.+0.00000000e+00j 100.-7.10542736e-15j 100.-2.84217094e-14j
 100.-4.26325641e-14j 100.-1.42108547e-14j 100.-2.13162821e-14j
 100.-2.13162821e-14j 100.-2.84217094e-14j 100.-2.48689958e-14j
 100.-3.19744231e-14j 100.-3.55271368e-14j 100.-2.48689958e-14j
 100.-3.55271368e-14j 100.-2.70354270e-14j 100.-1.68753900e-14j
 100.-2.48689958e-14j 100.-2.84217094e-14j 100.-1.77635684e-14j
 100.-1.77635684e-14j 100.+0.00000000e+00j 100.+0.00000000e+00j
 100.+0.00000000e+00j 100.+0.00000000e+00j 100.-7.10542736e-15j
 100.+2.13162821e-14j 100.+7.10542736e-15j 100.+1.42108547e-14j
 100.+1.42108547e-14j 100.+7.10542736e-15j 100.+7.10542736e-15j
 100.+2.13162821e-14j 100.+2.13162821e-14j 100.+2.84217094e-14j
 100.+3.19744231e-14j 100.+1.77635684e-14j 100.+2.48689958e-14j
 100.+1.77635684e-14j 100.+1.95399252e-14j 100.+2.17203711e-14j
 100.+2.22044605e-14j 100.+1.42108547e-14j 100.+1.42108547e-14j
 100.+1.77635684e-14j 100.+3.55271368e-15j 100.+0.00000000e+00j
 100.+0.00000000e+00j 100.+0.00000000e+00j 100.-7.10542736e-15j
 100.-7.10542736e-15j 100.+0.00000000e+00j 100.-2.13162821e-14j
 100.-1.42108547e-14j 100.-1.42108547e-14j 100.-2.13162821e-14j
 100.-2.13162821e-14j 100.-2.84217094e-14j 100.-4.26325641e-14j
 100.-2.84217094e-14j 100.-3.19744231e-14j 100.-2.48689958e-14j
 100.-2.48689958e-14j 100.-2.66453526e-14j 100.-3.64153152e-14j
 100.-2.30786243e-14j 100.-1.68753900e-14j 100.-2.48689958e-14j
 100.-2.13162821e-14j 100.-2.13162821e-14j 100.-1.06581410e-14j
 100.-2.13162821e-14j 100.-7.10542736e-15j 100.+0.00000000e+00j
 100.+7.10542736e-15j 100.+1.42108547e-14j 100.+7.10542736e-15j
 100.+0.00000000e+00j 100.+7.10542736e-15j 100.+2.13162821e-14j
 100.+1.42108547e-14j 100.+1.42108547e-14j 100.+1.42108547e-14j
 100.+2.13162821e-14j 100.+1.42108547e-14j 100.+2.13162821e-14j
 100.+3.19744231e-14j 100.+0.00000000e+00j 100.+1.77635684e-15j
 100.+3.73034936e-14j 100.+3.87037907e-14j]
```

```
In [30]: # Inverse three-phase system

ia = Im*np.cos(2*np.pi*f*t)
ib = Im*np.cos(2*np.pi*f*t + 2/3*np.pi)
ic = Im*np.cos(2*np.pi*f*t + 4/3*np.pi)

i_alpha_beta = 2/3*(ia + ib*e**(2j*pi/3) + ic*e**(4j*pi/3))

plot(i_alpha_beta)
```



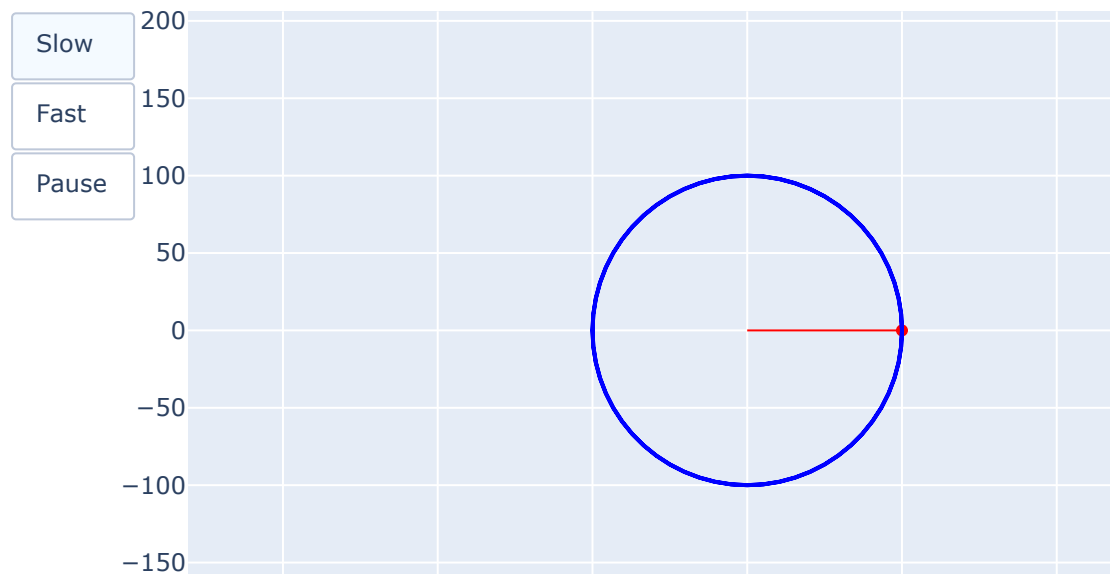
```
In [31]: # in the d-q rotating reference frame we have

wdq = 2*np.pi*f
i_dq = i_alpha_beta * e**(-1j*wdq*t)

# we inspect i_dq
print(i_dq)

# its plot will be revolving at twice the speed!
plot(i_dq)
```

```
[ 1.00000000e+02+1.89478063e-14j  9.94521895e+01-1.04528463e+01j
 9.78147601e+01-2.07911691e+01j  9.51056516e+01-3.09016994e+01j
 9.13545458e+01-4.06736643e+01j  8.66025404e+01-5.00000000e+01j
 8.09016994e+01-5.87785252e+01j  7.43144825e+01-6.69130606e+01j
 6.69130606e+01-7.43144825e+01j  5.87785252e+01-8.09016994e+01j
 5.00000000e+01-8.66025404e+01j  4.06736643e+01-9.13545458e+01j
 3.09016994e+01-9.51056516e+01j  2.07911691e+01-9.78147601e+01j
 1.04528463e+01-9.94521895e+01j  1.42108547e-14-1.00000000e+02j
-1.04528463e+01-9.94521895e+01j -2.07911691e+01-9.78147601e+01j
-3.09016994e+01-9.51056516e+01j -4.06736643e+01-9.13545458e+01j
-5.00000000e+01-8.66025404e+01j -5.87785252e+01-8.09016994e+01j
-6.69130606e+01-7.43144825e+01j -7.43144825e+01-6.69130606e+01j
-8.09016994e+01-5.87785252e+01j -8.66025404e+01-5.00000000e+01j
-9.13545458e+01-4.06736643e+01j -9.51056516e+01-3.09016994e+01j
-9.78147601e+01-2.07911691e+01j -9.94521895e+01-1.04528463e+01j
-1.00000000e+02+1.28245723e-14j -9.94521895e+01+1.04528463e+01j
-9.78147601e+01+2.07911691e+01j -9.51056516e+01+3.09016994e+01j
-9.13545458e+01+4.06736643e+01j -8.66025404e+01+5.00000000e+01j
-8.09016994e+01+5.87785252e+01j -7.43144825e+01+6.69130606e+01j
-6.69130606e+01+7.43144825e+01j -5.87785252e+01+8.09016994e+01j
-5.00000000e+01+8.66025404e+01j -4.06736643e+01+9.13545458e+01j
-3.09016994e+01+9.51056516e+01j -2.07911691e+01+9.78147601e+01j
-1.04528463e+01+9.94521895e+01j  2.84217094e-14+1.00000000e+02j]
```



In [32]: # Variable amplitude system

```
ia = Im*(1+t*f)*np.cos(2*np.pi*f*t)
ib = Im*(1+t*f)*np.cos(2*np.pi*f*t - 2/3*np.pi)
```

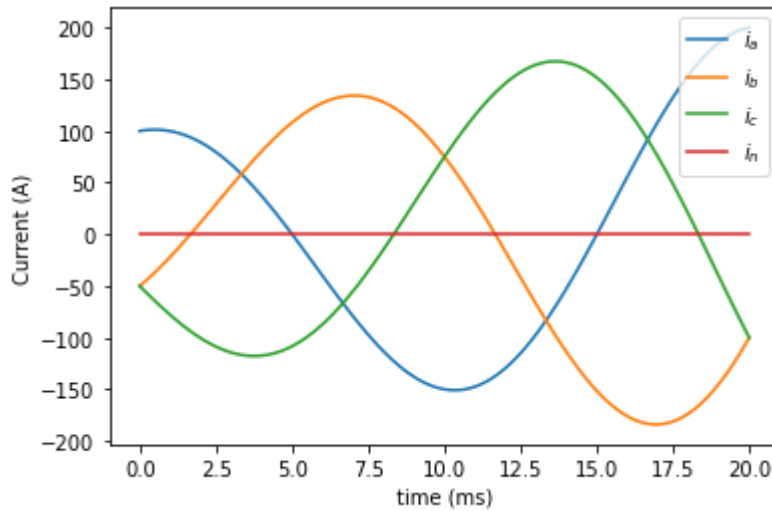
```

ic = Im*(1+t*f)*np.cos(2*np.pi*f*t - 4/3*np.pi)

plt.plot(t*1000, ia, label="$i_a$")
plt.plot(t*1000, ib, label="$i_b$")
plt.plot(t*1000, ic, label="$i_c$")
plt.plot(t*1000, ia+ib+ic, label="$i_n$")

plt.xlabel('time (ms)')
plt.ylabel('Current (A)')
plt.legend(ncol=1);

```

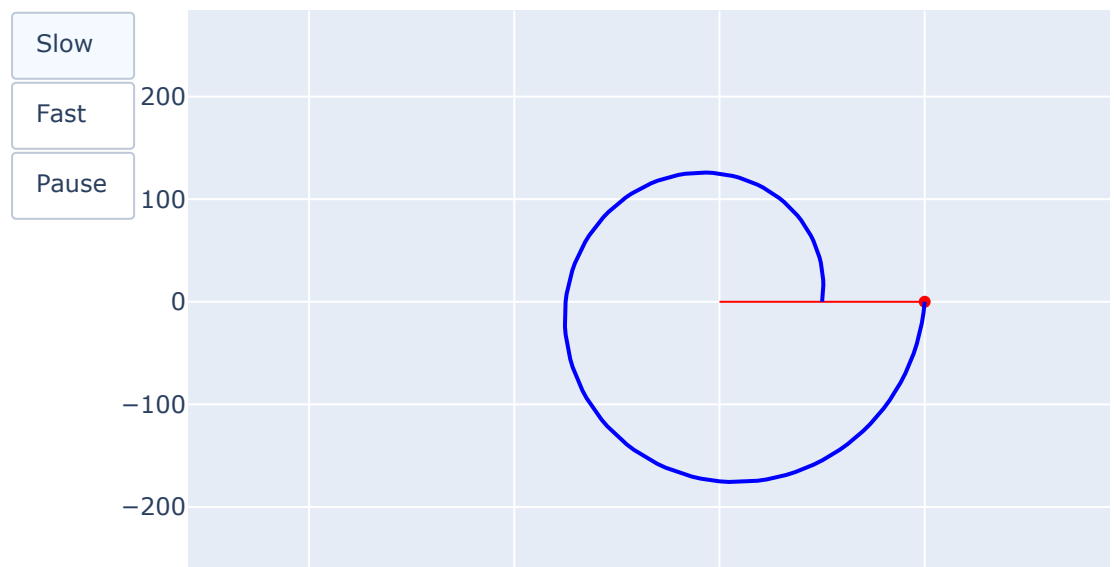


In [33]: *# we consider now the space vectors*

```

i_alpha_beta = 2/3*(ia + ib*e**(2j*pi/3) + ic*e**(4j*pi/3))
plot(i_alpha_beta)

```



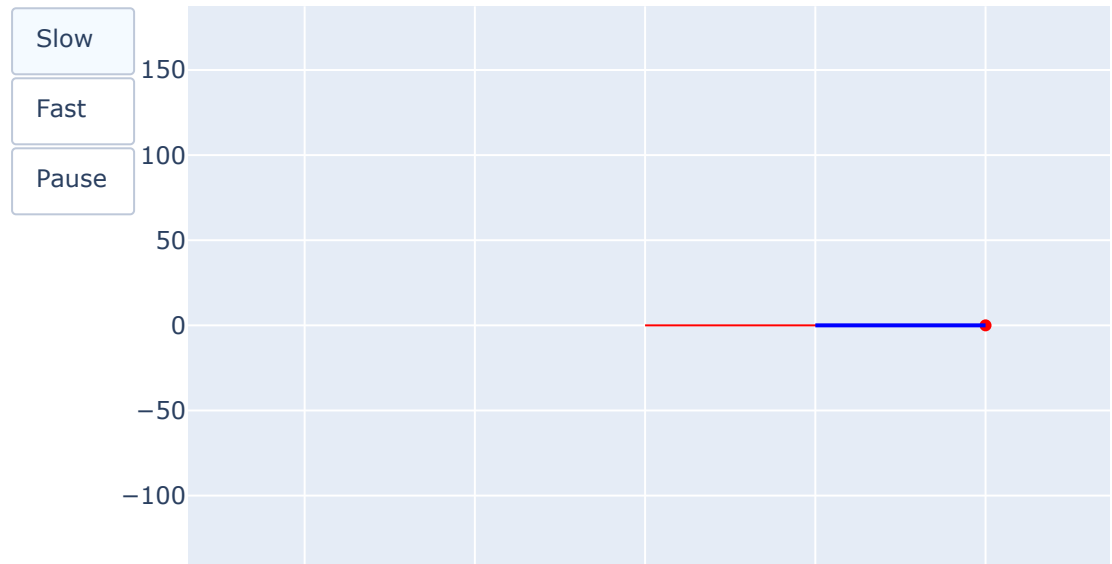
```
In [34]: # in the d-q rotating reference frame we have

wdq = 2*np.pi*f
i_dq = i_alpha_beta * e**(-1j*wdq*t)

# we inspect i_dq
print(i_dq)

# its plot will have constant angle and changing magnitude
plot(i_dq,220,-100,100,-100)
```

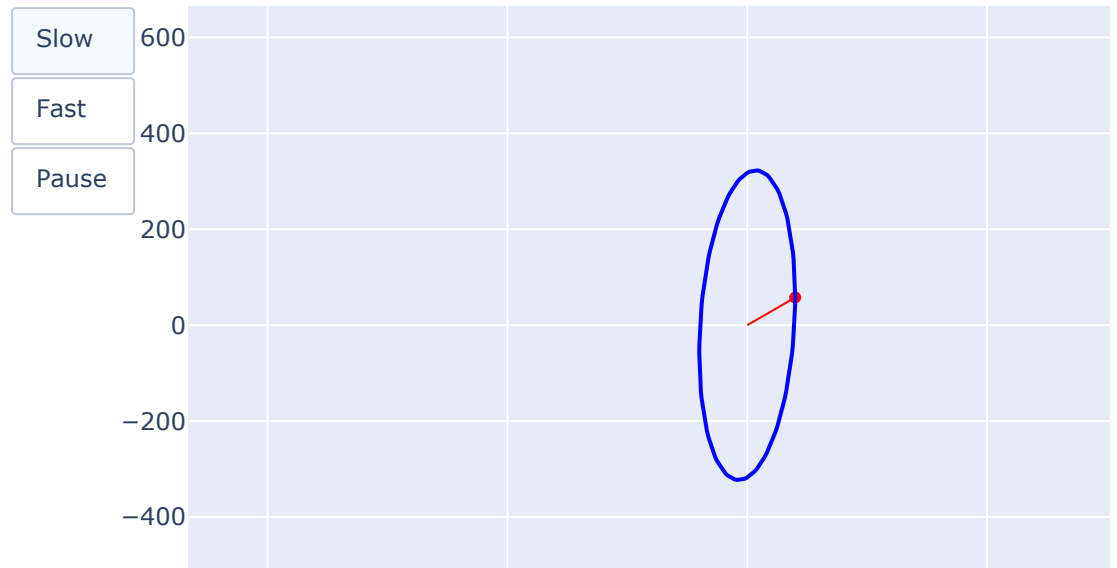
```
[100.          +1.89478063e-14j 100.83333333+2.13162821e-14j
101.66666667-3.55271368e-15j 102.5          +1.95399252e-14j
103.33333333+1.06581410e-14j 104.16666667+7.10542736e-15j
105.          +3.55271368e-15j 105.83333333+0.00000000e+00j
106.66666667+1.42108547e-14j 107.5          -2.13162821e-14j
108.33333333-1.42108547e-14j 109.16666667+0.00000000e+00j
110.          +0.00000000e+00j 110.83333333-2.84217094e-14j
111.66666667-2.13162821e-14j 112.5          -7.10542736e-15j
113.33333333-2.84217094e-14j 114.16666667-2.84217094e-14j
115.          -2.84217094e-14j 115.83333333-2.84217094e-14j
116.66666667-2.84217094e-14j 117.5          -2.84217094e-14j
118.33333333-2.13162821e-14j 119.16666667-2.84217094e-14j
120.          -3.55271368e-14j 120.83333333-3.90798505e-14j
121.66666667-3.55271368e-14j 122.5          -3.90798505e-14j
123.33333333-4.08562073e-14j 124.16666667-2.84217094e-14j
125.          -3.49785217e-14j 125.83333333-2.48689958e-14j
126.66666667-3.73034936e-14j 127.5          -3.19744231e-14j
```



In [35]: *# Balanced not symmetrical system*

```
Ia = Im          ; a = 0;
Ib = 2.75*Im      ; b = - np.pi/2
ia = Ia*np.cos(2*np.pi*f*t + a)
ib = Ib*np.cos(2*np.pi*f*t + b)
ic = -ia-ib

i_alpha_beta = 2/3*(ia + ib*e**(2j*pi/3) + ic*e**(4j*pi/3))
plot(i_alpha_beta)
```



In this case we can elaborate the expression of the space vector in the following way.

At first we generalize the expression for the three phase system as:

$$i_a(t) = I_a \Re [e^{(j\omega t + a)}]$$

$$i_b(t) = I_b \Re [e^{(j\omega t + b)}]$$

$$i_c(t) = -I_a \Re [e^{(j\omega t + a)}] - I_b \Re [e^{(j\omega t + b)}]$$

We define now the space vector associated to the three current as:

$$\begin{aligned}
\vec{i} &= \frac{2}{3} \left\{ I_a \Re [e^{(j\omega t+a)}] + I_b \Re [e^{(j\omega t+b)}] \cdot e^{j\frac{2\pi}{3}} - I_a \Re [e^{(j\omega t+a)}] \cdot e^{j\frac{4\pi}{3}} - I_b \Re [e^{(j\omega t+b)}] \cdot e^{j\frac{4\pi}{3}} \right\} \\
&= \frac{2}{3} \left\{ I_a \Re [e^{(j\omega t+a)}] \left(1 - e^{j\frac{4\pi}{3}} \right) + I_b \Re [e^{(j\omega t+b)}] \left(e^{j\frac{2\pi}{3}} - e^{j\frac{4\pi}{3}} \right) \right\} \\
&= \frac{2}{3} \left\{ I_a \Re [e^{(j\omega t+a)}] \left(\frac{3}{2} + j\frac{\sqrt{3}}{2} \right) + I_b \Re [e^{(j\omega t+b)}] (j\sqrt{3}) \right\} \\
&= \frac{2}{3} \left\{ I_a \Re [e^{(j\omega t+a)}] \left(\sqrt{3} e^{j\frac{\pi}{6}} \right) + I_b \Re [e^{(j\omega t+b)}] \left(\sqrt{3} e^{j\frac{\pi}{2}} \right) \right\} \\
&= \frac{2}{3} \left\{ I_a \frac{[e^{(j\omega t+a)} + e^{-(j\omega t+a)}]}{2} \left(\sqrt{3} e^{j\frac{\pi}{6}} \right) + I_b \frac{[e^{(j\omega t+b)} + e^{-(j\omega t+b)}]}{2} \left(\sqrt{3} e^{j\frac{\pi}{2}} \right) \right\} \\
&= \frac{1}{\sqrt{3}} \left\{ e^{j\omega t} \left[I_a e^{j\left(\frac{\pi}{6}+a\right)} + I_b e^{j\left(\frac{\pi}{2}+b\right)} \right] + e^{-j\omega t} \left[I_a e^{j\left(\frac{\pi}{6}-a\right)} + I_b e^{j\left(\frac{\pi}{2}-b\right)} \right] \right\}
\end{aligned}$$

```

In [36]: # the direct component coefficient
dc = (Ia * e**( 1j*(pi/6+a) ) + Ib * e**( 1j*(pi/2+b) ) )/sqrt(3)

# the inverse component coefficient
ic = (Ia * e**( 1j*(pi/6-a) ) + Ib * e**( 1j*(pi/2-b) ) )/sqrt(3)

print('dc = ',dc)
print('ic = ',ic)

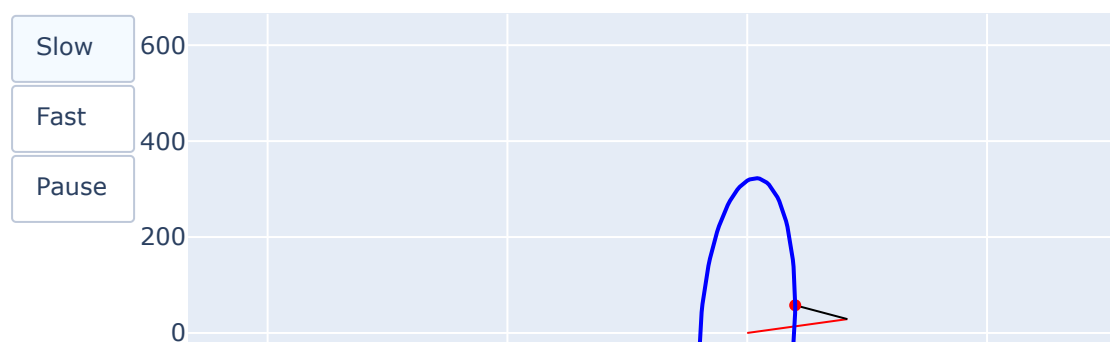
dc_vet = dc*e**(1j*2*np.pi*f*t)
ic_vet = ic*e**(-1j*2*np.pi*f*t)

dc = (208.77132402714713+28.867513459481287j)
ic = (-108.77132402714709+28.867513459481305j)

```

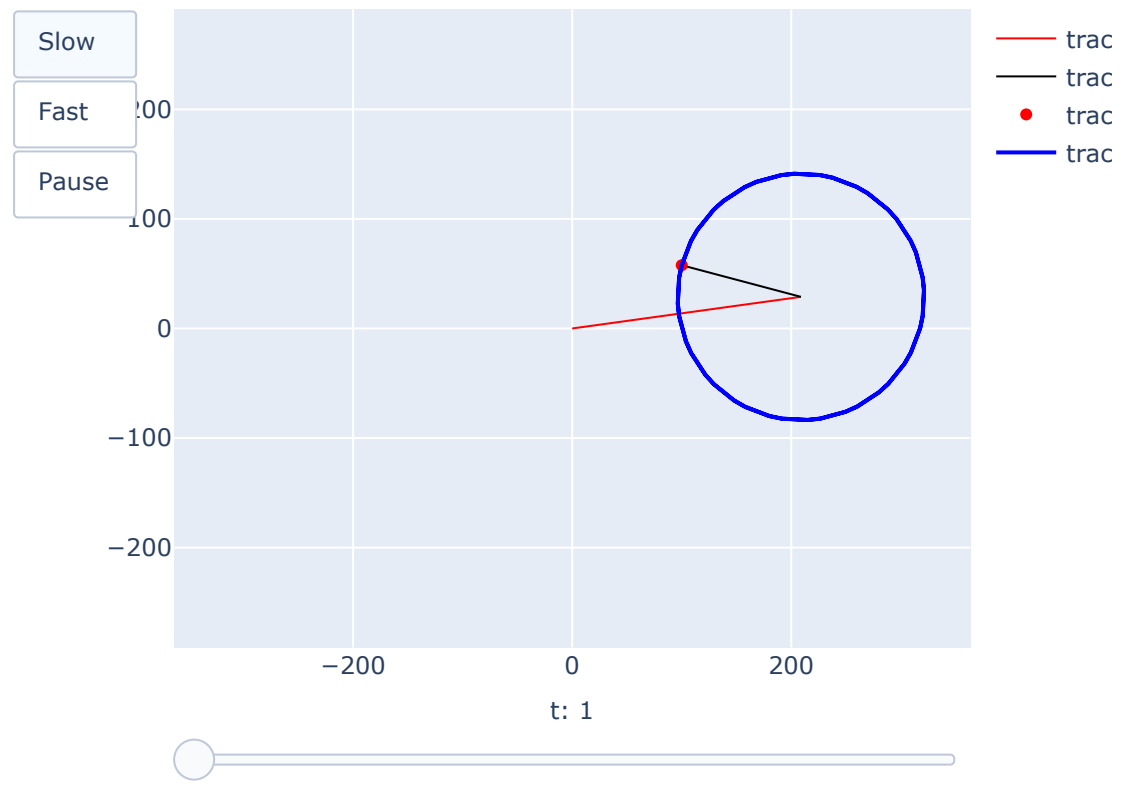
We can also plot the two components to better understand how they form the space vector trajectory in the time

In [37]:



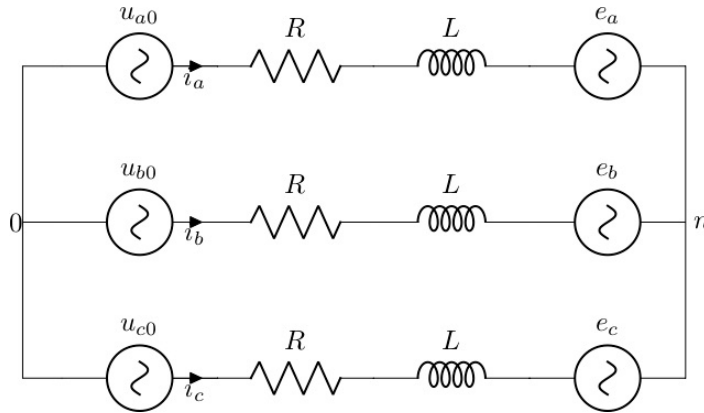
```
In [38]: # if we transform now in the d-q reference frame we will have:  
# the direct component at standstill  
# the inverse component at twice the speed.  
  
wdq = 2*np.pi*f  
i_dq = i_alpha_beta * e**(-1j*wdq*t)  
  
# we inspect i_dq  
#print(i_dq)
```

In [39]:



RLE circuit

We consider now the following circuit where the load is a balanced circuit composed by the resistors R , the inductances L and a set of three back EMF e .



The time domain equations describing the circuit are:

$$\begin{aligned} u_a &= Ri_a + L \frac{di_a}{dt} + e_a = u_{a0} - u_{n0} \\ u_b &= Ri_b + L \frac{di_b}{dt} + e_b = u_{b0} - u_{n0} \\ u_c &= Ri_c + L \frac{di_c}{dt} + e_c = u_{c0} - u_{n0} \end{aligned}$$

Multiplying the first equation for $\frac{2}{3}$, the second for $\frac{2}{3}e^{j2/3\pi}$ and the third for $\frac{2}{3}e^{j4/3\pi}$ and applying the definition of space vector we can obtain the following relation:

$$\vec{u} = R\vec{i} + L\frac{d\vec{i}}{dt} + \vec{e}$$

The homopolar component is missing since it has been lost in the transformation. Nevertheless the approach is still valid and general since, due to the connection of the load, even if a homopolar component would be present in the voltages, no homopolar component is allowed in the current.

The aim of the analysis is to compute the current response \vec{i} given the other circuit parameters, i.e. R , L , \vec{u} and \vec{e} .

Some examples are now presented.

Case #1: pure resistive load, $L=0$, $\vec{e} = 0$

In this case the equation becomes

$$\vec{u} = R\vec{i}$$

so that the voltage and current space vectors are related by the real constant R . They describe the same trajectory in the complex plane, being only their length properly scaled considering the actual unit of measurements.

Case #2: pure inductive load, $R=0$, $\vec{e} = 0$

In this case the equation becomes

$$\vec{u} = L \frac{d\vec{i}}{dt}$$

It can be rewritten as:

$$\frac{d\vec{i}}{dt} = \frac{\vec{u}}{L}$$

allowing to compute the current variation as a space-vector parallel to voltage.

If a balanced voltage set is considered, the space-vector \vec{u} has constant amplitude and rotate at constant speed, i.e.:

$$\vec{u} = U_m e^{j\omega t}$$

The current response is then computed as:

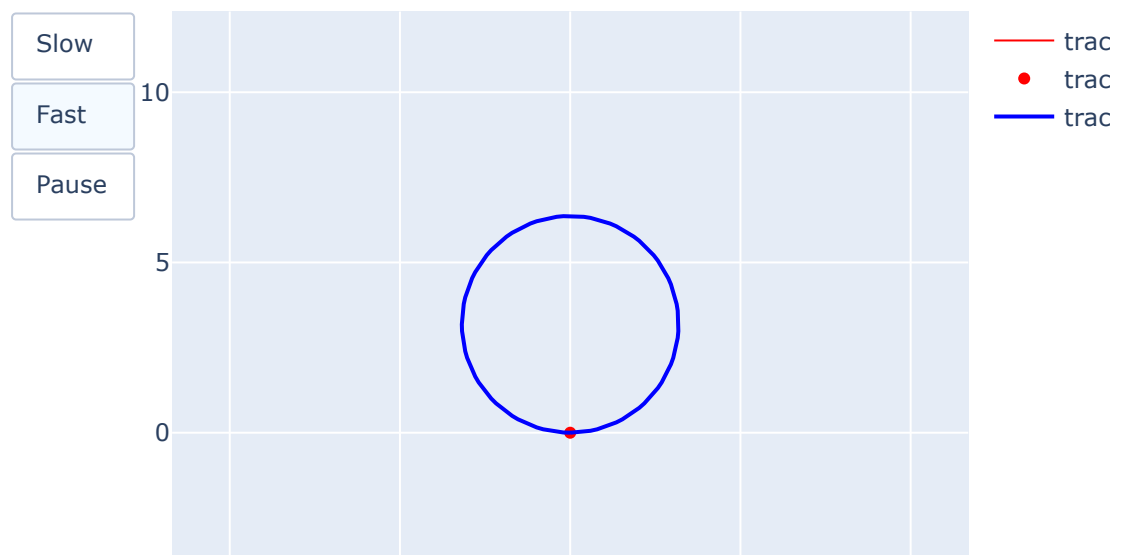
$$\begin{aligned}\vec{i} &= \vec{i}(0) + \frac{1}{L} \int_0^t U_m e^{j\omega t} dt \\ &= \vec{i}(0) + \frac{U_m}{j\omega L} (e^{j\omega t} - 1) \\ &= \vec{i}(0) + \frac{\vec{u} - U_m e^{-j\pi/2}}{j\omega L}\end{aligned}$$

```
In [133]: Um = 100
          L = 0.1
```

```
ua = Um*np.cos(2*np.pi*f*t)
ub = Um*np.cos(2*np.pi*f*t - 2/3*np.pi)
uc = Um*np.cos(2*np.pi*f*t - 4/3*np.pi)

u_sv = 2/3*(ua + ub*e**(2j*pi/3) + uc*e**(4j*pi/3))
#plot(u_sv)

i_sv = (u_sv - Um)/(2*np.pi*f*L) * e**(-1j*pi/2)
plot(i_sv,5,-5,8,-2)
```



Case 3: $\vec{e} = 0$, $\vec{u} = U_m e^{j\omega t}$

The system equation in this case is:

$$\vec{u} = R \vec{i} + L \frac{d\vec{i}}{dt}$$

This is a differential equation whose solution can be written in terms of the natural and forced response as:

$$\begin{aligned} \vec{i} &= \vec{i}_n + \vec{i}_f \\ &= \bar{A} e^{-t/\tau} + \frac{U_m}{R + j\omega L} e^{j\omega t} \end{aligned}$$

Where the time constant is $\tau = L/R$ and the complex constant \bar{A} is computed from the initial condition $\vec{i}(0) = 0$ achieving:

$$\bar{A} = -\frac{U_m}{R + j\omega L}$$

The complete current response is hence:

$$\vec{i} = -\frac{U_m}{R + j\omega L} e^{-t/\tau} + \frac{U_m}{R + j\omega L} e^{j\omega t}$$

In [9]:

```
Um = 100
L = 0.4
R = 20
tau = L/R # tau = L/R = 4e-3

t = np.linspace(0,4/f,201) # we take 2 periods

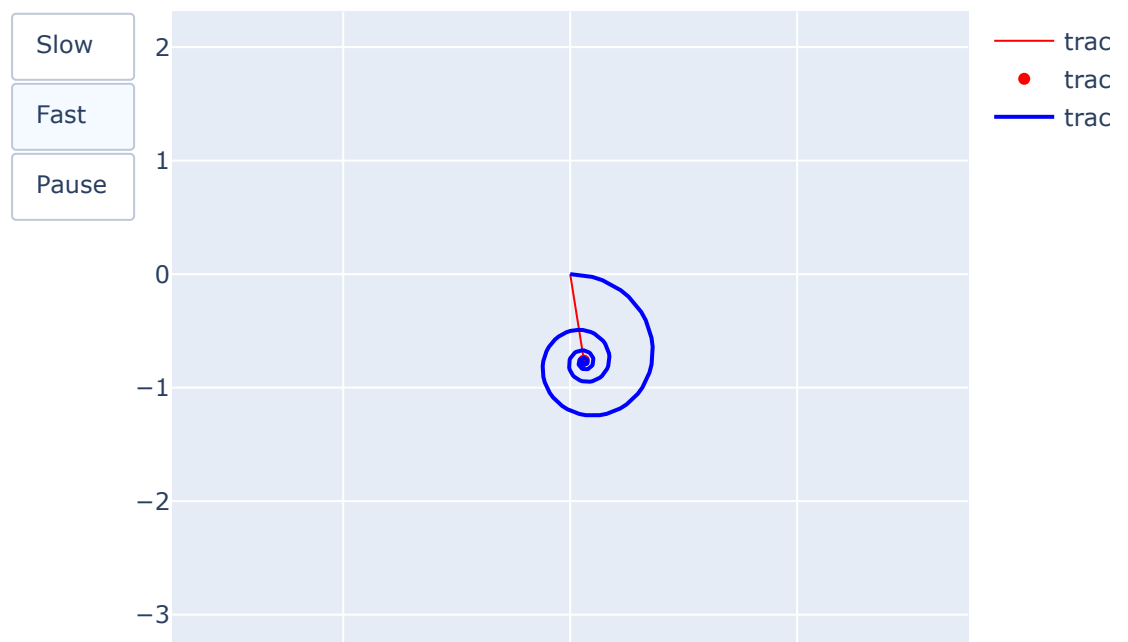
ua = Um*np.cos(2*np.pi*f*t)
ub = Um*np.cos(2*np.pi*f*t - 2/3*np.pi)
uc = Um*np.cos(2*np.pi*f*t - 4/3*np.pi)

u_sv = 2/3*(ua + ub*e**(2j*pi/3) + uc*e**(4j*pi/3))
#plot(u_sv)

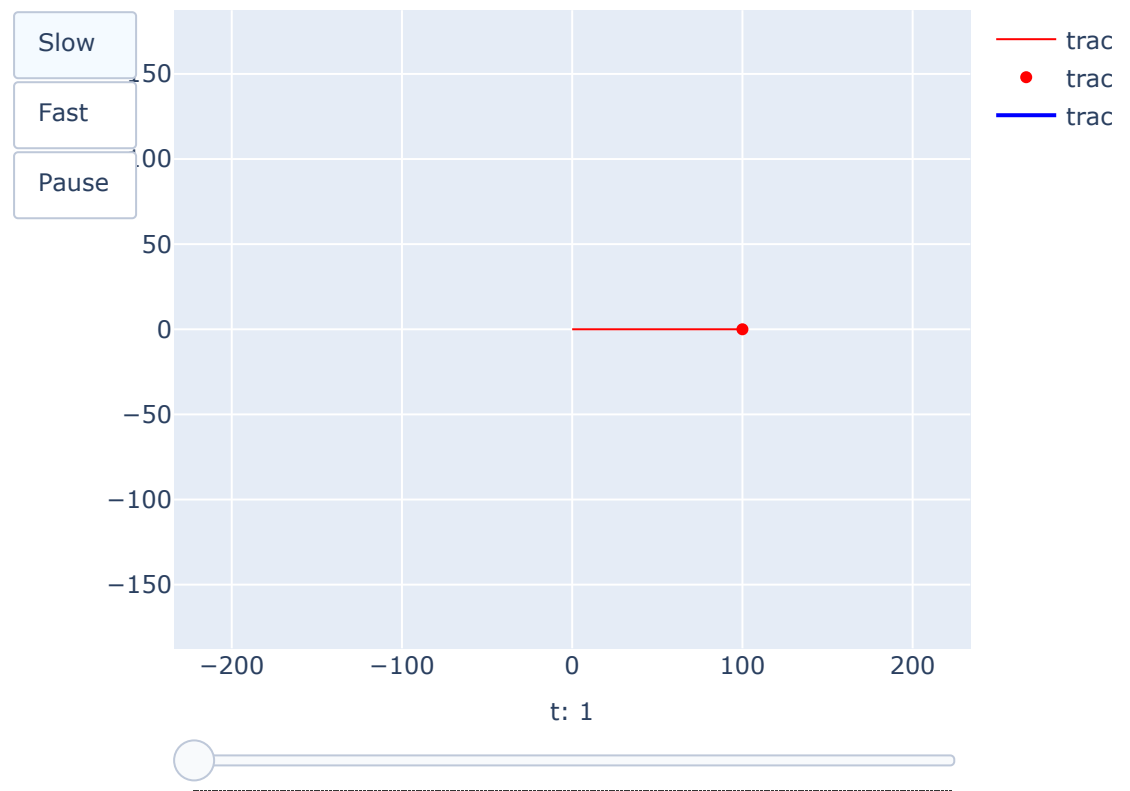
i_sv = - Um/(R + 2j*np.pi*f*L) * e**(-t/tau) + Um/(R + 2j*np.pi*f*L)
plot(i_sv,2,-2,2,-2)
```

```
In [10]: # we can consider the same current response  
# in the d-q reference frame
```

```
wdq = 2*np.pi*f  
i_dq = i_sv * e**(-1j*wdq*t)  
u_dq = u_sv * e**(-1j*wdq*t)  
  
plot(i_dq,1,-1,1,-2)
```



```
In [11]: #plot(u_sv)
plot(u_dq,100,-100,100,-100)
```



```
In [12]: atan(2*np.pi*f*L/R)*180/np.pi
```

```
Out[12]: (80.95693892096232+0j)
```

```
In [ ]:
```