Functional Analysis- teacher A. Cesaroni

Exam- Janauary 23, 2026- (90 minutes)

Exercise 1.

(1) Give the definition of absolutely continuous measure and of singular measure (with respect to the Lebesgue measure in $\mathbb{R}$). Recall the Lebesgue Radon Nikodym decomposition of a measure.

(2) Consider the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x^3 + 1 & 0 \leq x < 1 \\ 4 & x \geq 1 \end{cases}$$

and let μ_F the Borel measure associated to this function. Write the singular part of this measure and the density of the absolutely continuous part of this measure, if there exists any.

Exercise 2.
Let $M^2(\Omega, \mathbb{P}, \mathcal{F})$ the space of random variables with bounded second moment. For $X \in M^2$ denote as $\sigma(X)$ the minimal σ-algebra contained in $\mathcal{F}$ which contains all the sets $A_x = \{\omega \in \Omega, \mid X(\omega) \leq x\}$, for $x \in \mathbb{R}$ (so $\sigma(X)$ is the minimal σ-algebra such that X is measurable).

(1) Let $Y \in M^2(\Omega, \mathbb{P}, \mathcal{F})$. Define the conditional expectation $\mathbb{E}(Y|X)$ in terms of an orthogonal projection. State the orthogonal projection theorem.

(2) If Y is independent of X what is the conditional expectation $\mathbb{E}(Y|X)$?

Exercise 3. Let $\mathcal{P}(\mathbb{R})$ the space of probability measures on $\mathbb{R}$ (that is the space of all laws of real valued random variables).

(1) Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$. Recall the definition of coupling between μ and ν.

(2) Recall the definition of Wasserstein distance $W_2(\mu, \nu)$ among μ and ν.

(3) Let $\mu = \frac{1}{2}\delta_{1/3} + \frac{1}{2}\delta_{2/3}$ and $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. What is the Wasserstein distance W_2 among μ and ν?

(4) Let μ the measure associated to the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

and $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. What is the Wasserstein distance W_2 among μ and ν?

Sketch of solutions

Solution 1.

(2) F is continuous in $\mathbb{R} \setminus 0, 1$. Moreover $F'(x) = f(x) = 3x^2\chi_{(0,1)}$ and F is constant in $x < 0, x > 1$. Therefore the singular part of μ_F is given by $(F(0) - F(0^-))\delta_0 + (F(1) - F(1^-))\delta_1 = \delta_0 + 2\delta_1$ and the absolutely continuous part has density $f(x) = 3x^2\chi_{(0,1)}$. In particular for every Borel set A, $\mu_F(A) = \delta_0(A) + 2\delta_1(A) + \int_{A\cap[0,1]} 3x^2 dx$.

Solution 2.

(1) $\mathbb{E}(Y|X)$ is the orthogonal projection of Y onto the space $M^2(\Omega, \mathbb{P}, \sigma(X))$, that is the element $Z \in M^2(\Omega, \mathbb{P}, \sigma(X))$ which has minimal M^2 distance from Y. In particular $Z = h(X)$ for some $h : \mathbb{R} \to \mathbb{R}$ measurable and $Y - Z$ is orthogonal to $M^2(\Omega, \mathbb{P}, \sigma(X))$. Finally Y can be written in a unique way as an element of $M^2(\Omega, \mathbb{P}, \sigma(X))$ and an element orthogonal to $M^2(\Omega, \mathbb{P}, \sigma(X))$.

(2) Note that $\mathbb{E}(Y) \in M^2(\Omega, \mathbb{P}, \sigma(X))$, since it is a constant. Moreover $Y - \mathbb{E}(Y)$ is orthogonal to $M^2(\Omega, \mathbb{P}, \sigma(X))$, by independence: indeed $\mathbb{E}(X(Y - \mathbb{E}(Y))) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0$. By uniqueness of the orthogonal projection, it holds $\mathbb{E}(Y|X) = \mathbb{E}(Y)$.

Solution 3.

(3) Observe that there exists two deterministic couplings between the two measures: one associated to ψ_1 sending 1/3 in 0 and 2/3 in 1 and the other associated to ψ_2 sending 1/3 in 1 and 2/3 in 0 . The first one is monotone, since $\psi_1(0) < \psi_1(1/2)$ and it is the optimal one.

$$W_2(\mu,\nu)^2 = \int_{\mathbb{R}} |x - \psi_1(x)|^2 d\mu(x) = \frac{1}{2}|1/3 - \psi_1(1/3)|^2 + \frac{1}{2}|2/3 - \psi_1(2/3)|^2 =$$

$$= \frac{1}{2}|1/3|^2 + \frac{1}{2}|2/3 - 1|^2 = \frac{1}{2}\frac{1}{9} + \frac{1}{2}\frac{1}{9} = \frac{1}{9}.$$

(4) Observe that μ is absolutely continuous with respect to Lebesgue with density $f(x) = 1\chi_{[0,1]}(x)$. The cumulative distribution function associated to ν is given by

$$F_\nu(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

We compute for $t \in (0, 1)$,

$$F_\nu^-(t) = \inf\{x | F_\nu(x) \ge t\} = \begin{cases} 0 & t < \frac{1}{2} \\ 1 & t \ge \frac{1}{2}. \end{cases}$$

The optimal coupling is a deterministic coupling given by

$$(F_\nu^- \circ F_\mu)(x) := \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$$

that is $(F_\nu^- \circ F_\mu)_\sharp \mu = \nu$. Therefore

$$W_2(\mu,\nu)^2 = \int_{\mathbb{R}} |x - F_\nu^- \circ F_\mu(x)|^2 d\mu(x) = \int_0^1 |x - F_\nu^-(x)|^2 dx =$$

$$= \int_0^{1/2} |x|^2 dx + \int_{1/2}^1 |x-1|^2 dx = \int_0^{1/2} x^2 dx + \int_{1/2}^1 x^2 - 2x + 1 dx =$$

$$= [\frac{1}{3}x^3]_0^{1/2} + [\frac{1}{3}x^3 - x^2 + x]_{1/2}^1 = \frac{1}{3}\frac{1}{2^3} + \frac{1}{3} - 1 + 1 - \frac{1}{3}\frac{1}{2^3} + \frac{1}{2^2} - \frac{1}{2}$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$