density

$\mathbb{R}^n$ $\quad k < n$

$E \in \mathcal{B}(\mathbb{R}^n)$ $\quad$ (E a measurable set for $\mathcal{H}^k$)

$\mathcal{H}^k(E) > 0$

density $\forall x \in \mathbb{R}^n$

$$\lim_{r \to 0^+} \frac{\mathcal{H}^k(E \cap B(x,r))}{\omega_k r^k} = f(x)$$

if E is k-rectifiable $\quad f(x) = 0$ for $\mathcal{H}^k$-a.e. $x \in \mathbb{R}^n \setminus E$

$f(x) = 1$ for $\mathcal{H}^k$ a.e. $x \in E$.

if E is not k-rect. $\quad 0 < c \le \limsup_{r \to 0^+} \cancel{\mathcal{H}^k}(\ ) \le 1$

Back to Radon measures

$\boxed{U \text{ open set}} \subseteq \mathbb{R}^n$ $\qquad M(U) =$ Radon measures (in $\mathbb{R}^n$) restricted to U

$$C_c(U) = \{ f : U \to \mathbb{R} \quad f \in \mathcal{C}(U)$$
supp f is a compact set inside U }

$\Big(f_n \to f$ in $\mathcal{C}_c(U)$ if $\exists K \subset\subset U$
supp f_n, supp $f \subseteq K$
$\|f_n - f\|_\infty \to 0 \Big)$

Def : $I : \mathcal{C}_c(U) \longrightarrow \mathbb{R}$

$f \longmapsto I(f) \in \mathbb{R}$.

- I <u>linear</u> $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$
  $\forall f, g \in C_c(U)$ $\alpha, \beta \in \mathbb{R}$

- I is <u>positive</u> [if $f \geq 0$ ($f(x) \geq 0$ $\forall x \in U$)) then $I(f) \geq 0$]

Proposition Let I be a <u>positive</u> <u>linear</u> functional on $\mathcal{C}_c(U)$ then

$\forall K \subset\subset U$ $\exists C_K > 0$ $|I(f)| \leq C_K \|f\|_\infty$
(K compact inside U) $\forall f \in \mathcal{C}_c(U)$ supp $f \subseteq K$

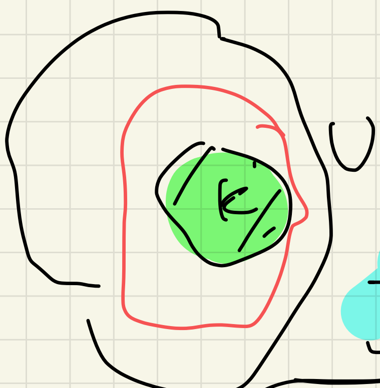
Obs: if $f_n \to f$ in $C_c(U)$ then $I(f_n) \to I(f)$

proof: Fix $K \subset\subset U$

$0 \le \phi \le 1$ $\phi = 1$ on K

$\phi = 0$ on $\mathbb{R}^n \setminus U$

$\mathrm{supp}(\phi)$ is a compact set inside U.



$\forall f \in C_c(U)$ $\mathrm{supp}\, f \subseteq K$

$-\|f\|_\infty \phi(x) \le f(x) \le \|f\|_\infty \cdot \phi(x)$ $\forall x \in \overline{U}$

by linearity and positivity

$$\boxed{-\|f\|_\infty I(\phi) \le I(f) \le \|f\|_\infty I(\phi)}$$

$$f(x) + \|f\|_\infty \, \phi(x) \geq 0 \qquad \forall x \in U$$

positivity

$$I\left(f + \|f\|_\infty \, \phi\right) \geq 0$$

linearity

$$I(f) + \|f\|_\infty \, I(\varphi) \geq 0$$

$$I(f) \geq -\|f\|_\infty \, I(\phi)$$

$$C_K := I(\phi) \qquad \text{for } \phi \qquad \phi \equiv 1 \text{ on } K$$

$$0 \leq \phi \leq 1 \qquad \phi \in \mathcal{C}_c(U).$$

Riesz representation theorem

(Riesz 1909, Markov 1930, Kakutani 1941)

Let $I : C_c(U) \to \mathbb{R}$ be a positive linear functional.

There exists a unique μ Radon measure such that $\forall f \in C_c(U)$

$$I(f) = \int_{\mathbb{R}^n} f(x)\, d\mu = \int_U f(x)\, d\mu .$$

$\forall A \subseteq U$ open

$$\mu(A) = \sup \left\{ I(f) \; , \; \begin{array}{l} f \in C_c(U) \\ \operatorname{supp} f \subset\subset A \\ 0 \le f \le 1 \end{array} \right\}$$

$\forall K \subset\subset U$ K compact

$$\mu(K) = \inf \left\{ I(f) \; , \; \begin{array}{l} f \in C_c(U) \\ f(x) \ge \chi_K(x) \quad \forall x \in U \end{array} \right\}$$

Obs.

1) if μ is Radon then $C_c(U)$, $f \longrightarrow \int_U f \, d\mu$ is linear, positive

2) μ is constructed starting from open sets $A \subseteq U$, A open

$$\mu(A) = \sup \left\{ I(f) \ , \ f \in C_c(U) \ , \ 0 \leq f \leq 1 \ , \ \operatorname{supp} f \subset\subset A \right\}$$

$\forall E$ subset of U

$$\mu^*(E) = \inf \left\{ \mu(A) \ , \ A \supseteq E \quad A \text{ open} \right\}$$

$\downarrow$ outer measure $\to$ Carathéodory criterium.

$U = \mathbb{R}^n$

space of Radon measure in $\mathbb{R}^n$ has the structure of "dual space of $C_c(\mathbb{R}^n)$ (separable space)

$\boxed{\text{CONVERGENCE}}$

μ_k, μ Radon measures in $\mathbb{R}^n$

$\mu_k \xrightarrow{*} \mu$ (μ_k converges weakly* to μ if $\forall f \in \mathcal{C}_c(\mathbb{R}^n)$

$$\underbrace{\int_{\mathbb{R}^n} f(x)\, d\mu_k}_{} \longrightarrow \int_{\mathbb{R}^n} f(x)\, d\mu(x)$$

COMPACTNESS theorem

If $\forall F \subset\subset \mathbb{R}^n$ (compact) $\exists C_F > 0$

$$\underbrace{0 \le \mu_k(F)} \le C_F \quad \forall k \in \mathbb{N}$$

$\exists$ subsequence μ_{k_j} and μ Radon measure such that

$$\mu_{k_j} \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathbb{R}^n$$

$U \subseteq \mathbb{R}^n$ open

I linear positive functional on $C_c(U)$

$\downarrow$

μ Radon measure

$$\mu(U) = \sup\{ I(f) \quad 0 \le f \le 1 \quad f \in C_c(U) \}$$

$$I(f) = \int_U f(x)\, d\mu$$

if $\mu(U) < +\infty \Rightarrow I$ can be extended to a continuous linear f. $I : (C_0(U), \|\cdot\|_\infty) \to \mathbb{R}$

$$\boxed{\left(C_0(U), \|\cdot\|_\infty\right) \text{ is the closure of } C_c(U), \text{ with respect to } \|\cdot\|_\infty}$$

I is a continuous linear op on $\left(C_0(U), \|\cdot\|_\infty\right)$

$\downarrow$

$\mu(U) = \|I\|$ operator norm.

$$\boxed{\forall f \in C_0(U) \quad |I(f)| \leq \mu(U) \cdot \|f\|_\infty}$$