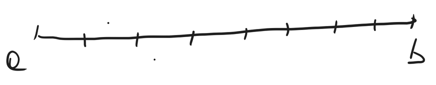
INTEGRAZIONE NUMERICA

$$\int_a^b f(x)\,dx = \lim_{M\to+\infty} \sum_{i=0,M} f\left(a+\frac{(b-a)}{M}i\right)\frac{(b-a)}{M}$$



N intervalli $x_0=a$ e $x_N=b$

$h=\frac{b-a}{N}$

$$\int_{a=x_0}^{b=x_N} f(x)\,dx \approx \sum_{0,i} f_i h$$

METODO DEI RETTANGOLI NAIF

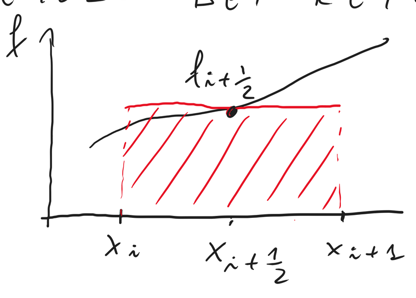
$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \int_{x_i}^{x_{i+1}} (f_i+O(h))\,dx = f_i h + O(h^2)$$

$$\int_{x_0}^{x_N} f(x)\,dx = \sum_{i=0,N-1} f_i h + N O(h^2) \qquad h=\frac{b-a}{N}$$

$$= \sum_{i=0,N-1} f_i h + N O\left(\frac{1}{N^2}\right) = \sum_{i=0,N-1} f_i h + O\left(\frac{1}{N}\right)$$

METODO DEI RETTANGOLI

idea



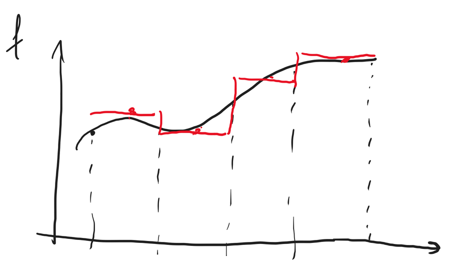
$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \int_{x_i}^{x_{i+1}} \left[f_{i+\frac{1}{2}} + f^1_{i+\frac{1}{2}}(x-x_{i+\frac{1}{2}}) + O(h^2)\right] dx$$

$$= \int_{-h/2}^{h/2} \left(f_{i+\frac{1}{2}} + f^1_{i+\frac{1}{2}} y + O(h^2)\right) dy = f_{i+\frac{1}{2}} h + O(h^3)$$

$$\int_{x_0}^{x_N} f(x)\,dx = \sum_{i=0,N-1} f_{i+\frac{1}{2}} h + \underbrace{N O(h^3)}_{O\left(\frac{1}{N^2}\right)}$$

METODO DEI RETTANGOLI

(è un metodo 'aperto')



METODO DEI TRAPEZI

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \int_{x_i}^{x_{i+1}} \left[f_i + \left(\frac{f_{i+1}-f_i}{h} + O(h)\right)(x-x_i) + O(h^2)\right] dx$$

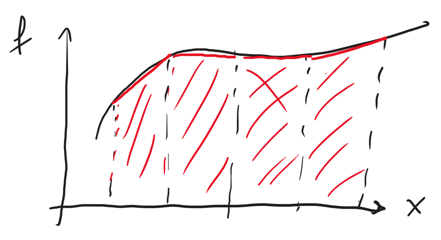
$$= \int_0^h \left(f_i + \frac{f_{i+1}-f_i}{h} y + O(h) y + O(h^2)\right) dy$$

$$= f_i h + \left(\frac{f_{i+1}-f_i}{2h}\right) h^2 + O(h^3) = \left(\frac{f_i+f_{i+1}}{2}\right) h + O(h^3)$$

$$\int_{x_0}^{x_N} f(x)\,dx = \sum_{i=0,N-1} \frac{f_i+f_{i+1}}{2} h + O\left(\frac{1}{N^2}\right)$$

Vediamo che $f_i + \left(\frac{f_{i+1}-f_i}{h}\right)(x-x_i)$

è una retta che passa per (x_i, f_i) e (x_{i+1}, f_{i+1})



METODO DI SIMPSON

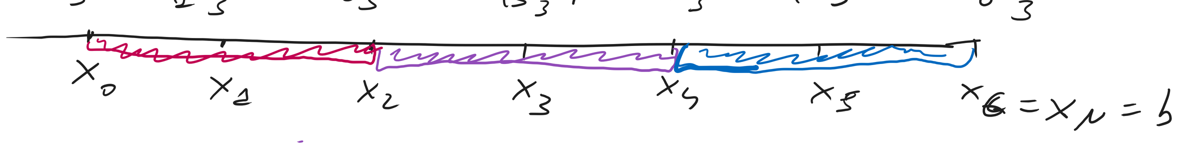
$$\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} \left[f_i + \left(\frac{f_{i+1}-f_{i-1}}{2h} + O(h^2)\right)(x-x_i) + \frac{1}{2}\left(\frac{f_{i+1}-2f_i+f_{i-1}}{h^2} + O(h^2)\right)(x-x_i)^2 + \frac{f_i^3}{3!}(x-x_i)^3 + O(h^4)\right] dx$$

$$= \int_{-h}^{h} \left[f_i + \frac{1}{2}\left(\frac{f_{i+1}-2f_i+f_{i-1}}{h^2} + O(h^2)\right) y^2 + O(h^4)\right] dy$$

$$= f_i 2h + \cancel{\frac{1}{2}} \frac{f_{i+1}-2f_i+f_{i-1}}{h^2} \frac{\cancel{2}}{3} h^3 + O(h^5)$$

$$= h\left(f_i\left(2-\frac{2}{3}\right) + f_{i+1}\frac{1}{3} + f_{i-1}\frac{1}{3}\right) = f_{i+1}\frac{h}{3} + \frac{4}{3} h f_i + f_{i-1}\frac{h}{3}$$

$f_0 \cdot \frac{1}{3}h$ $f_1 \frac{4}{3}h$ $f_2 \frac{2}{3}h$ $f_3 \frac{4}{3}h$ $f_4 \frac{2}{3}h$ $f_5 \frac{4}{3}h$ $f_6 \frac{1}{3}h$



l'accuratezza del metodo di Simpson $O\left(\frac{1}{N^4}\right)$

Ci accorgiamo che integrare con Simpson equivale ad interpolare con polinomio di secondo grado

$$\tilde{f}(x) = f_i + \frac{f_{i+1}-f_{i-1}}{2h}(x-x_i) + \frac{1}{2}\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}(x-x_i)^2$$

$\tilde{f}(x_i) = f_i$ $\qquad \tilde{f}(x_{i-1}) = f_{i-1}$

$\tilde{f}(x_{i+1}) = f_{i+1}$