
UNIVERSITÀ
DEGLI STUDI
DI PADOVA

# Network Science

A.Y. 24/25

ICT for Internet & multimedia, Data science, Physics of data

# Course overview

Network science 24/25

# Lecturer

tomaso.erseghe@unipd.it
room 217, DEI/A


Tomaso Erseghe

lectures: wed 8:30-10:00 & fri 16:30-18:00
www.dei.unipd.it

office hours: contact me by email

# In this course you'll also meet



# Prerequisites

## Basic requirements (that you already satisfy)

Calculus and linear algebra
Familiarity with a programming language
(Python, R, MatLab, C, Java, etc.)
Probability theory / Statistics

## Other useful knowledge

Networking processes in economics, telecommunications, semantics, etc …
Otpimization, machine learning, deep learning, etc …

# Which programming language?

- Python

  very good at scraping data (e.g., via Twitter APIs), polishing, plotting graphs, implementing algorithms

- R

  very good for memory storage, plotting graphs, implementing algorithms

- MatLab

  An alternative for algorithms and graph plotting

  *University license available*

  https://www.ict.unipd.it/servizi/servizi-utenti-istituzionali/contratti-software-e-licenze/matlab

# What about you?

Why did you pick the course?

Which is your background?
Who knows about deep learning?

Do you know Python?
and CoLab?

What do you expect
from this course?

Do you have a laptop?

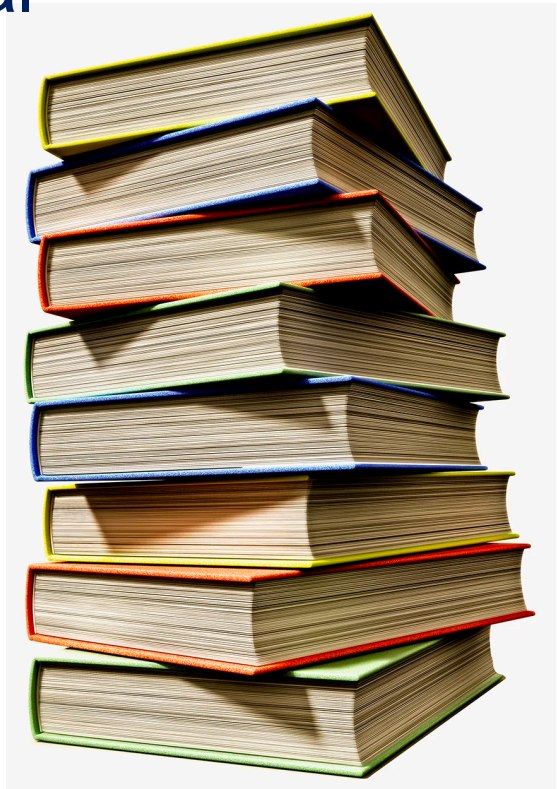
Are you interested in an
interdisciplinary work?

# Textbooks

## No textbook! ☺

Slides/videos & additional material available

@ stem.elearning.unipd.it



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# A few useful books

but we will not cover them completely

- A.L. Barabási, «Network science»

  http://barabasi.com/networksciencebook

  (these slides = Ch.1 "Introduction")

- J. Lescovec, «Machine learning with graphs»

  http://web.stanford.edu/class/cs224w

- M. Newman, «Networks: an introduction»

  Oxford University Press, 2010

- R. van der Hofstad, «Random graphs and complex networks»

  http://www.win.tue.nl/~rhofstad/NotesRGCN.html

# Project based exam

- Written exam
  - multiple choice questions (30 min)
  - 2 open questions (30+30 min)

- Project
  - extract network analytics using your preferred programming language(s)
  - oral presentation: slides + code
  - 10 min presentation (slides)
  - 5 min for questions

Final grade: 40% written exam, 60% project

# Exam sessions

- Feb 3, 2025 (Mon) - 14:00 (written) 15:30 (oral), Le
- IP day Feb 12, 2025 (Wed) - 9:00 (oral), tbd
- Feb 19, 2025 (Wed) - 14:00 (written) 15:30 (oral), Ae
- July 1, 2025 (Tue) - 10:00 (written) 11:30 (oral), Le
- Sep 9, 2025 (Tue) - 14:00 (written) 15:30 (oral), Le

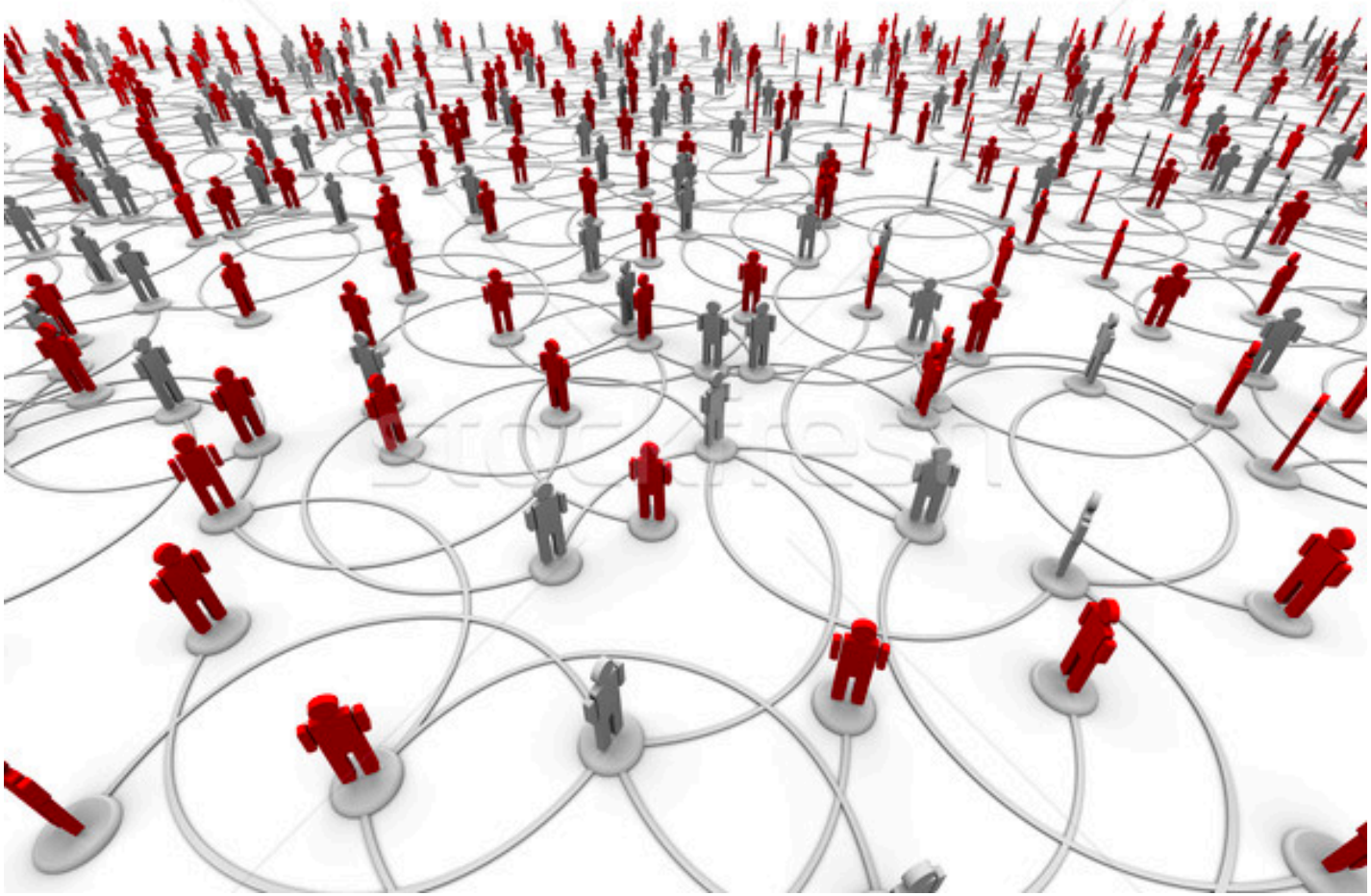
PS: You will be asked to enrol in

www.uniweb.unipd.it

# Contents

a brief overview

# This course is about networks

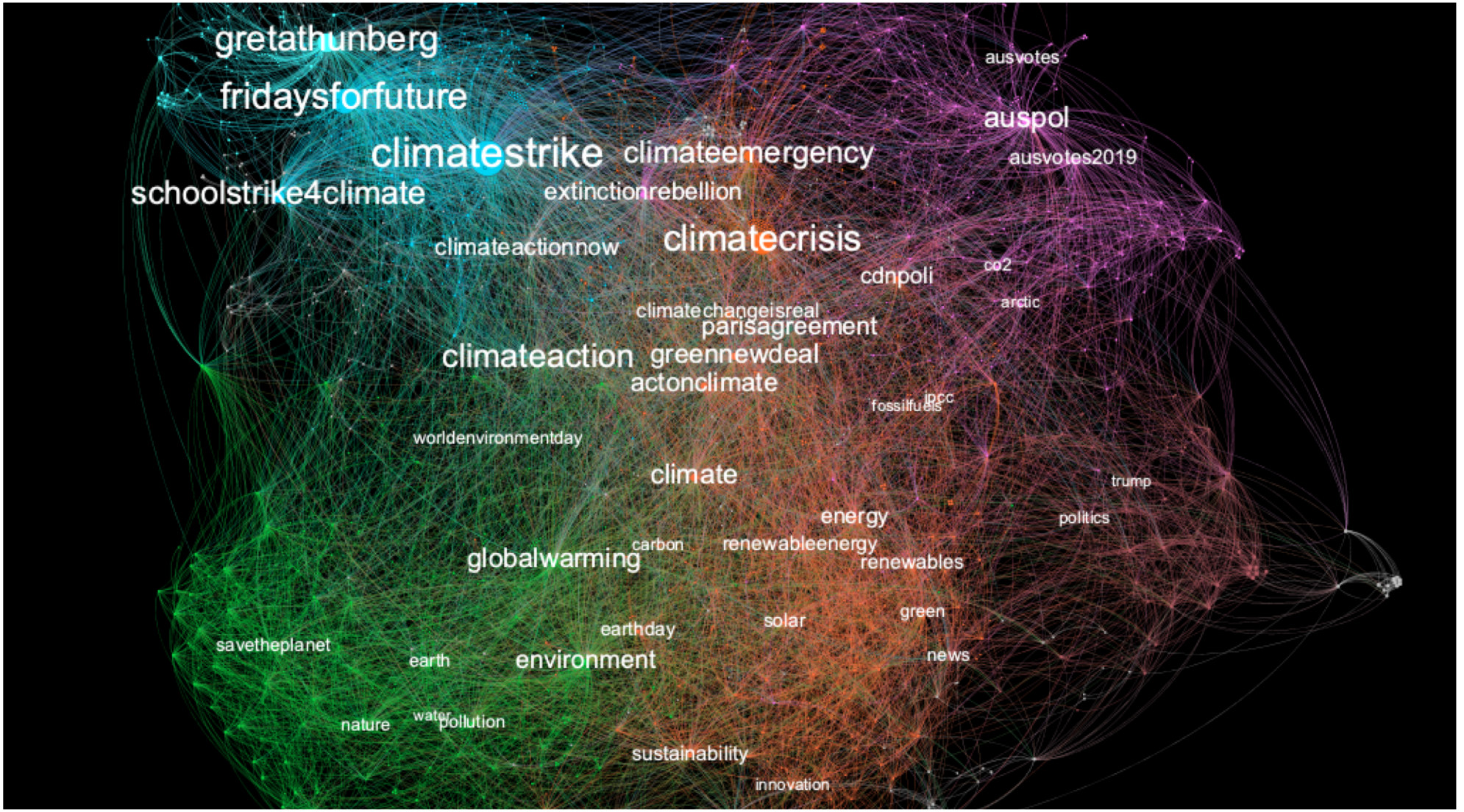


Network = anything that interconnects

e.g., people sharing friendship in a social network platform

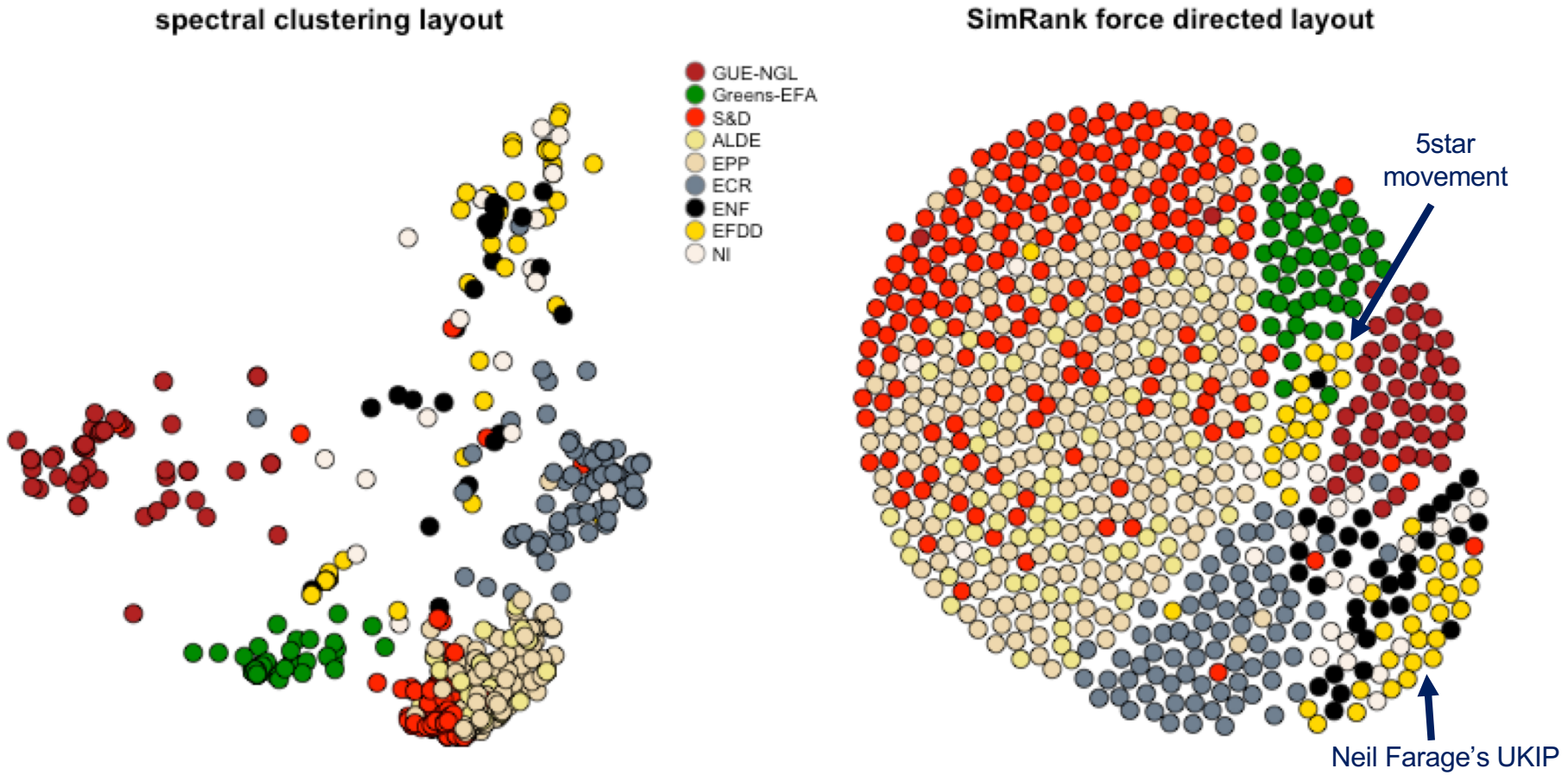
# Network example

2019 hashtag network related to #climatechange
from Twitter, after #gretathunberg



# Network examples (cont'd)

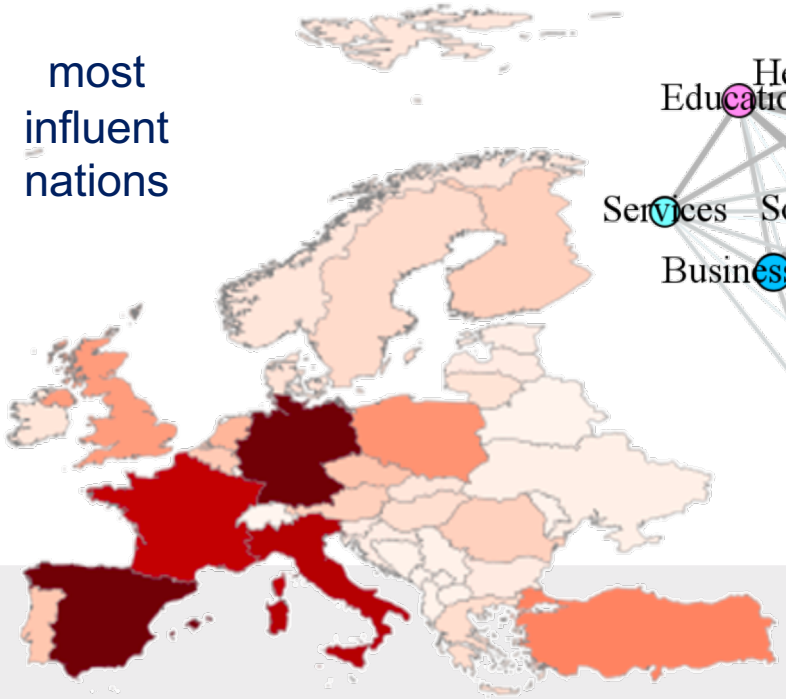
April-May 2016 political network (votes at the EU parliament)



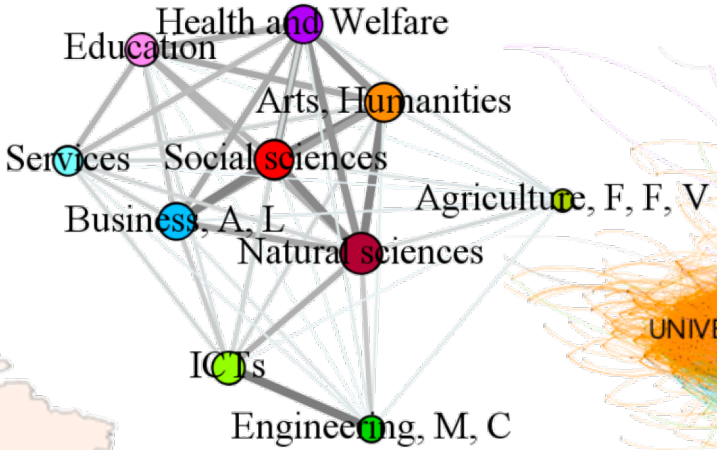
# Network examples (cont'd)

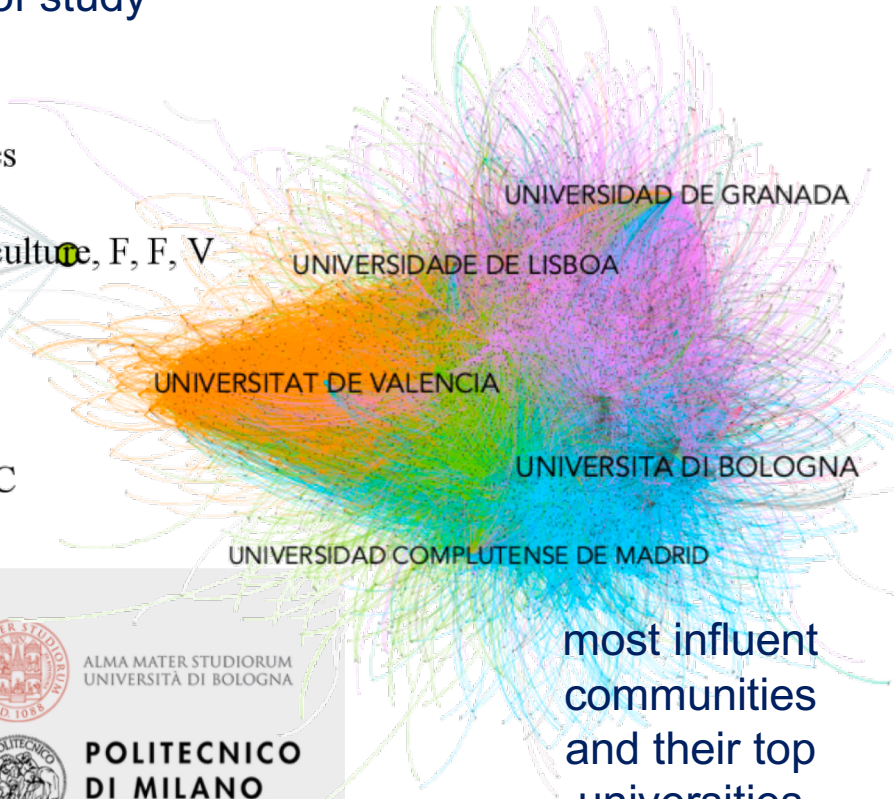
## Erasmus exchanges network 2019

most influent nations



fields of study




most influent communities and their top universities

Which are the more connected Italian institutions?

01

ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

POLITECNICO DI MILANO

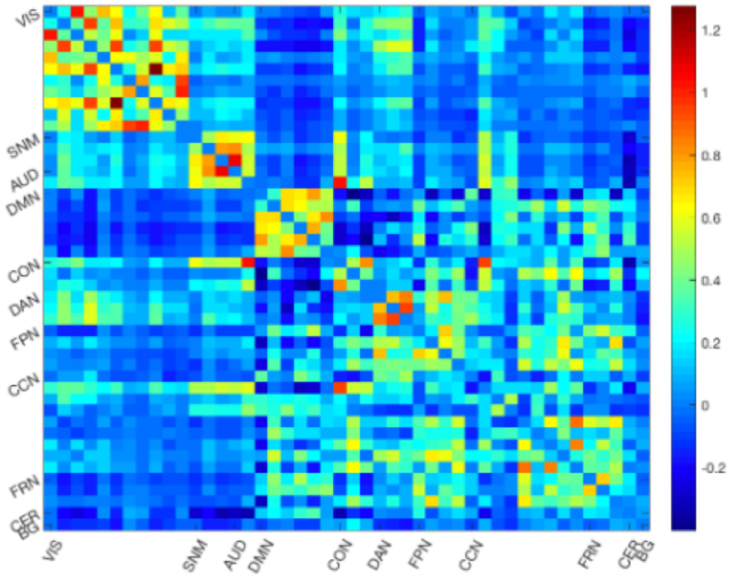
UNIVERSITÀ DEGLI STUDI DI PADOVA

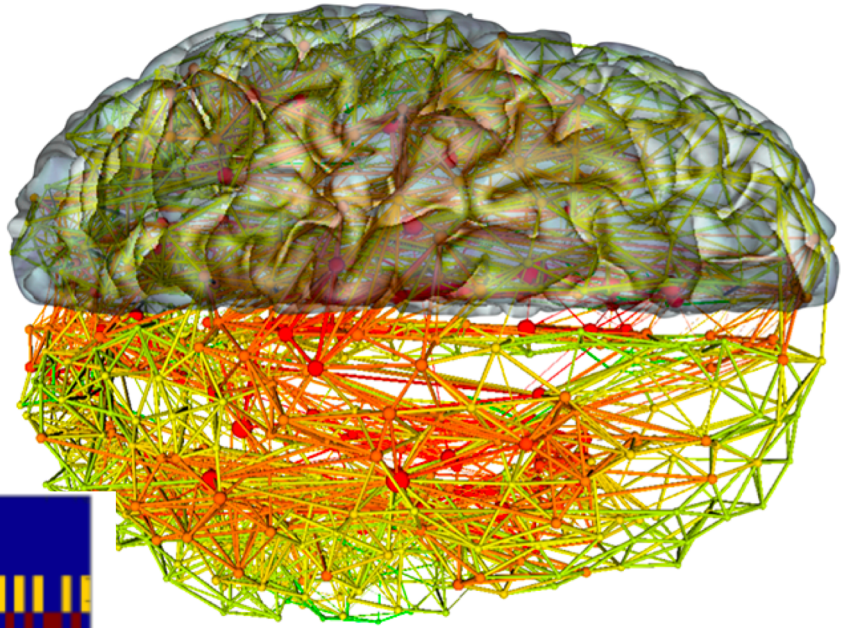
SAPIENZA UNIVERSITÀ DI ROMA

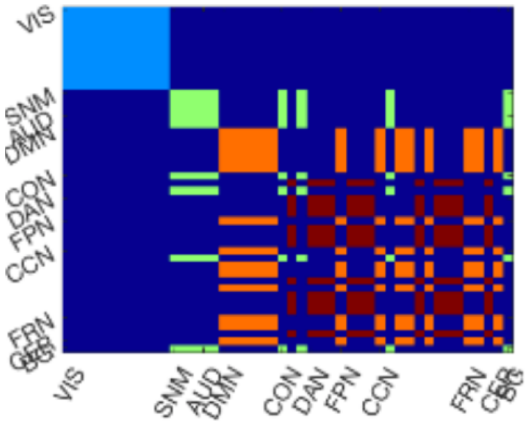
UNIVERSITÀ DEGLI STUDI FIRENZE

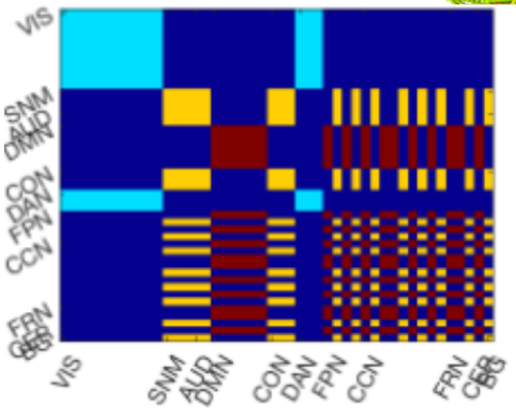
# Network examples (cont'd)

## the brain network – functional connectivity





# What is then network science?

## Network science

From Wikipedia, the free encyclopedia

*For other uses, see Network (disambiguation).*

**Network science** is an academic field which studies complex networks such as telecommunication networks, computer networks, biological networks, cognitive and semantic networks, and social networks, considering distinct elements or actors represented by *nodes* (or *vertices*) and the connections between the elements or actors as *links* (or *edges*). The field draws on theories and methods including graph theory from mathematics, statistical mechanics from physics, data mining and information visualization from computer science, inferential modeling from statistics, and social structure from sociology. The United States National Research Council defines network science as "the study of network representations of physical, biological, and social phenomena leading to predictive models of these phenomena."[1]

# … what about social network analysis?

## Social network analysis

From Wikipedia, the free encyclopedia

**Social network analysis** (**SNA**) is the process of investigating social structures through the use of networks and graph theory.[1] It characterizes networked structures in terms of *nodes* (individual actors, people, or things within the network) and the *ties*, *edges*, or *links* (relationships or interactions) that connect them. Examples of social structures commonly visualized through social network analysis include social media networks,[2][3] memes spread,[4] information circulation,[5] friendship and acquaintance networks, business networks, knowledge networks,[6][7] difficult working relationships,[8] social networks,

Social network analysis has emerged as a key technique in modern sociology. It has also gained a significant following in anthropology, biology,[12] demography, communication studies,[3][13] economics, geography, history, information science, organizational studies,[6][8] political science, public health,[14][7] social psychology, development studies, sociolinguistics, and computer science[15] and is now commonly available as a consumer tool (see the list of SNA software).[16][17][18][19]

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# And how do we study networks?

With a holistic character

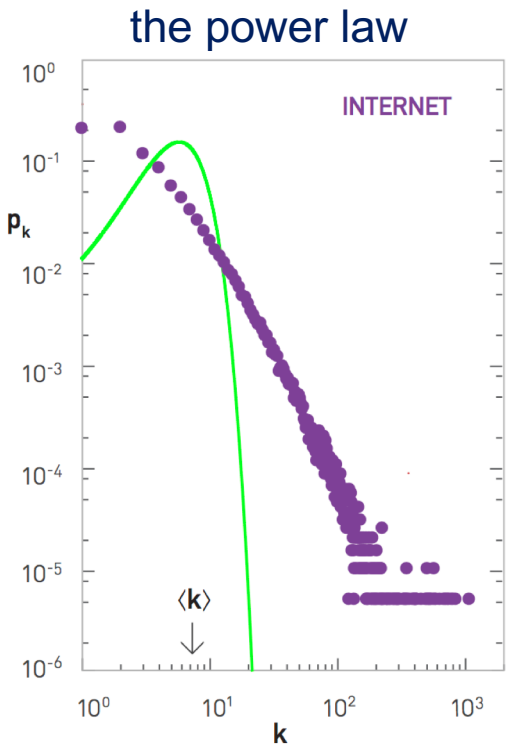
(the whole is greater than the sum of its parts)

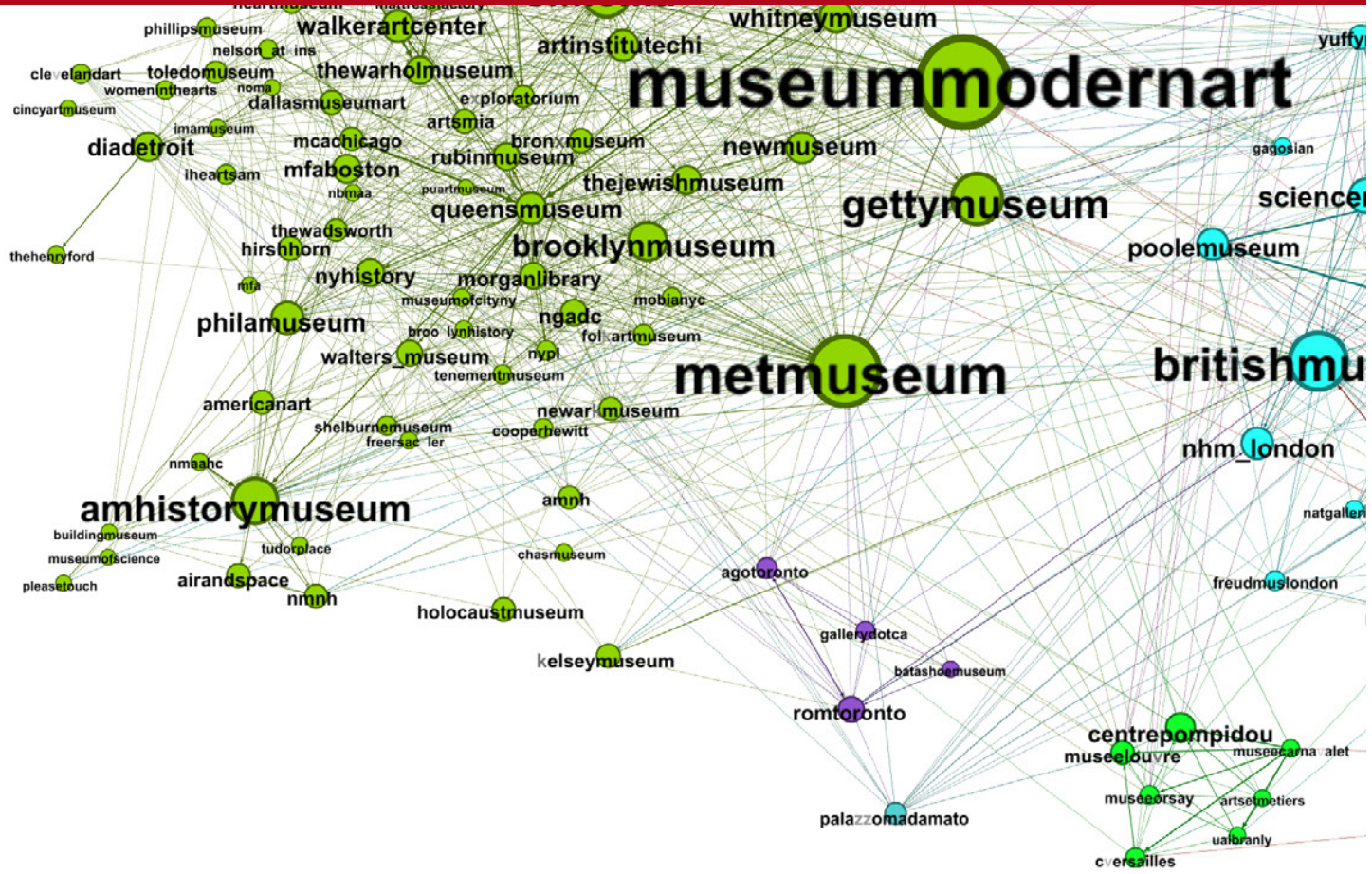
With mathematical rigour

The approach is

empirical (driven by concrete data),
precise (requires a proper formalism),
interdisciplinary (can be applied to several fields), and
challenging (in data size and in objectives)

# And what do we study?

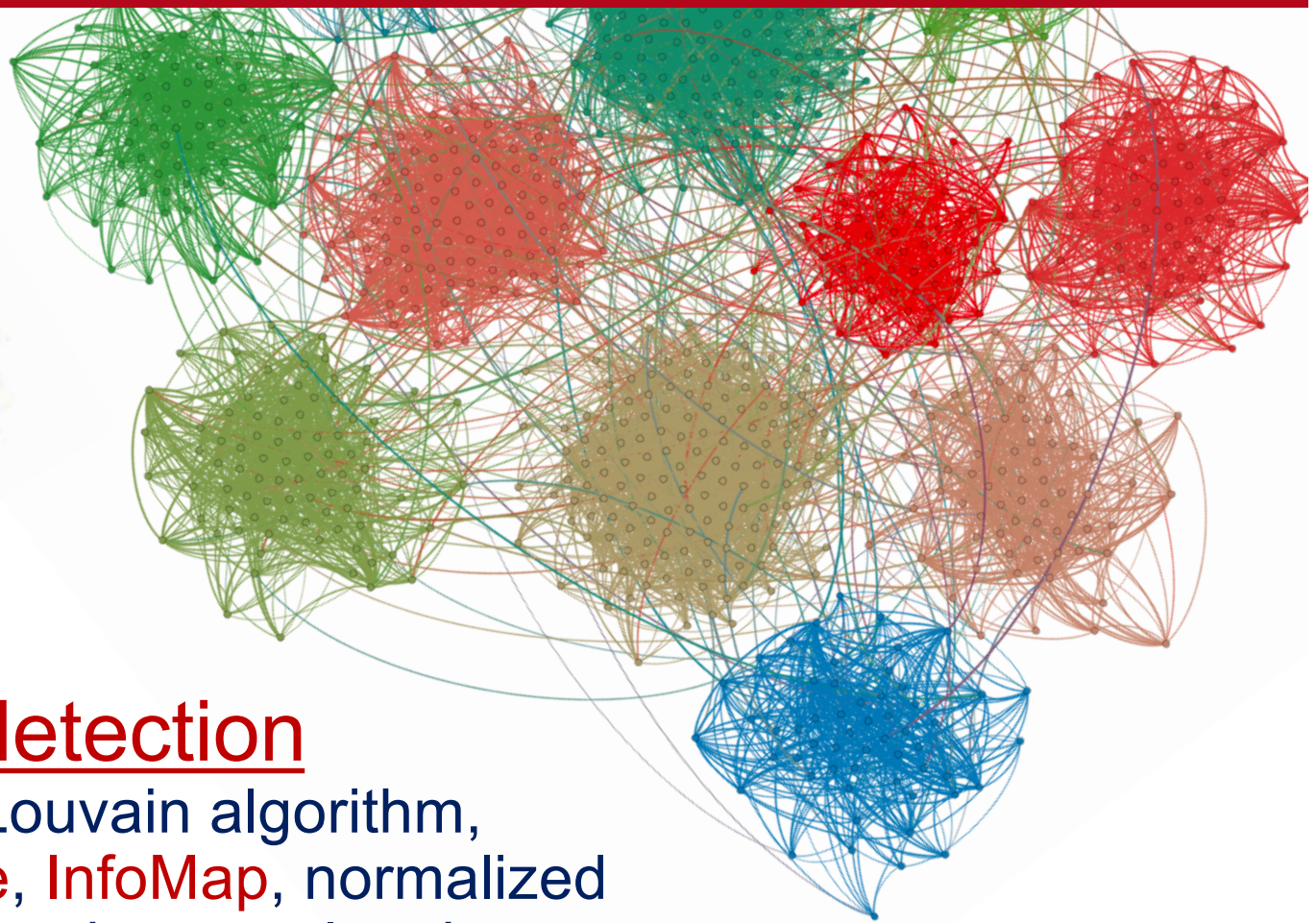



many network analytics, e.g., centrality

- degree, PageRank, HITS, betweenness, etc.

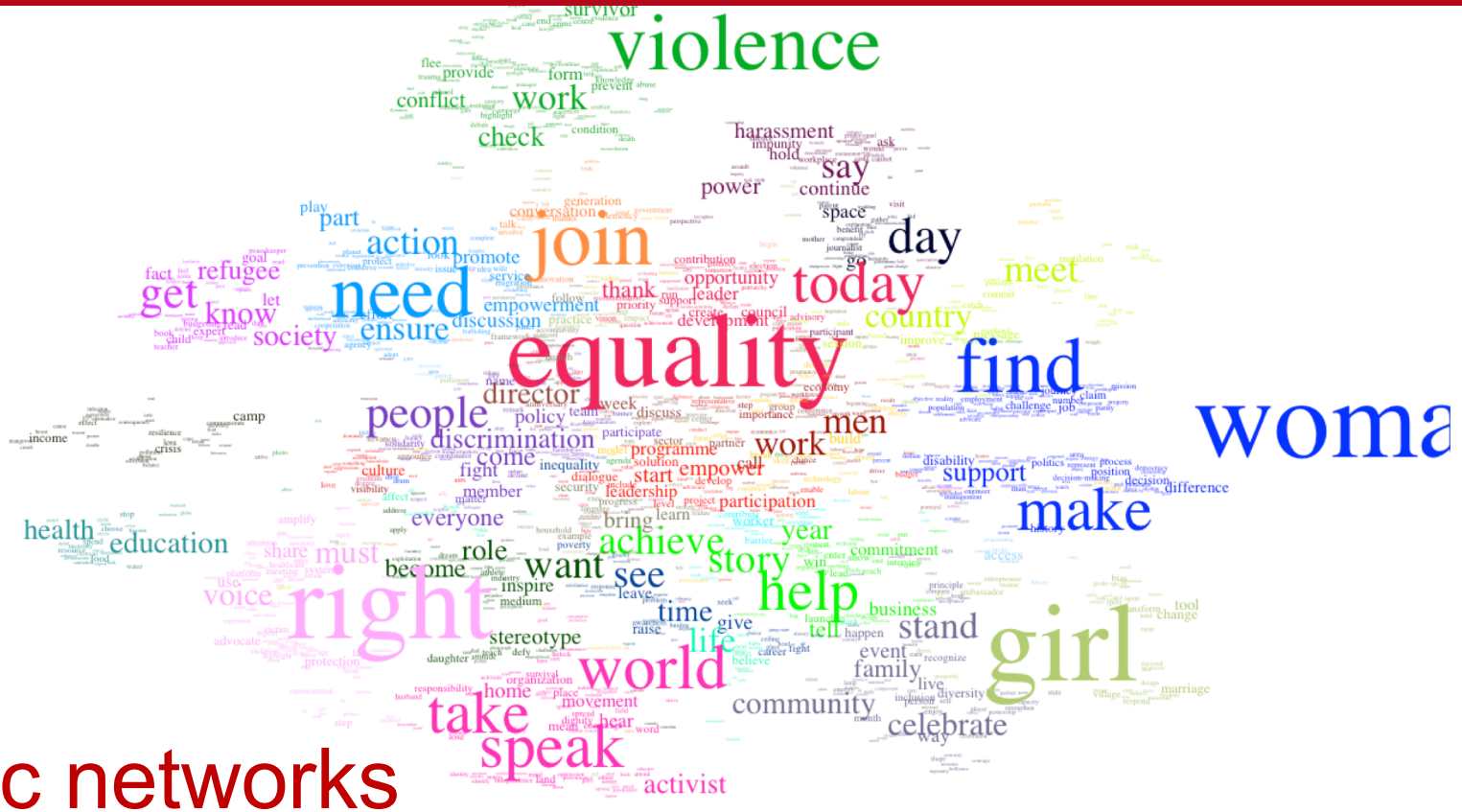
# And what do we study? (cont'd)



## community detection

- modularity, Louvain algorithm, conductance, InfoMap, normalized mutual information, overlapping communities, BigCLAM, stochastic block models

# And what do we study? (cont'd)




## semantic networks

- network layout, data collection, sentiment analysis, BERTAgent, topic detection, latent Dirichlet allocation, variational autoencoders, BERTopic and the Transformer architecture

# Project

a brief overview

# What about the project?

create your own group (1 to 3 people)

choose your dataset (possibly create your own dataset)

apply the ideas learned during the course

show that you can do clever things

try extracting meaningful measures/analytics that describe an interesting aspect of your network

write good code

each contributor to the group should focus on a different aspect (no everything together)

present the project in a clear and convincing way, using clear and convincing plots

# What about interdisciplinary projects?

mainly related to semantic networks

in collaboration with the twin course of Social Network Analysis @ Communication Strategies


SNA students suggest research questions

NS students conceive appropriate algorithmic solutions

in brainstorming sessions
the instructor will help/give feedback ☺

MIME Master's degree ICT Internet Multimedia Engineering

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

DISLL DIPARTIMENTO DI STUDI LINGUISTICI E LETTERARI

Dipartimento di Psicologia dello Sviluppo e della Socializzazione

# INTERDISCIPLINARY PROJECTS PRESENTATION

Network Science & Social Networks Analysis

## AULA MAGNA LEPSCHY

DEI – VIA GRADENIGO 6 – PADOVA

Thu February, 8, 2024, 9:00

# IP examples from past years

on **Twitter**

- 2019 – Evolution of Climate Change Perception on Twitter - Focusing on Greta Thunberg Impact
- 2019 – UN Women Twitter profile's reaction to the MeToo movement
- 2020 – NBA and Premier League players around #blacklivesmatter and the racial issue on Twitter
- 2020 – Republicans vs Democracts on Twitter
- 2020 – Haters gonna (make you) hate - Semantic analysis of hate during 2019 European elections
- 2021 – Sports brands and eco-sustainability
- 2022 – Sexism in Politics
- 2022 – What is the perception around the world in terms of Menstruation Stigma in 2021?
- 2022 – Cancel culture on social media - Social network analysis on famous cases of cancellation

on **TikTok**

- 2022 – PoliTok: How do Italian politicians use TikTok as tool to promote their political ideas and influence the young generation during the 2022 elections?

**other**

- 2019 – Noodles and Spaghetti - How people make pasta in eastern countries
- 2021 – Erasmus+ Programme: a social network analysis study of the 2014-2019 exchanges
- 2021 – Nationality vs. movie prestige: from the Oscars to International Film Festivals

# Calendar

tentative

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# Calendar

## OCTOBER 2024

| SUN | MON | TUE | WED | THU | FRI | SAT |
|---|---|---|---|---|---|---|
| 29 | 30 | | 2 intro | 3 | 4 | 5 |
| 6 | | | 9 centrality | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 27 | | | 30 community detection | 31 | 1 | 2 |

## NOVEMBER 2024

| SUN | MON | TUE | WED | THU | FRI | SAT |
|---|---|---|---|---|---|---|
| 27 | 28 | 29 | 30 | 31 | X | 2 |
| 3 | 4 | 5 | X | 7 | X | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 seminar | 23 |
| 24 | 25 | 26 IP day | 27 | 28 | 29 layout | 30 |

## DECEMBER 2024

| SUN | MON | TUE | WED | THU | FRI | SAT |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 semantic networks | 4 | 5 | 6 lab1 | 7 |
| 8 | 9 | 10 lab2 | 11 | 12 | 13 lab3 | 14 |
| 15 | 16 | 17 lab4 | 18 | 19 | 20 | 21 |

## JANUARY 2025

| SUN | MON | TUE | WED | THU | FRI | SAT |
|---|---|---|---|---|---|---|
| 29 | 30 | 31 | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 | 10 robustness | 11 |
| 12 | 13 | 14 IP feedback | 15 | 16 | 17 | 18 |

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# Contents

- Misc (4 lectures)

introduction; graphs; graph layout: ForceAtlas2, Gephi, UMAP; robustness; homophily

- Centrality (5 lectures)

degree centrality, power law, preferential attachment, fitness, Bianconi-Barabasi model, scale-free regime, PageRank, convergence properties, Local PageRank, Approximate and signed PageRank, Row-normalized PageRank, HITS, closeness, betweenness, clustering coefficient

- Community detection (5 lectures)

modularity, Louvain algorithm, consensus clustering, Modularity for directed and signed networks and overlapping communities, Minimum cut criterion, spectral clustering, InfoMap, Normalized mutual information, F1 score, Dice correlation, BigCLAM, stochastic block models, Dendrograms, Girvan-Newman, HDBSCAN

- Semantic networks (3 lectures)

Reddit, cleaning steps: spaCy, LIWC, BERTAgent, semantic networks, TF-IDF, modularity, latent Dirichlet analysis, variational autoencoders, BERTopic, performance comparison

- Python labs (4 lectures)

- IP projects (1+1 lectures)

# To do list

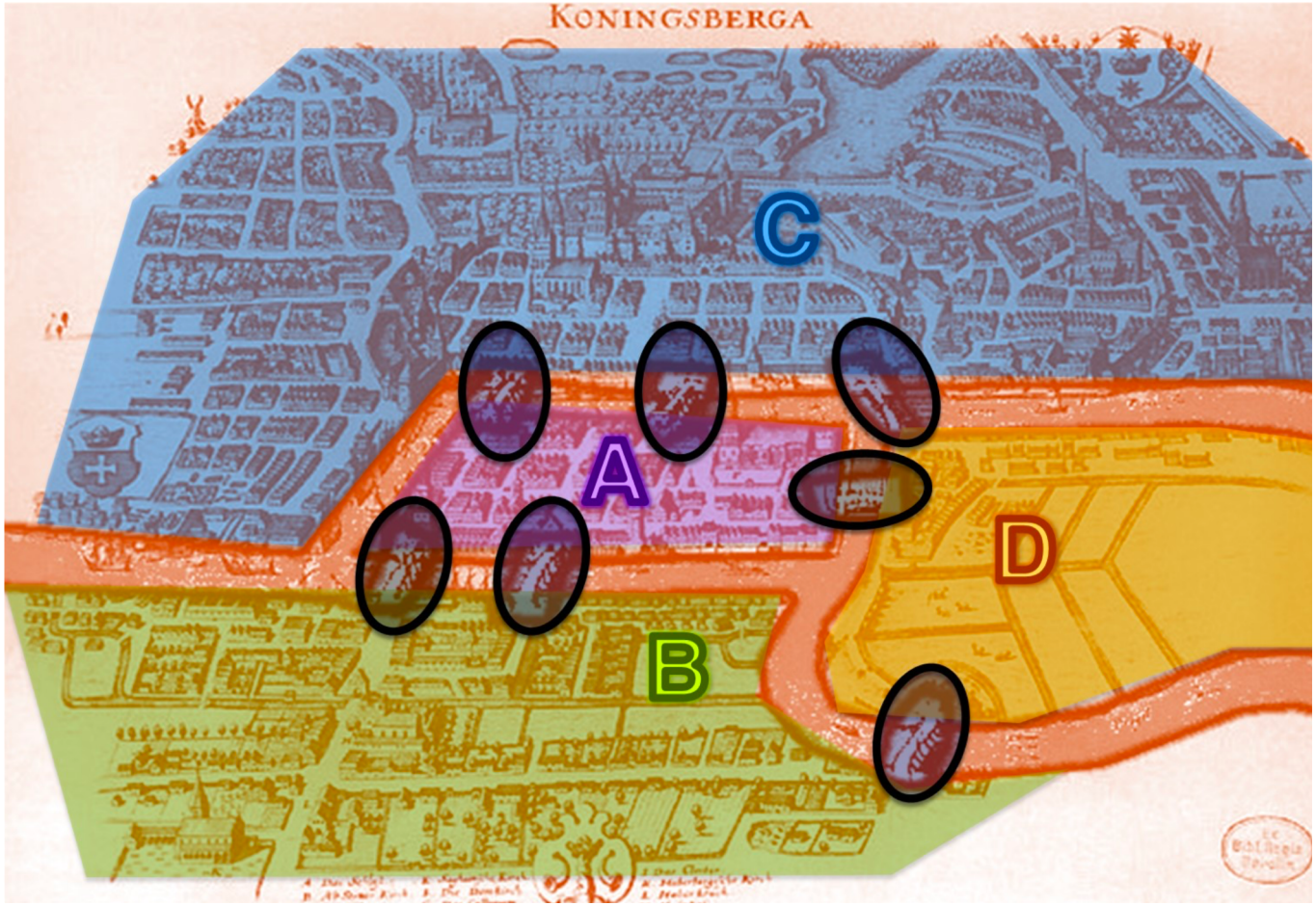

- Enrol @ stem.elearning.unipd.it ☺
- Have a laptop available
- Ensure you know Python's basics
- Activate a Google account (with the @unipd.it email)
  - → Google Drive
  - → Google CoLab
- Activate a Reddit account (using Google's account)
  - → Reddit apps https://www.reddit.com/prefs/apps
- Install Gephi on your laptop https://gephi.org/
- Review everything you know about deep learning and/or optimization
- Organize yourselves into working groups (max 3 people)

# Graphs

an introduction

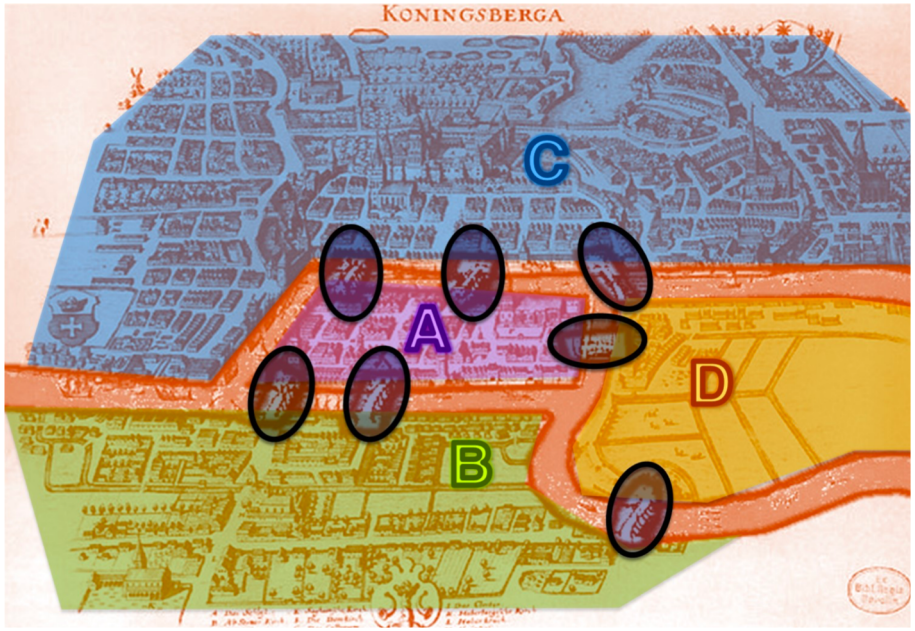
# Euler and the 7 bridges of Könisberg

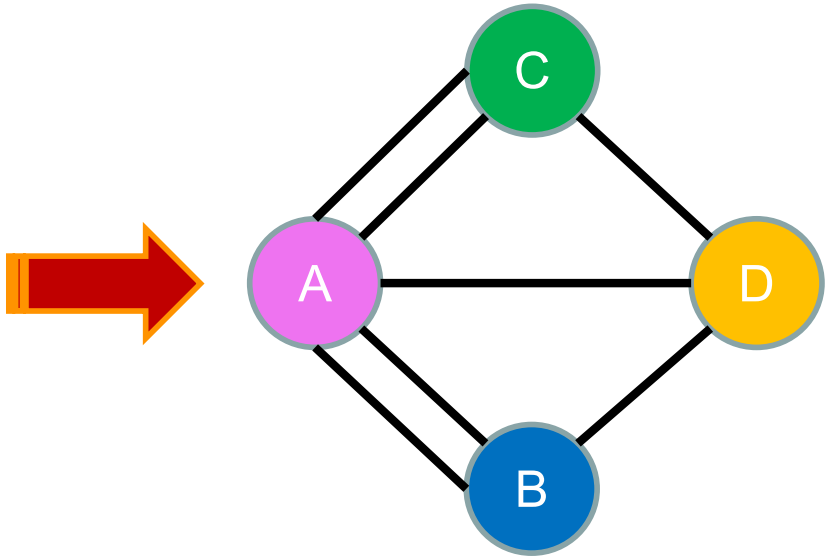
(Prussia, 1736) today Kaliningrad




How to walk through the city by crossing each bridge only once?

# Networks as graphs




Graph $\mathcal{G}$ ($\mathcal{V}$,$\mathcal{E}$): network

- Vertices (set $\mathcal{V}$) : nodes, people, concepts
- Edges (set $\mathcal{E}$): links, relations, associations

*mathematics* *technology* *social psychology* *social cognition*

# Directed versus undirected

- A connection relationship can have a privileged direction or can be mutual
  - Either a directed or an undirected link

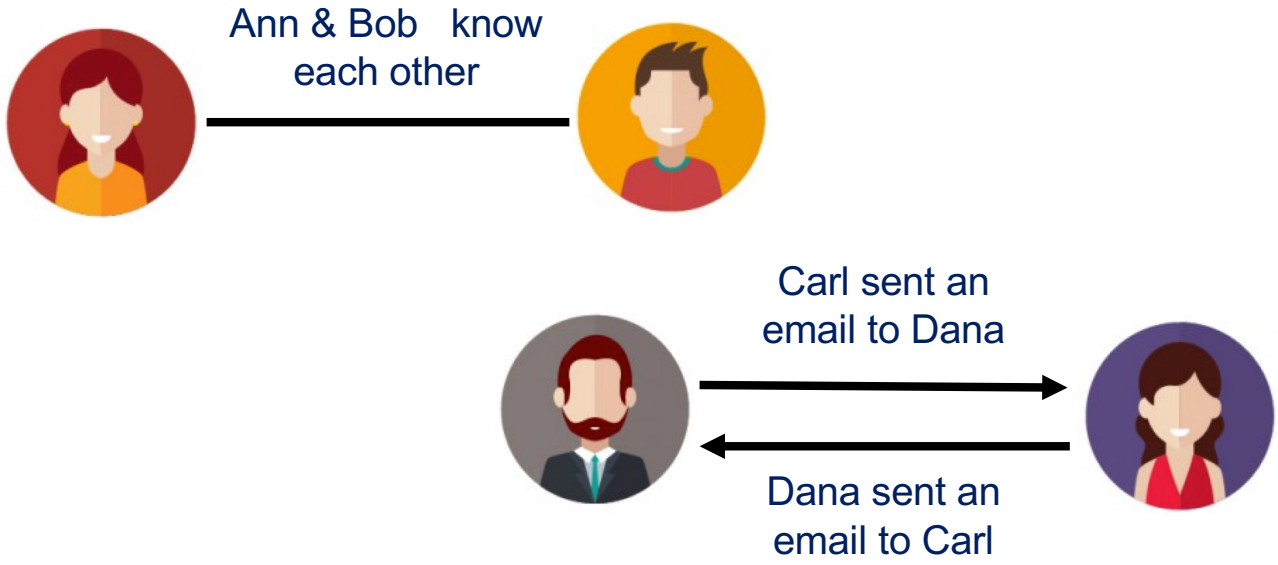

- If the network has only (un)directed links, it is also called itself (un)directed network
  - Certain networks can have both types

# Some examples

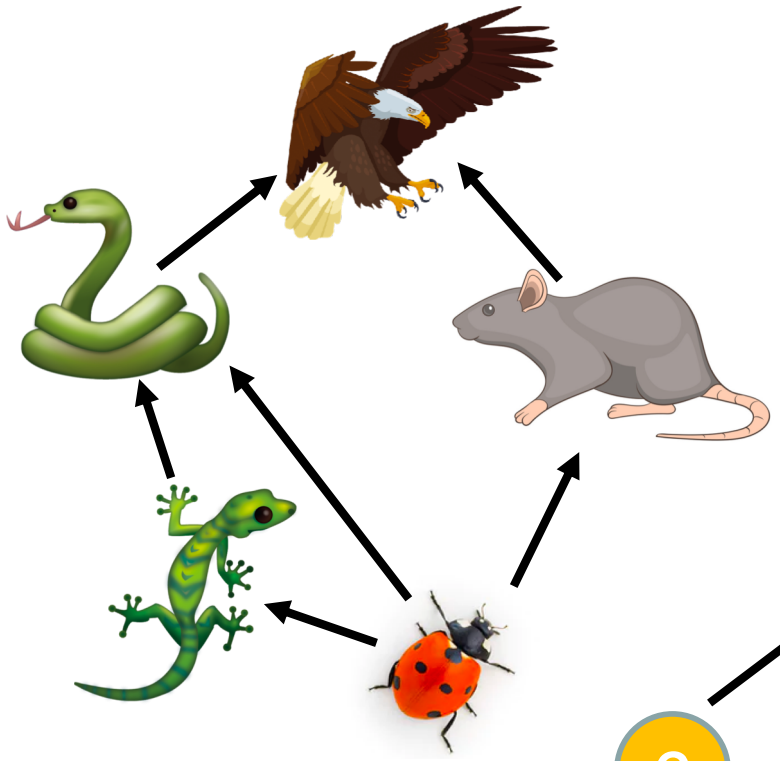
| network | nodes and links | type |
|---|---|---|
| Facebook | Profiles and friendship | undirected |
| Instagram | Accounts and followers | directed |
| the www | Webpages and links | directed |
| citation network | Papers and references | directed |
| social network | People and friends/acquaintances | undirected |
| movie network | Actors and co-starring | undirected |
| genealogy | People and parenthood | directed |

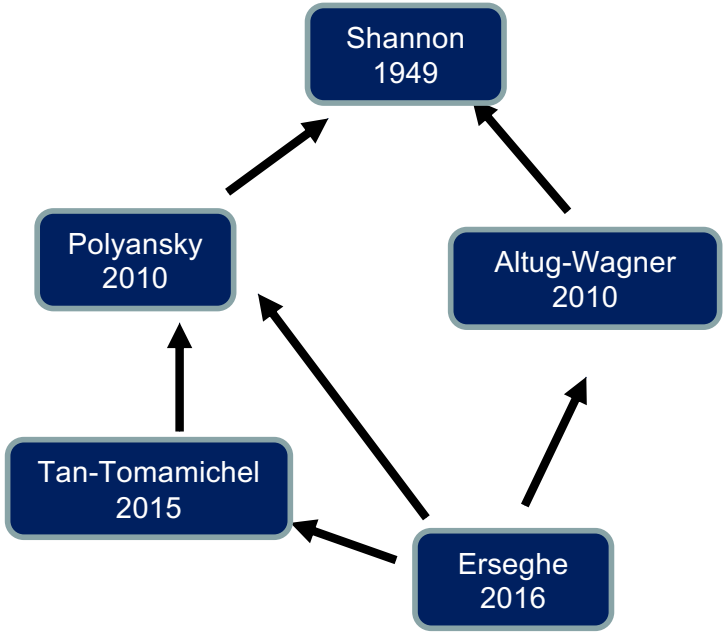
# Directed versus undirected

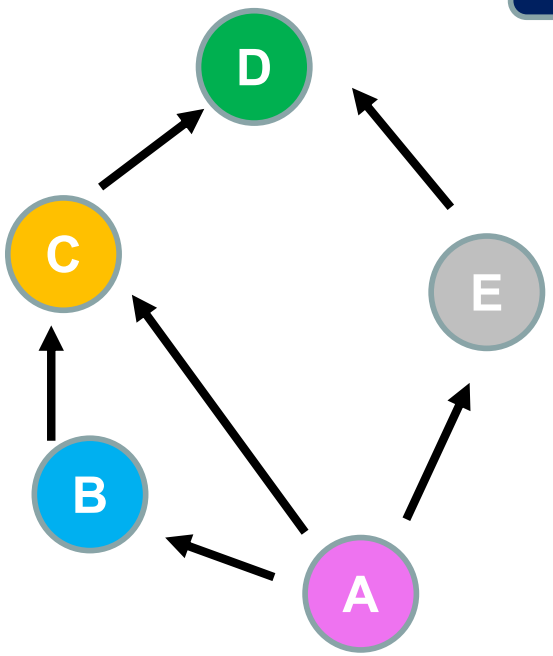
- At first glance undirected → directed by duplicating links, but not necessarily quite the same though



# Generality of representation





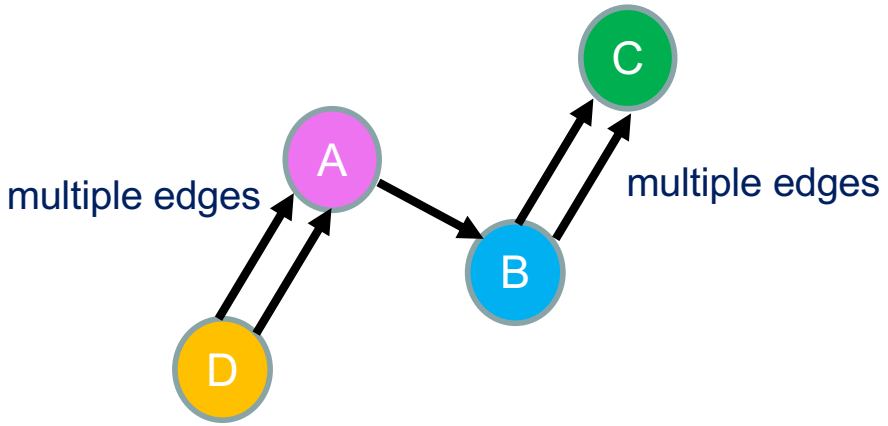



# Weighted graps

and adjacency matrix

# Multi-graphs

- Multi-graphs (or pseudo-graphs)
  - Some network representations require multiple links (e.g., number of citations from one author to another)

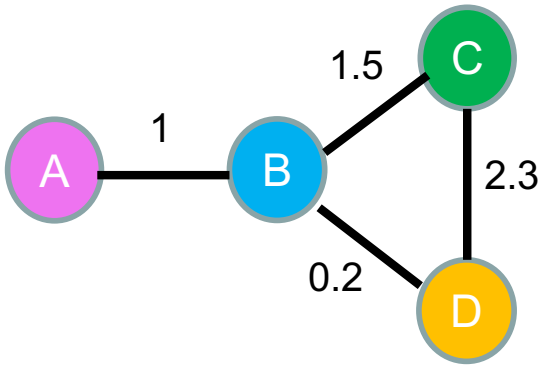


# Weighted graphs

## Weighted graph

Usually a weight $w_{ij}$ is associated to a link $(i,j) \in \mathcal{E}$ , e.g., to underline that the links are not identical (strong/weak relationships)

Can be seen as a generalization of multi-graphs (weight = # of links)

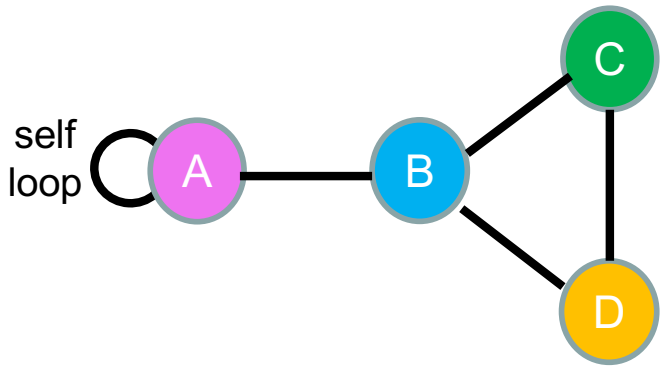


# Self interactions

- In many networks nodes do not interact with themselves

  if $i \in \mathcal{V}$ then $(i,i) \notin \mathcal{E}$

- To account for self-interactions, we add loops to represent them



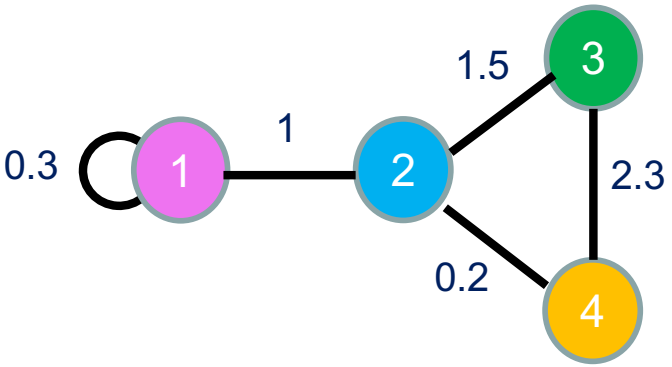
# Adjacency matrix

- An adjacency matrix $A = [a_{ij}]$ associated to graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ has

  entries $a_{ij} = 0$ for $(i,j) \notin \mathcal{E}$ (not a connection)

  if nodes $i$ and $j$ are connected then $a_{ij} \neq 0$

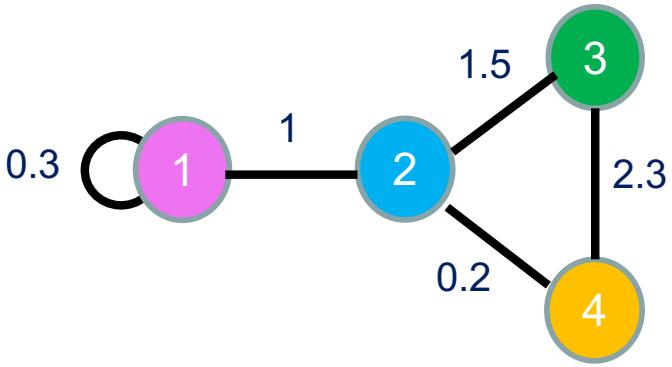
  in plain graphs $a_{ij} = 1$ for $(i,j) \in \mathcal{E}$




$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0.2 \\ 0 & 1.5 & 0 & 2.3 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

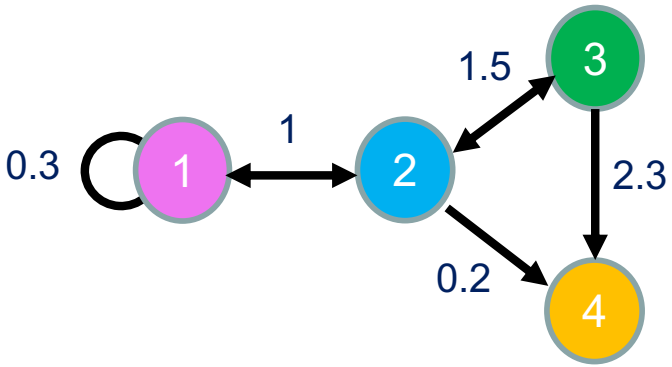
# Symmetries

- Undirected graph = symmetric matrix



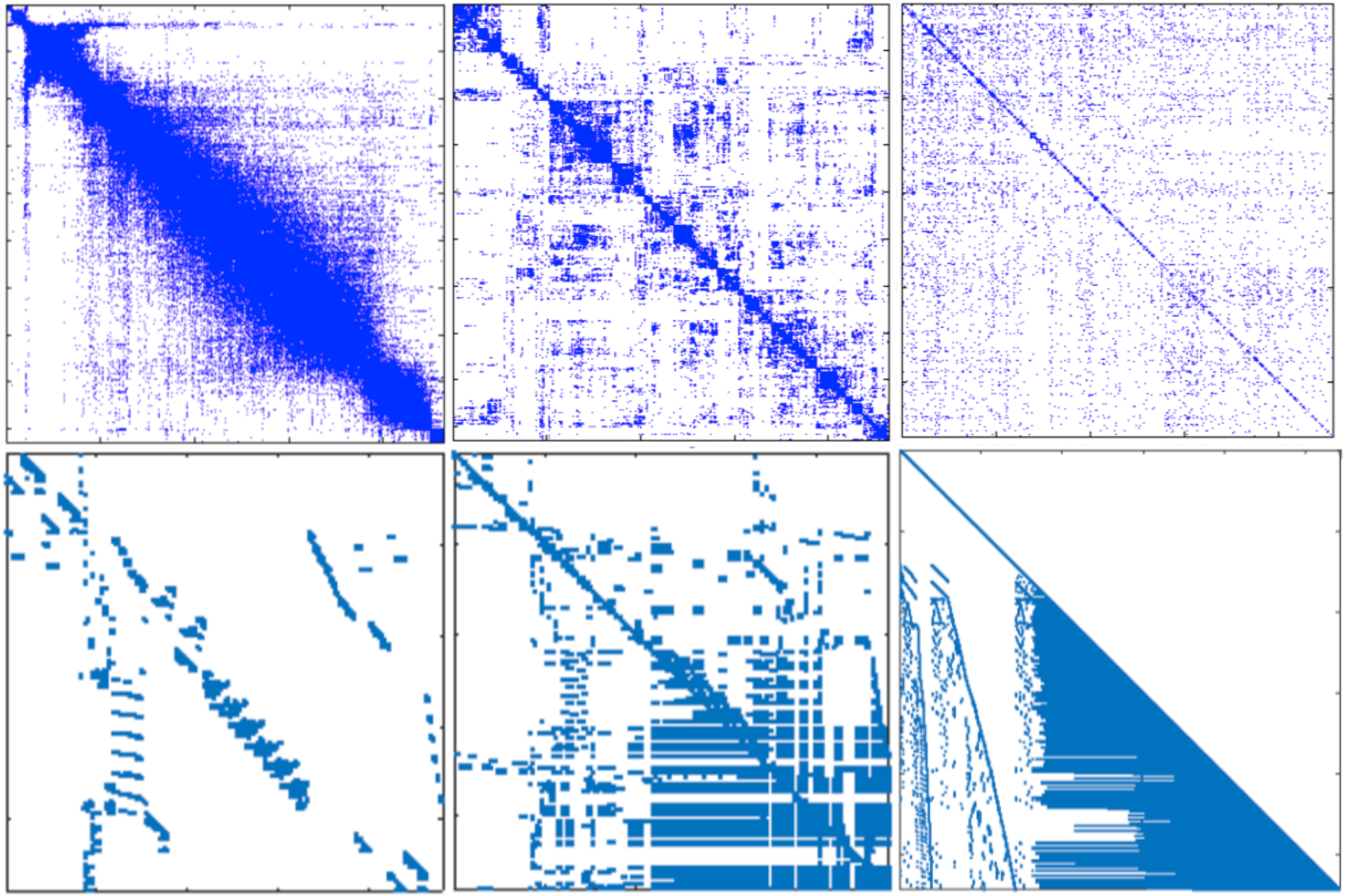

$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0.2 \\ 0 & 1.5 & 0 & 2.3 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

- Directed graph = asymmetric matrix



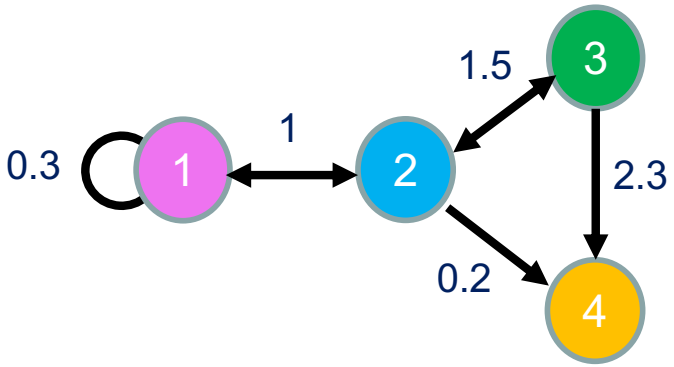

$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

# Symmetries



# Convention

- The weight $a_{ij}$ is associated to
  - *i* th row
  - *j* th column
  - directed edge *j* → *i* starting from node *j* and leading to node *i*



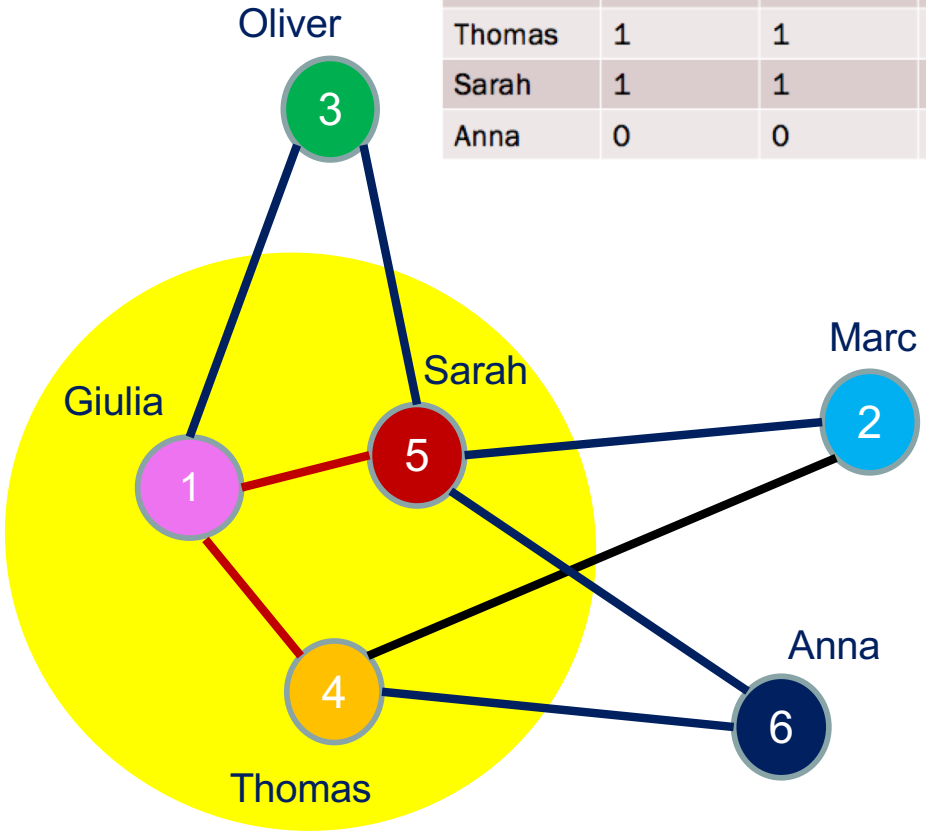

$$A = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 1 & 0 & 1.5 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0.2 & 2.3 & 0 \end{bmatrix}$$

$a_{24}$, $a_{34}$, $a_{42}$, $a_{43}$

# Question

which of these representations do you like best?

| | Giulia | Marc | Oliver | Thomas | Sarah | Anna |
|---|---|---|---|---|---|---|
| Giulia | X | | | | | |
| Marc | 0 | X | | | | |
| Oliver | 1 | 0 | X | | | |
| Thomas | 1 | 1 | 0 | X | | |
| Sarah | 1 | 1 | 1 | 0 | X | |
| Anna | 0 | 0 | 0 | 1 | 1 | x |




edge list

1 → 3
1 → 4
1 → 5
2 → 4
2 → 5
3 → 5
4 → 6
5 → 6

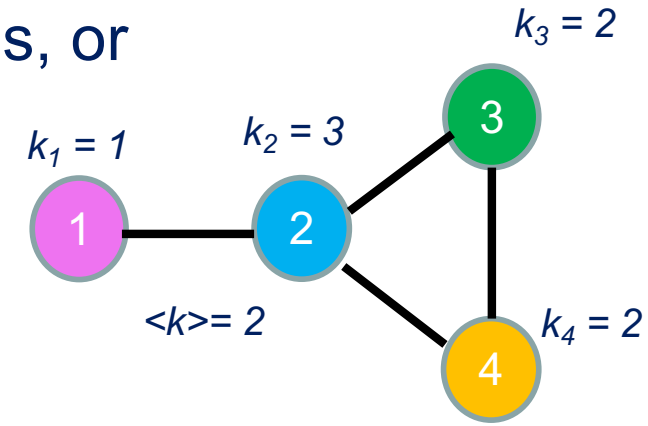
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

# Node degree

in directed and undirected networks

# Node degree
## undirected networks

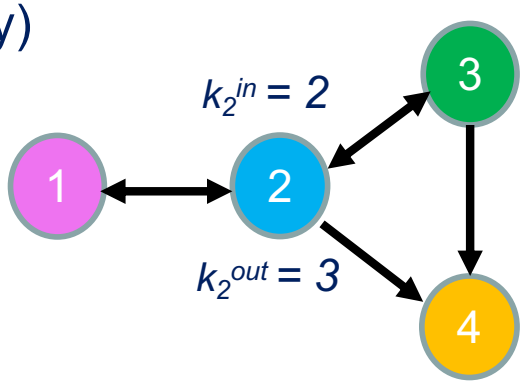
- The degree $k_i$ of node *i* in an undirected networks is
  - the # of links *i* has to other nodes, or
  - the # of nodes *i* is linked to




- The # of nodes is $N = |\mathcal{V}|$
- The # of edges is $L = |\mathcal{E}| = ½ \sum_i k_i$
- The average degree is $\langle k \rangle = \sum_i k_i / N = 2L / N$

# Node degree
## directed networks

- For directed networks we distinguish between
  - in-degree $k_i^{in}$ = # of entering links
  - out-degree $k_i^{out}$= # of exiting links
  - (undirected: $k_i^{in} = k_i^{out}$ due to the symmetry)

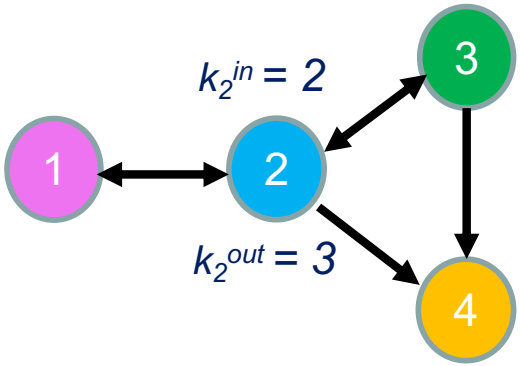



- The # of links is $L = \sum_i k_i^{in} = \sum_i k_i^{out}$
  (no need for factor ½)

  The average # of links is $\langle k \rangle = L / N$

# Adjacency matrix and degree

- The in (out) degree can be obtained by summing the adjacency matrix over rows (columns)




no self-loops in this case!!!

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$k_2^{in} = 2$

$k_2^{out} = 3$

- A few useful linear algebra expressions

$$\boldsymbol{k}^{in} = \boldsymbol{A} \cdot \boldsymbol{1}$$

$$\boldsymbol{k}^{out} = \boldsymbol{A}^T \cdot \boldsymbol{1} = (\boldsymbol{1}^T \cdot \boldsymbol{A})^T$$

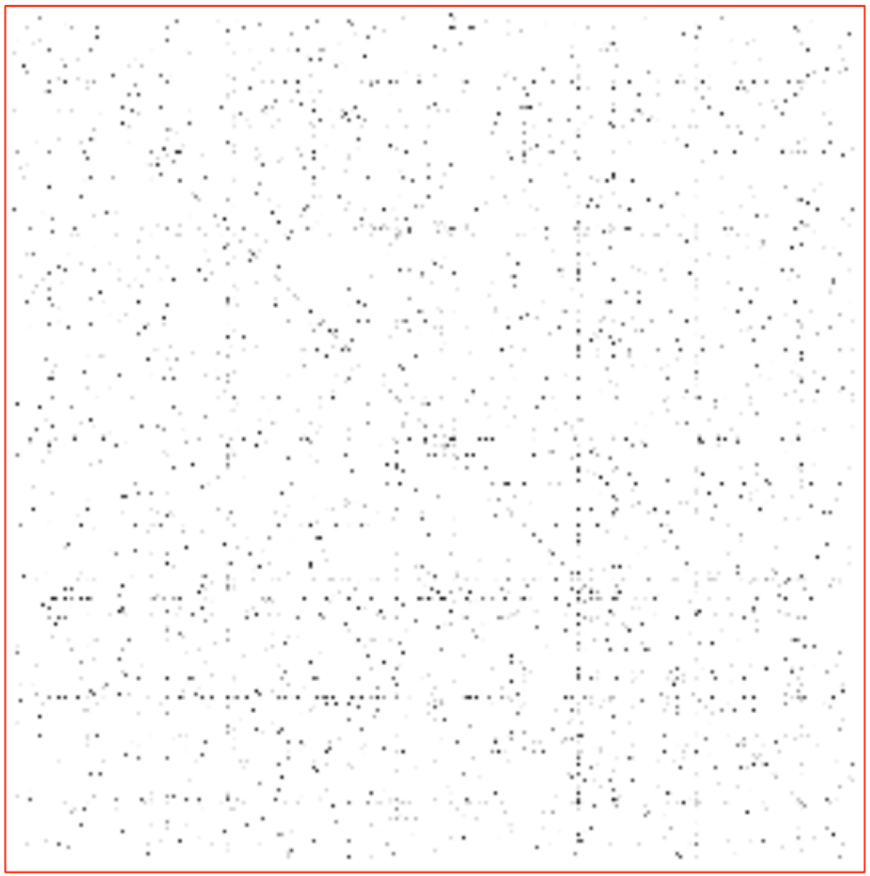
valid also for weighted graphs

# Real networks are sparse

- The adjacency matrix is typically sparse

  good for tractability !

$A =$



protein interaction network

# Real networks are sparse

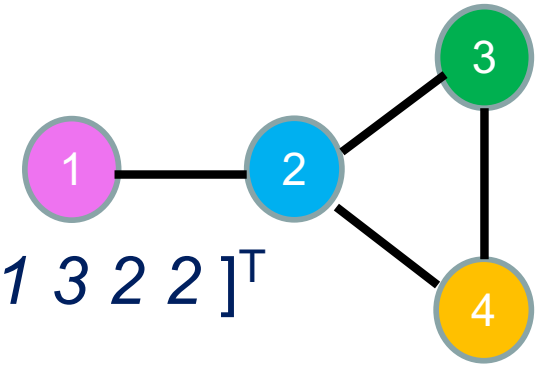
- The maximum degree is $\langle k \rangle_{max} = N - 1$
- In real networks $\langle k \rangle \ll N - 1$

| network | type | N | L | <k> |
|---|---|---|---|---|
| www | directed | $3.2 \times 10^5$ | $1.5 \times 10^6$ | 4.60 |
| Protein | directed | 1870 | 4470 | 2.39 |
| Co-authorships | undirected | 23133 | 93439 | 8.08 |
| Movie actors | undirected | $7 \times 10^5$ | $29 \times 10^6$ | 83.7 |

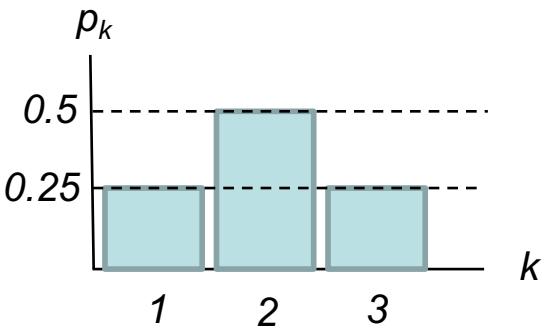
# Degree distribution

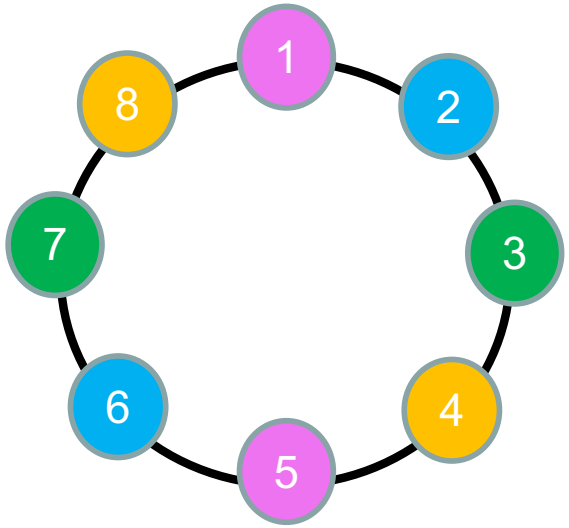
- Degree distribution $p_k$, a probability distr.

  $p_k$ is the fraction of nodes that have degree exactly equal to $k$ (i.e., # of nodes with that degree / $N$ )

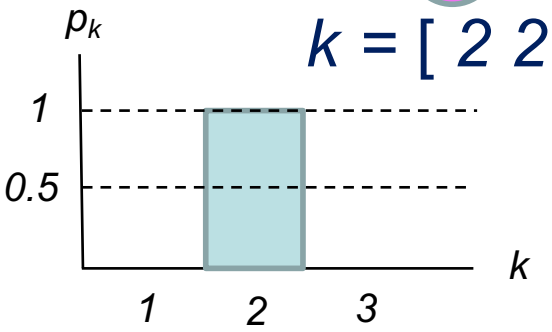



$k = [\ 1\ 3\ 2\ 2\ ]^T$




$k = [\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ ]^T$

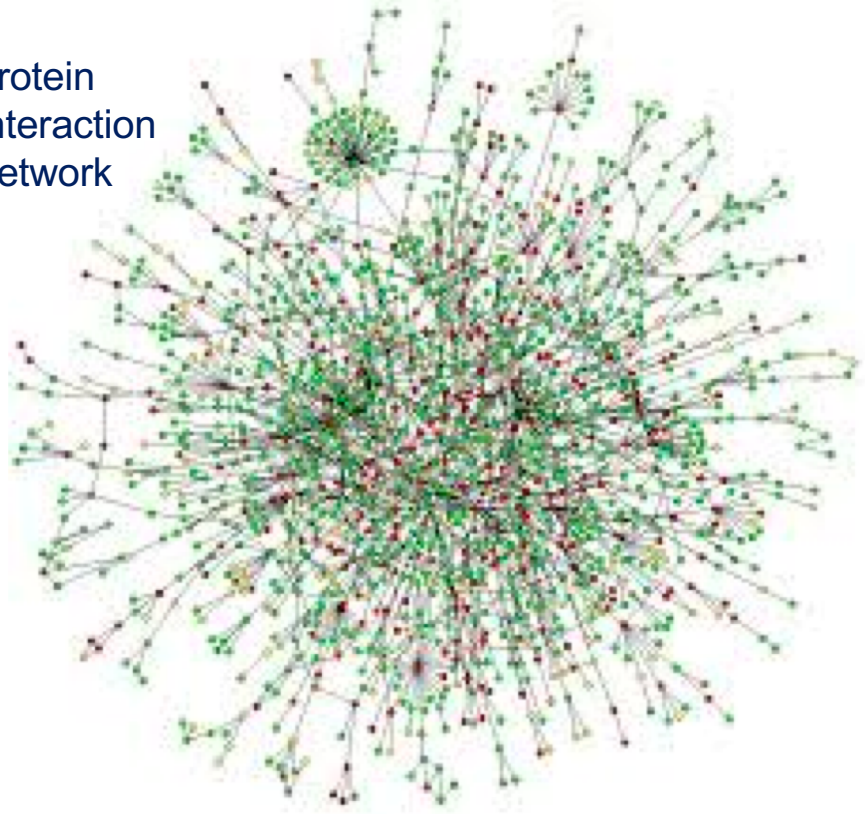


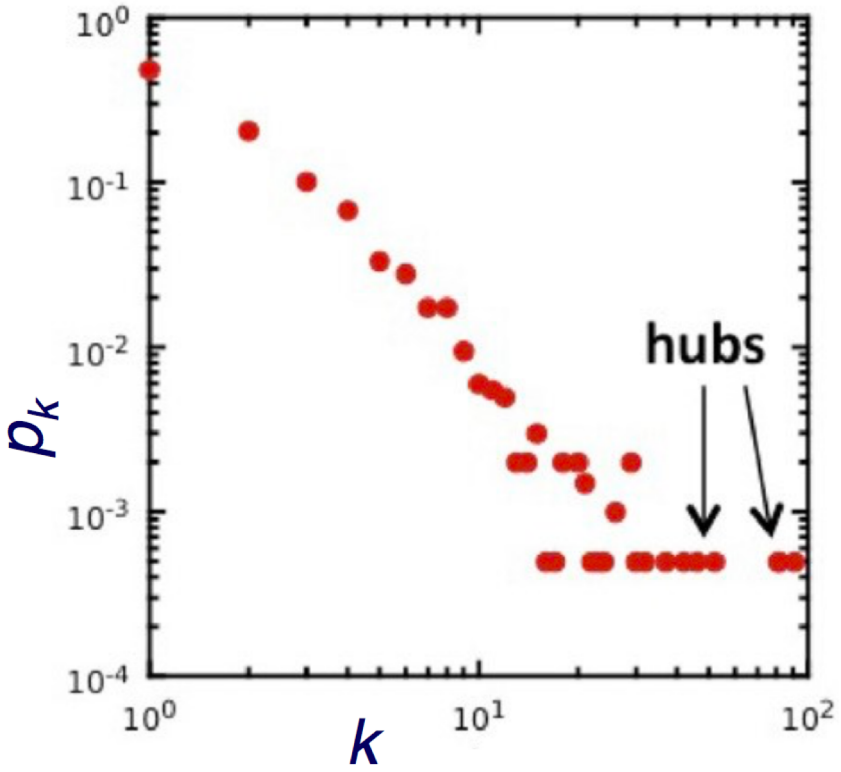
# Degree distribution

- In real world (large) networks, degree distribution is typically heavy-tailed

  nodes with high degree = hubs

protein interaction network



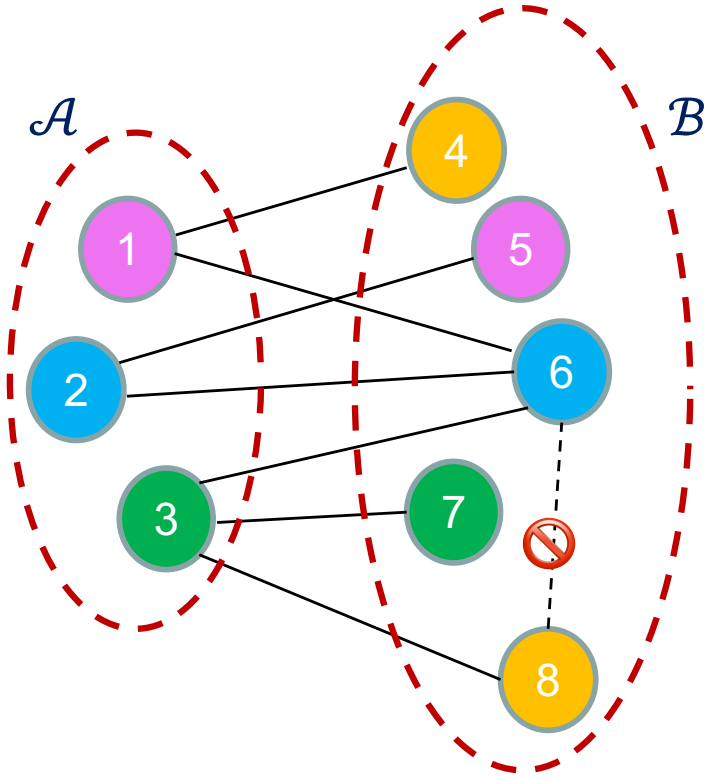


# Other graph types

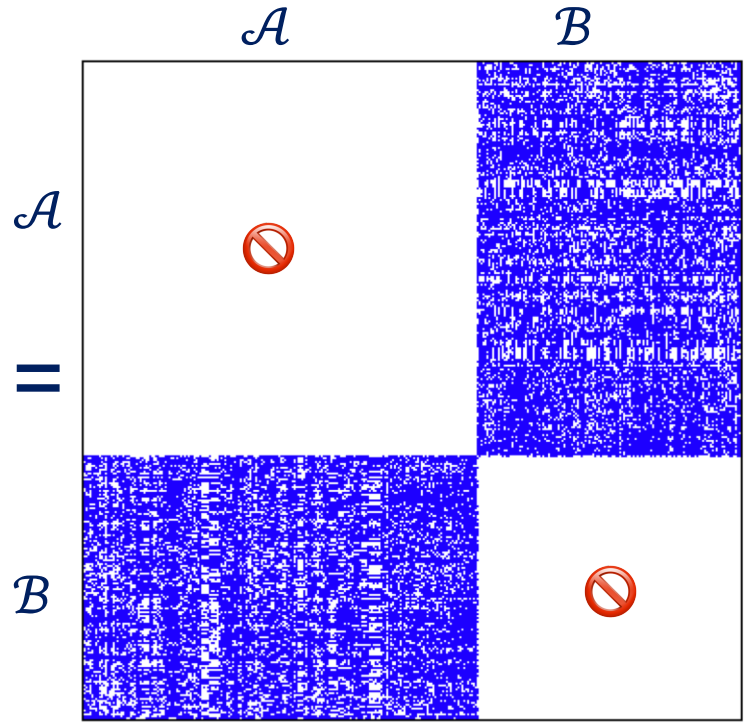
of interest to us

# Bipartite graphs

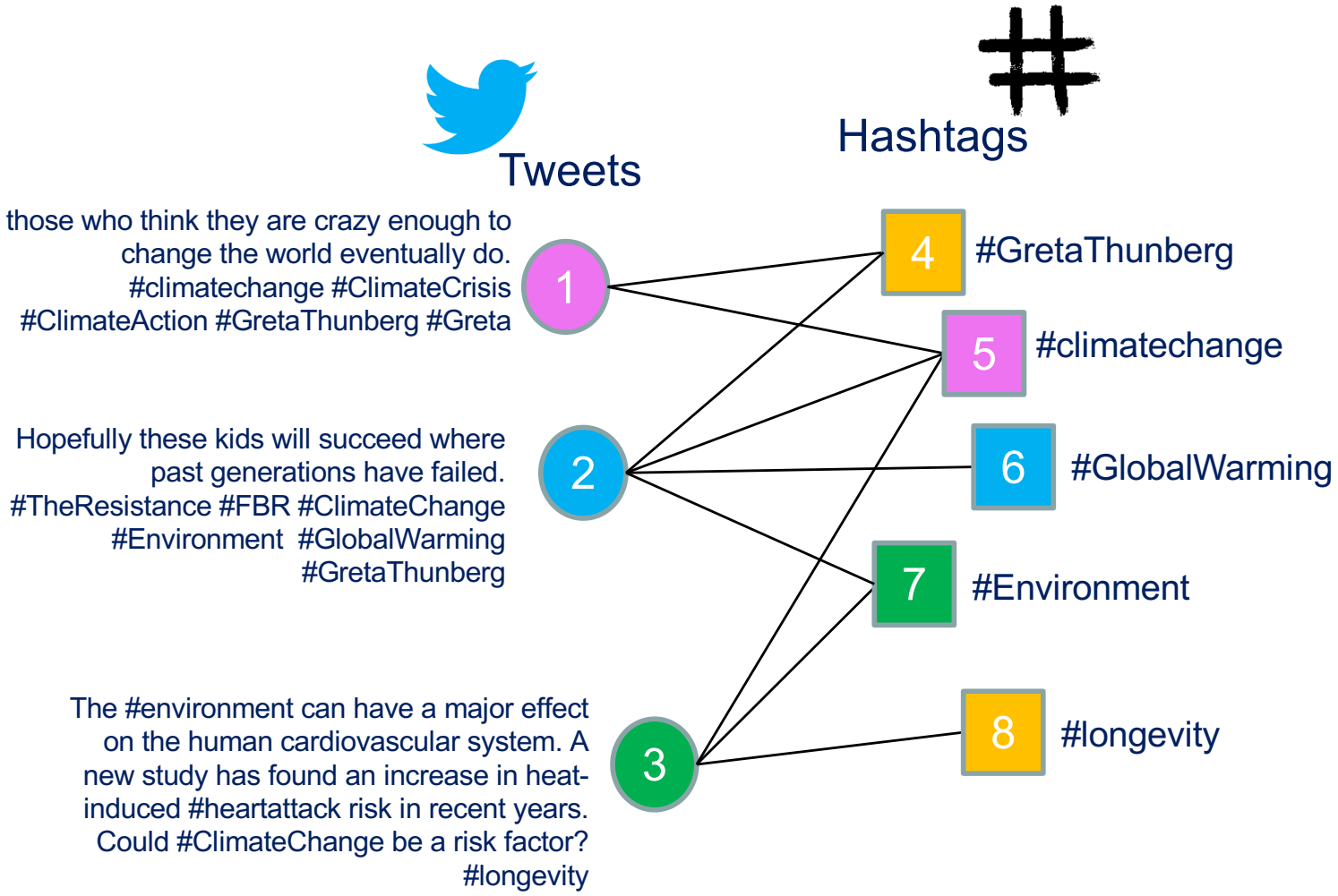
- Connections are available only between the groups $\mathcal{A}$ and $\mathcal{B}$




$A =$



# Bipartite graph example



# Meaning

- Bipartite graphs represent memberships/relationships, e.g., groups ($\mathcal{A}$) to which people ($\mathcal{B}$) belong

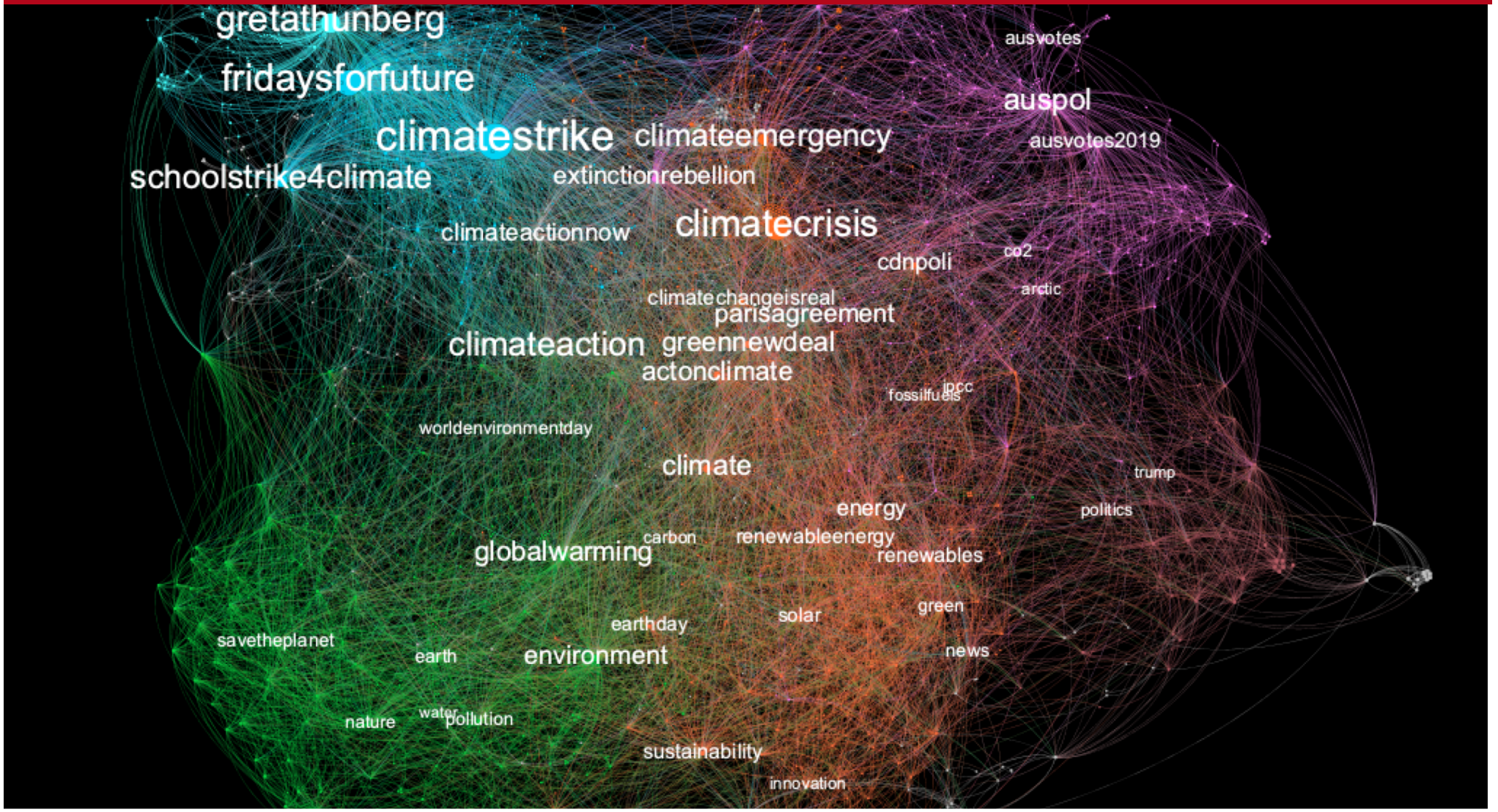
  examples: movies/actors, classes/students, conferences/authors

- We can build separate networks (projections) for $\mathcal{A}$ and $\mathcal{B}$ (sometimes this is useful)

  in the movies/actors example being linked can be interpreted in two ways: "actors in the same movie" (projection on $\mathcal{B}$), or "movies sharing the same actor" (projection on $\mathcal{A}$)

# Projection on a semantic network

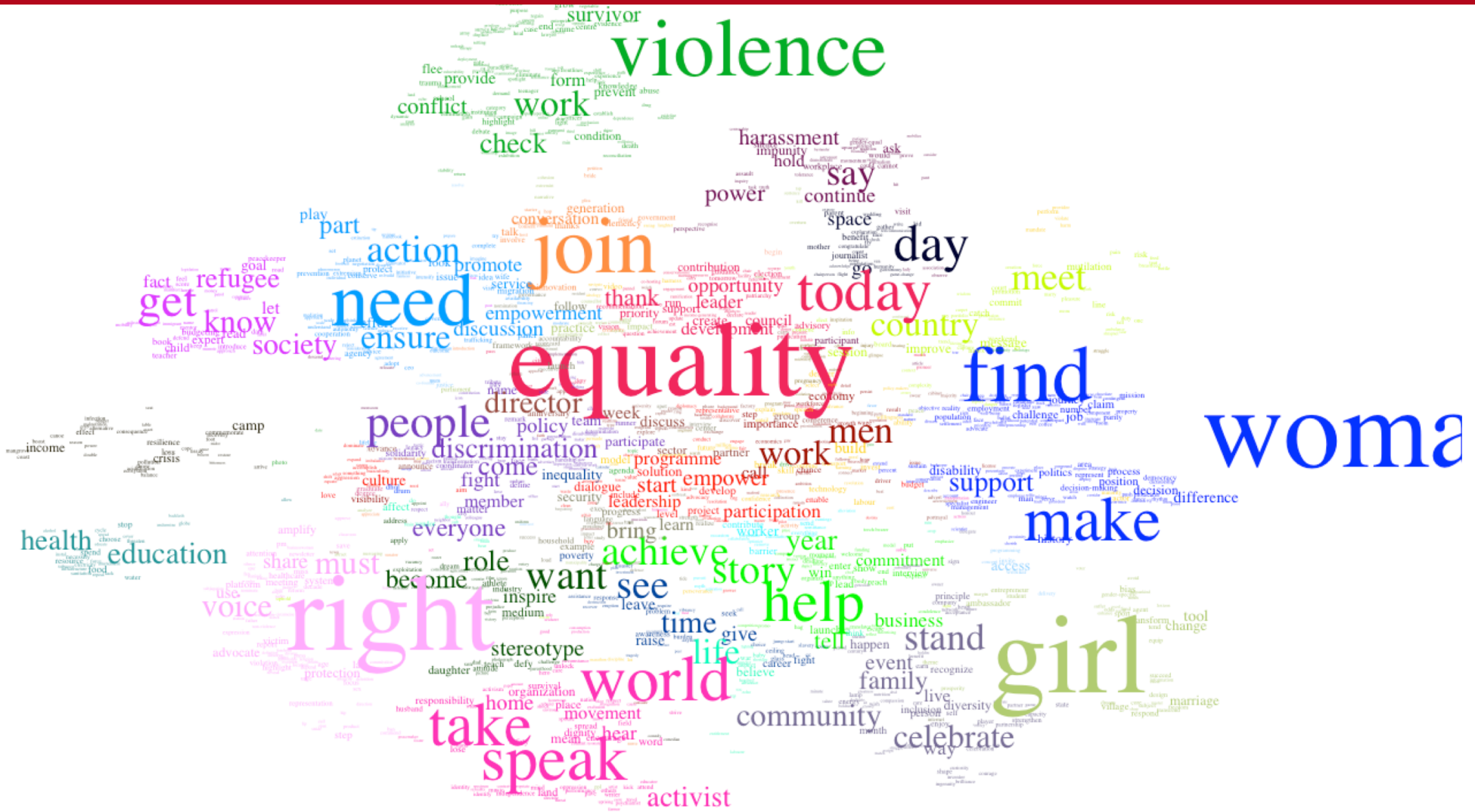
## #hashtags that appear in the same tweet are linked



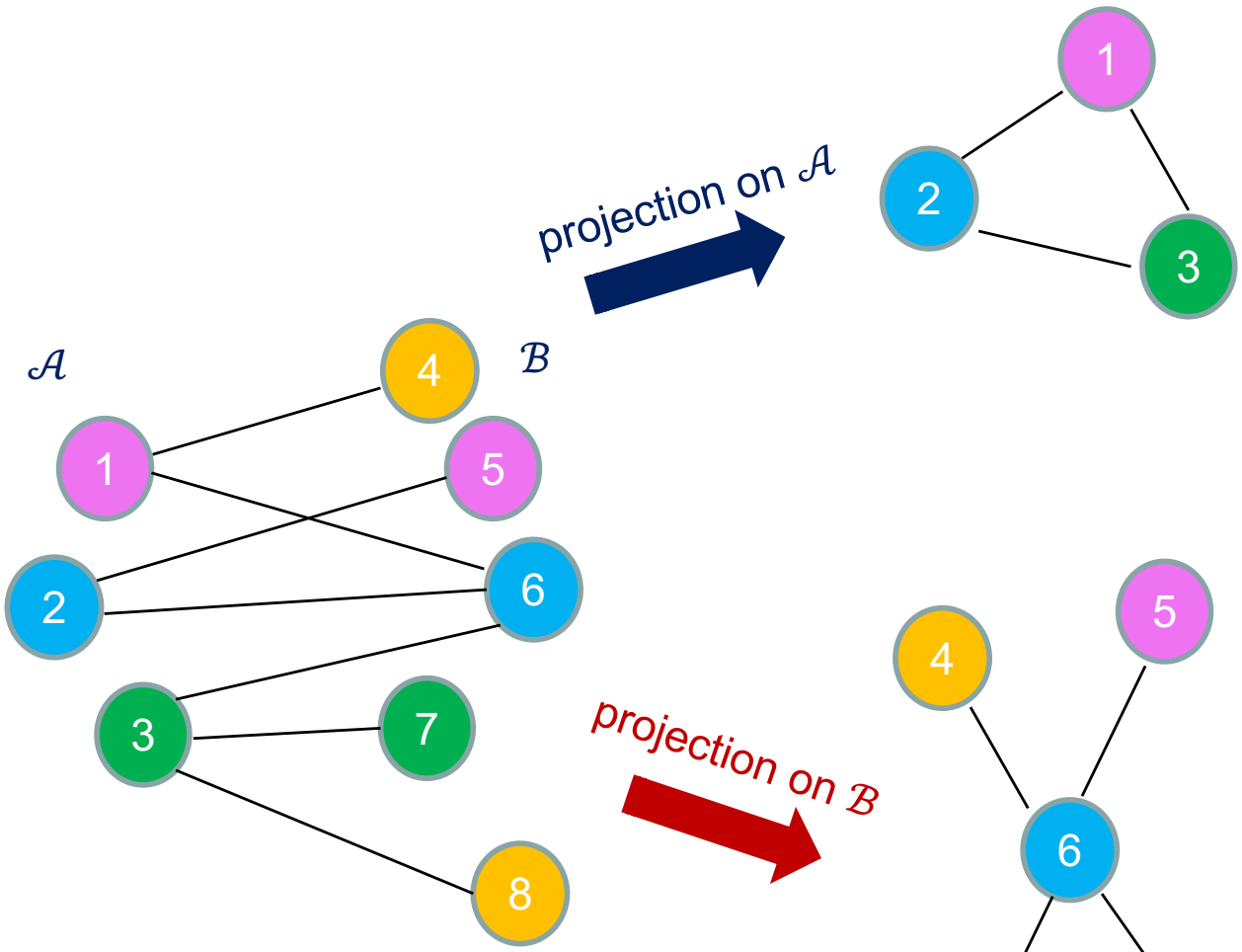

#climateaction tweets after Greta Thunberg

# Projection on a semantic network

words that appear in the same tweet are linked



# Abstract example



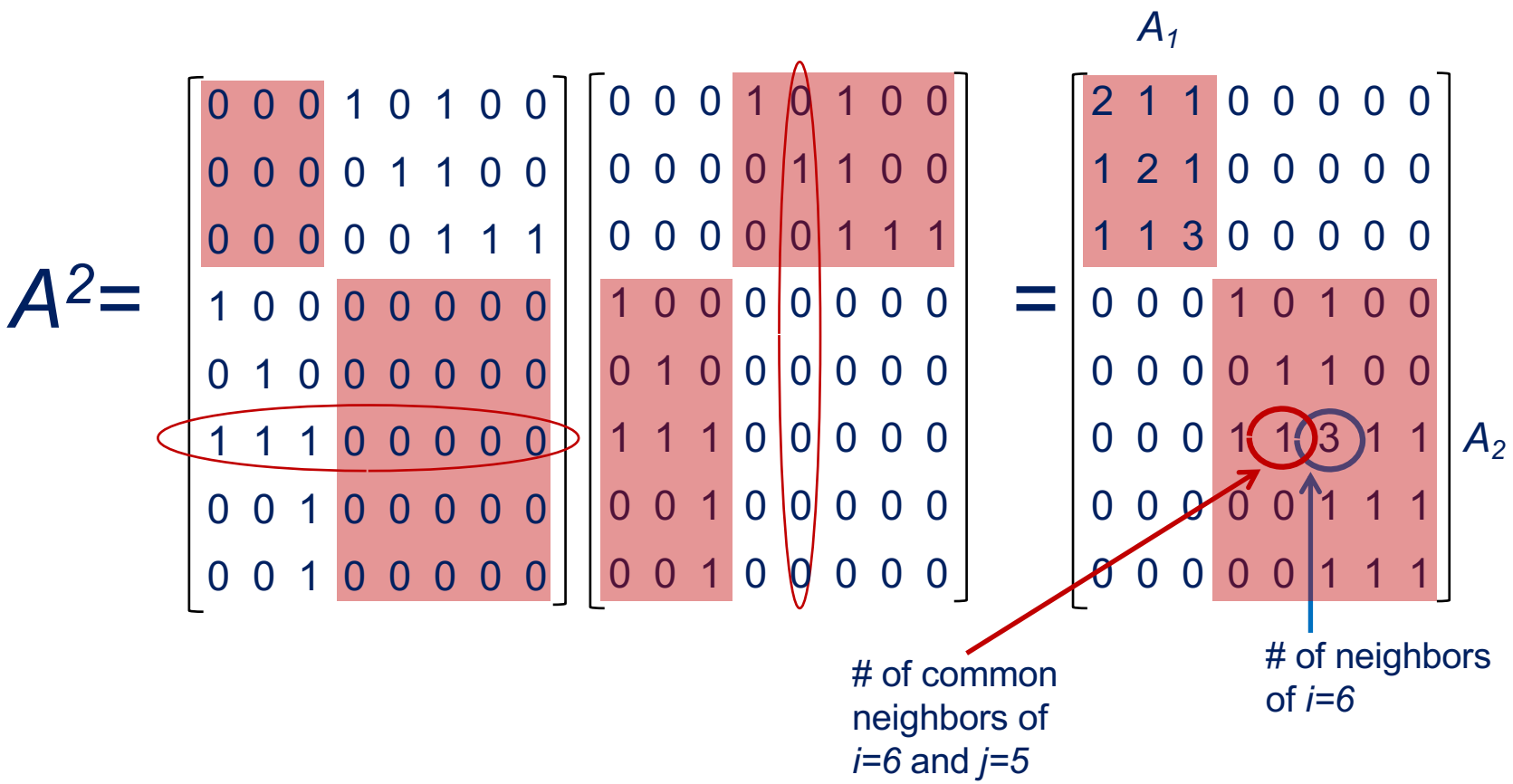

Nodes are linked if they have a common neighbour in $\mathcal{B}$

PS: we say that nodes *i* and *j* have a common neighbour *k* if both *i* and *j* are connected to *k*

Nodes are linked if they have a common neighbour in $\mathcal{A}$

# Projections

- The two projections on $\mathcal{A}$ and $\mathcal{B}$ can be obtained by inspecting the squared adjacency matrix $A^2$



# Tri-partite graphs

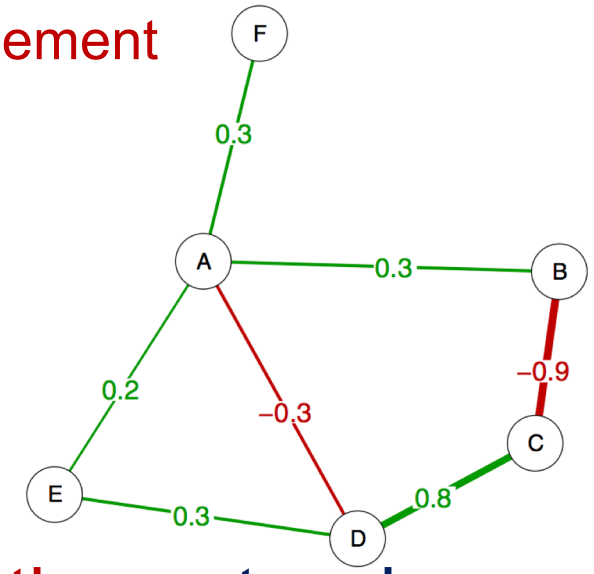
- Telecommunications
- Game theory
- Network science
- Signals & systems
- Channel coding
- Machine learning
- Computer vision
- Wireless comm.
- Human data analytics

- BS in ICT
- MiME
- MS computer science
- MS in data science

# Signed graphs

- Edges can have signed values

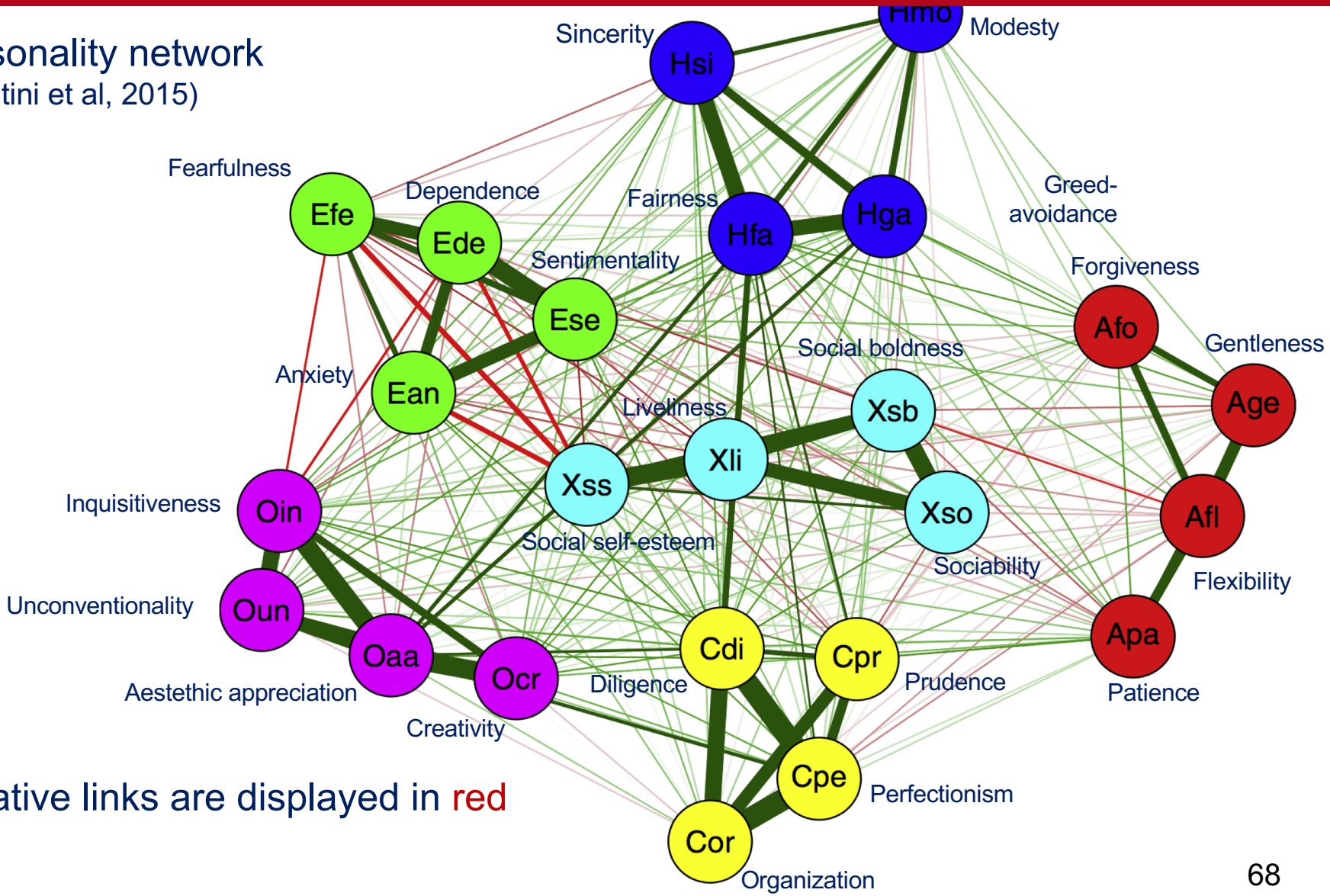
  positive if there is an agreement between nodes
  negative if there's a disagreement




- This is typical of correlation networks

# Signed graph example

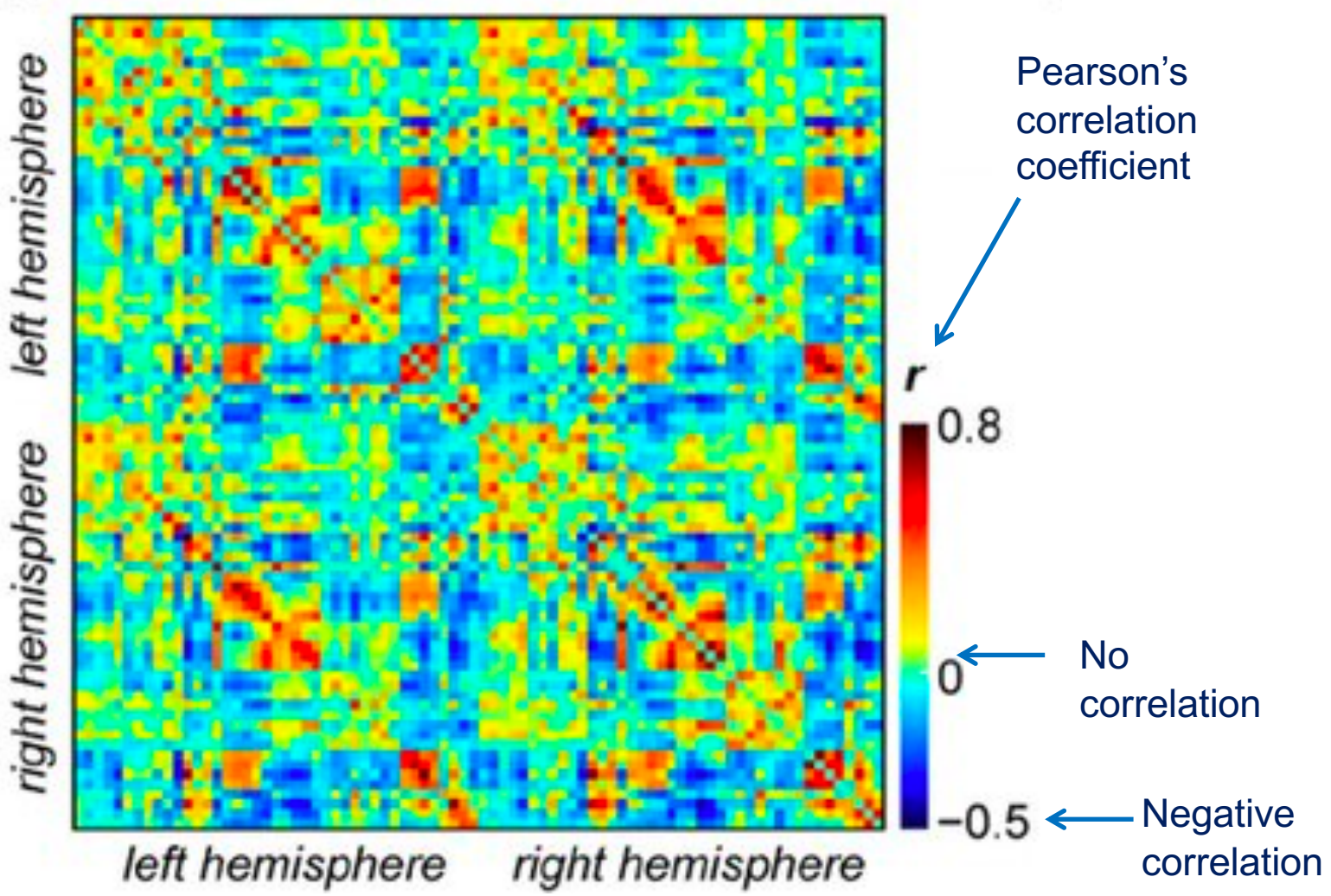
## A personality network
(Costantini et al, 2015)



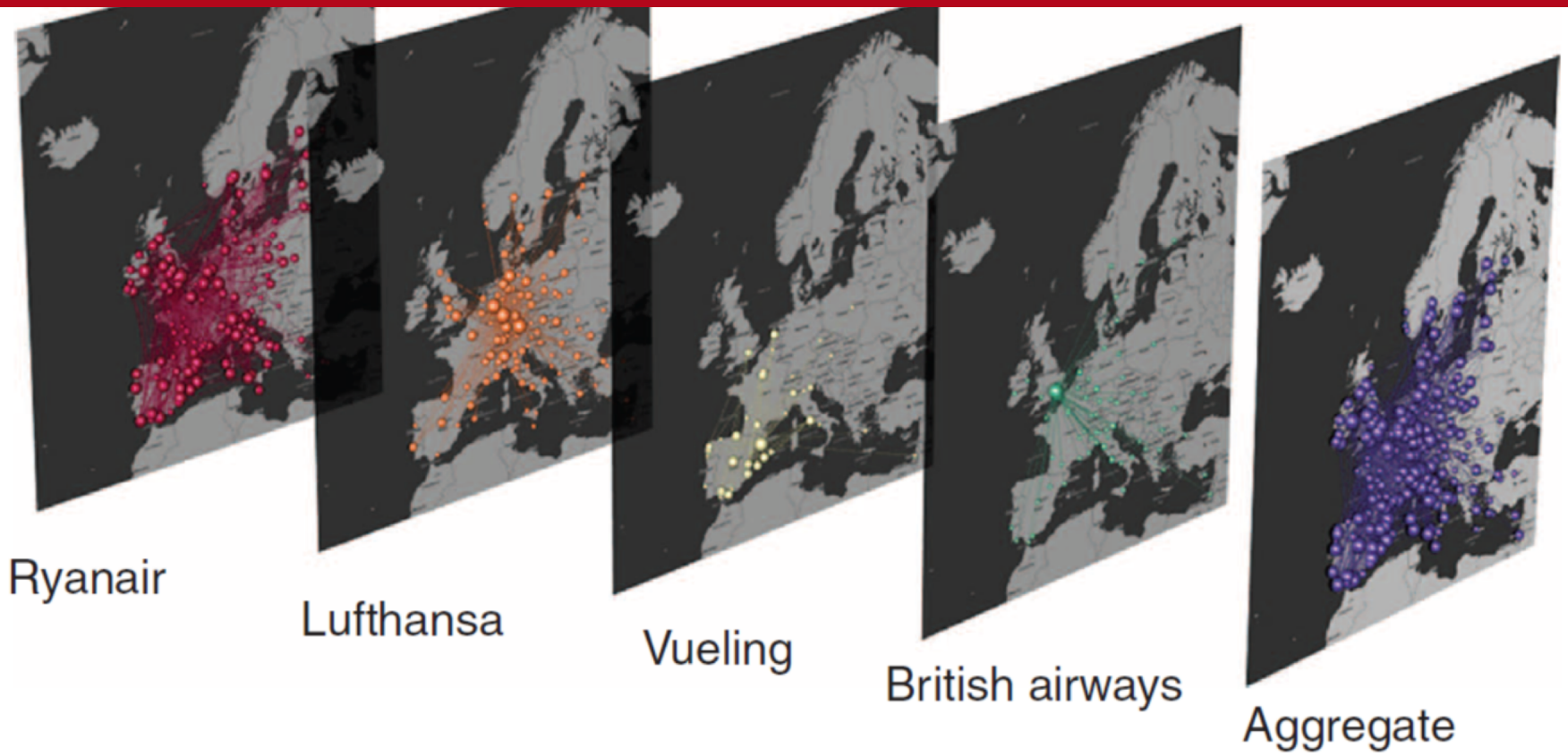

Negative links are displayed in red

# Signed graph example

An fMRI adjacency matrix (fMRI = functional magnetic resonance imaging)



# Multi-layer networks

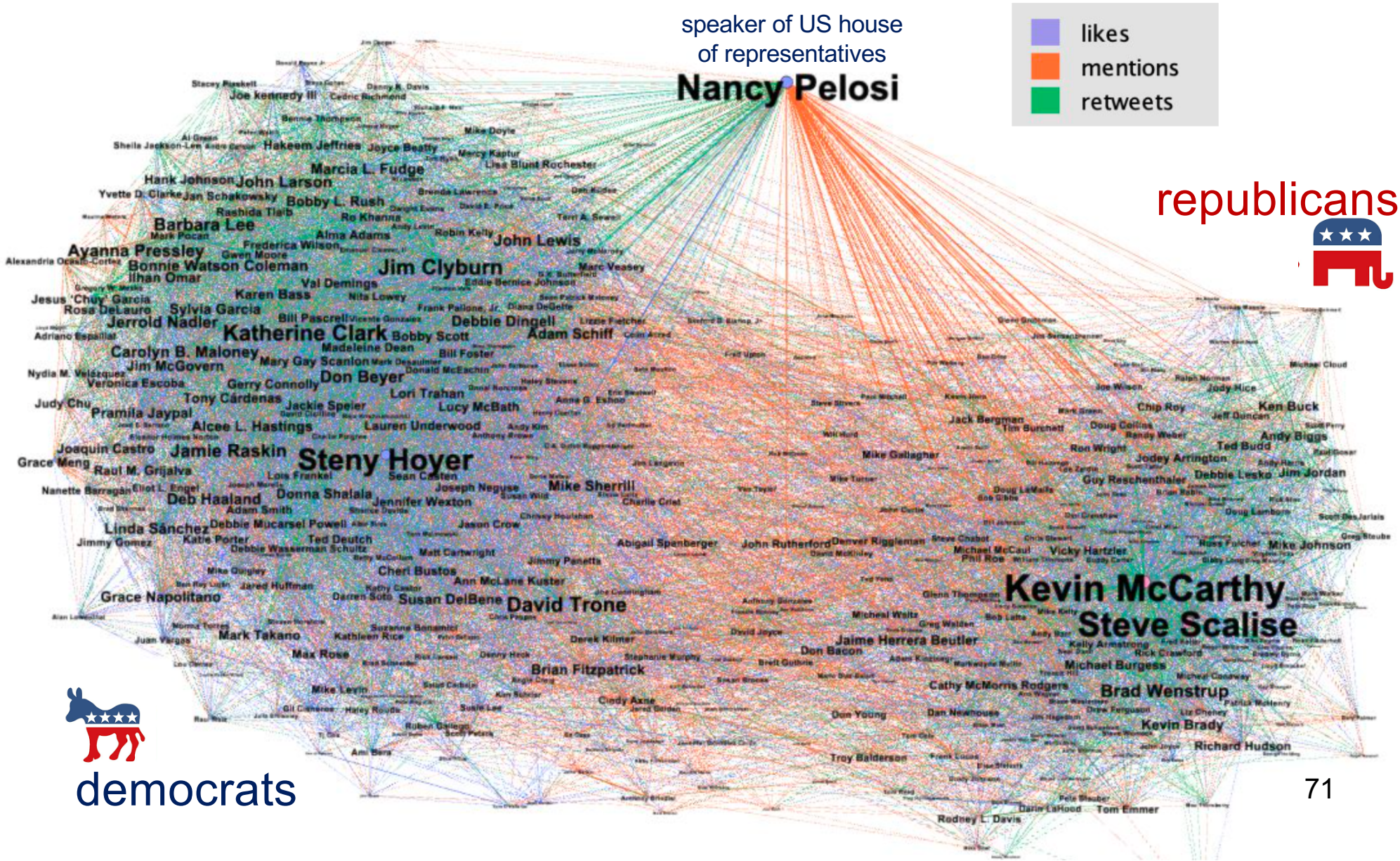



described by a set of adjacency matrices $\boldsymbol{A}_\ell$

average connection $\boldsymbol{A} = \sum_\ell \boldsymbol{A}_\ell$

# An example

## US republicans and democrats interactions on Twitter (2020)



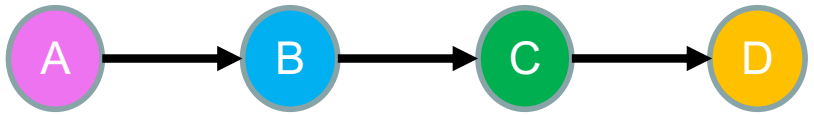
# Paths and connectivity

in graphs

# Paths

- **Path**

  a sequence of interconnected nodes (meaning each pair of nodes adjacent in the sequence are connected by a link)

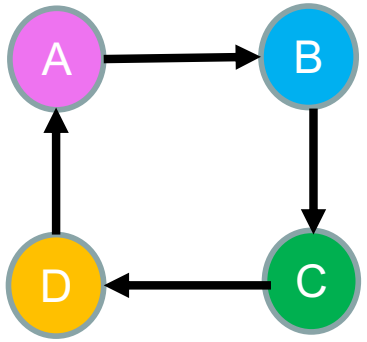



- **Path length**

  # of links involved in the path (if the path involves *n* nodes then the path link is *n-1*)

- **Cycle**

  path where starting and ending nodes coincide

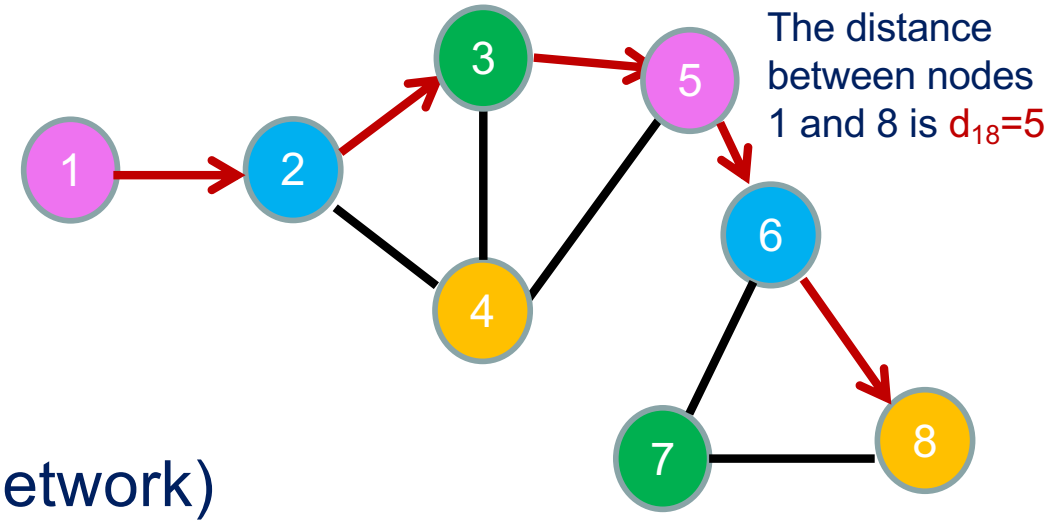


# Distances

- **Shortest path** (between any two nodes)

  the path with the minimum length, which is called the distance

  it is not unique!




- **Diameter** (of the network)

  the highest distance in the network

- **Algorithms**

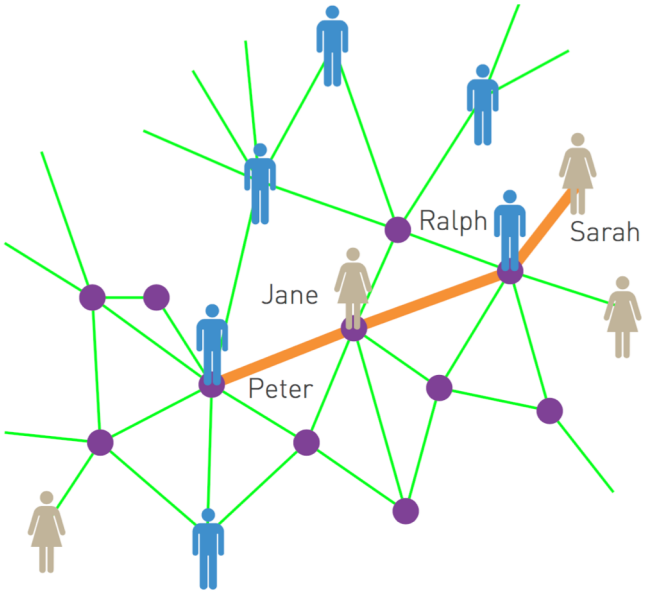
  available to compute distances: Dijkstra, Bellman-Ford, BFS

# Small world

- Average path length

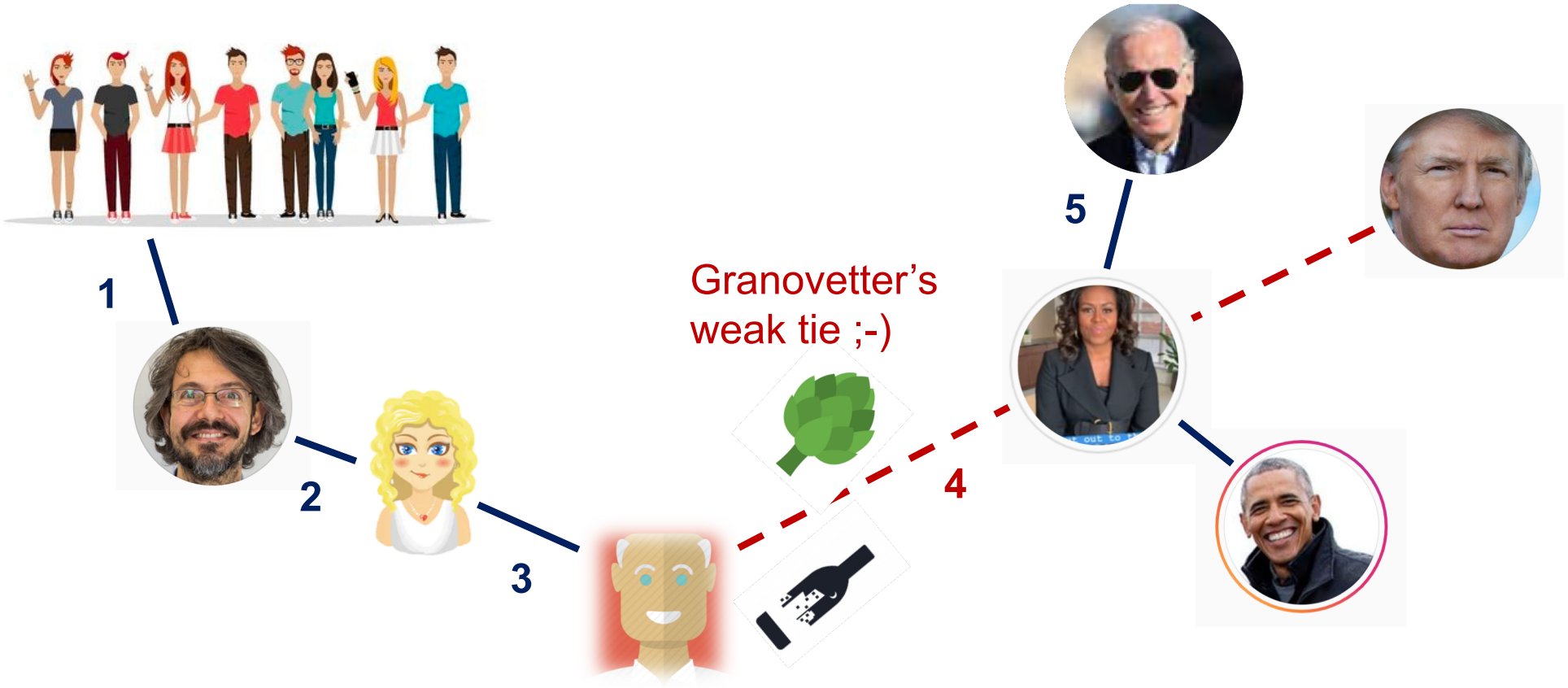
  average distance between all nodes pairs (apply an algorithm to all node couples, and take the average)

- In real networks distance between two randomly chosen nodes is generally short
- Milgram [1967]: *6 degrees of separation*




- What does this mean?
  We are more connected than we think

# Small world
## we and the US presidents



# Connectivity in undirected networks

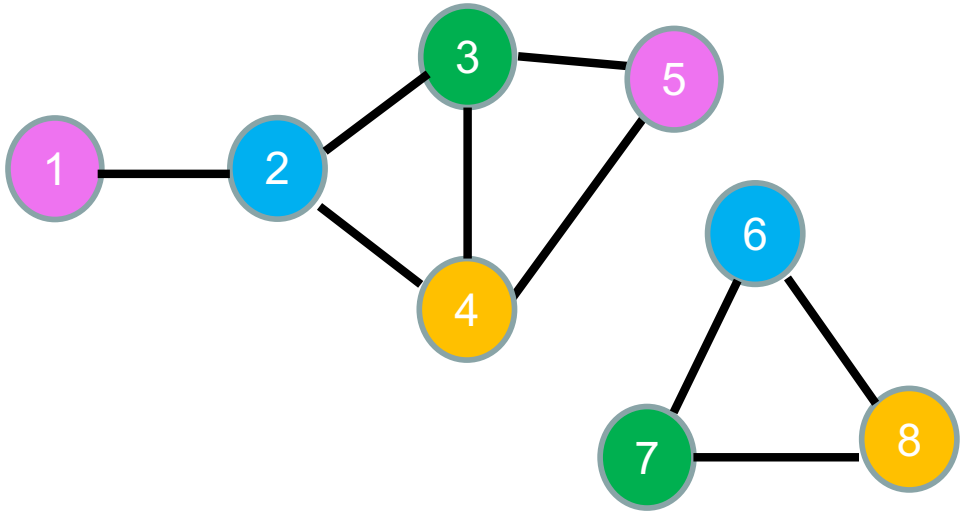
- Connected graph (undirected)

  for all couples (*i*,*j*) there exists a path connecting them

  if disconnected, we count the # of connected components (e.g., use BFS and iterate)

- Giant component (the biggest one)

- Isolates (the other ones)

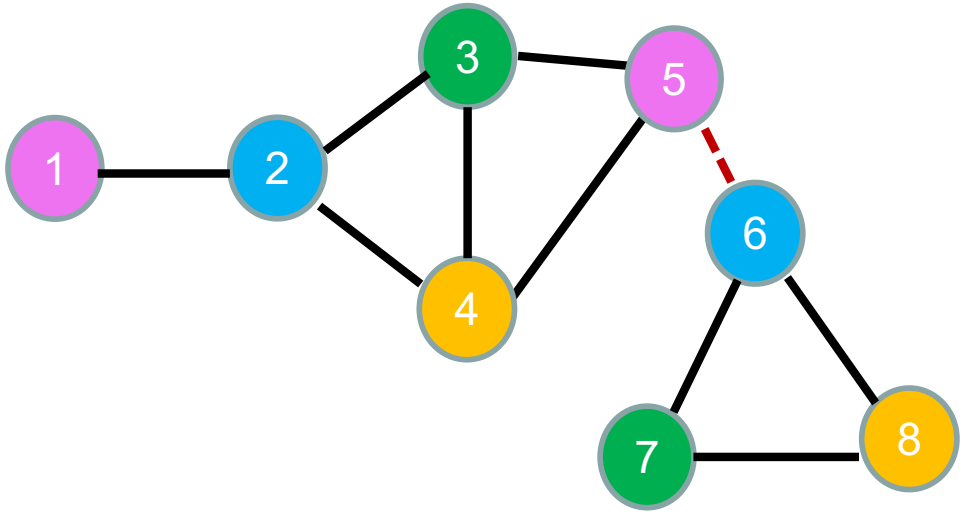



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

block-diagonal matrix

- A bridge is a link between two connected components

  its removal would make the network disconnected




$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

# Connectivity in directed networks

For directed networks we distinguish between

- Strongly connected components

  where $i \rightarrow j$ and $j \rightarrow i$ for all choices of $(i,j)$ in the component

- Weakly connected components

  connected in the undirected sense (i.e., disregard link directions)

# Condensation graph

- Strong connectivity induces a partition in disjoint strongly connected sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_K$
- By reinterpreting the sets as nodes we obtain a condensation graph $\mathcal{G}^*$ where $i \rightarrow j$ is an edge if a connection exists between sets $\mathcal{V}_i \rightarrow \mathcal{V}_j$