3.9.21. 2) Consider the following languages, defined over the alphabet $\Sigma = \{a, b, c\}$

$L_1 = \{ w \mid w = XuYvZ,\ X, Y, Z \in \Sigma,\ u, v \in \Sigma^*,\ X = Z,\ |u| = |v| \}$

$L_2 = \{ w \mid w = XuYvZ,\ X, Y, Z \in \Sigma,\ u, v \in \Sigma^*,\ X = Y = Z \}$

$L_3 = \{ w \mid w = XuYvZ,\ X, Y, Z \in \Sigma,\ u, v \in \Sigma^*,\ X = Y = Z,\ |u| = |v| \}$

State whether the above are regular languages, and provide a mathematical proof of your answers.

- L_2 is regular. $\longrightarrow$ $RE = (\underline{a}(a+b+c)^*\underline{a}(a+b+c)^*\underline{a}) + (\underline{b}(a+b+c)^*\underline{b}(a+b+c)^*\underline{b}) + (\underline{c}(a+b+c)^*\underline{c}(a+b+c)^*\underline{c})$

$\longrightarrow L(RE) = L_2$

- If L_1 were regular, then there would be a constant n such that $|w| \geq n$ we could break w into 3 parts, $w = xyz$ such that:
  - $|y| \geq 1 \Rightarrow y \neq \varepsilon$
  - $|xy| \leq n$
  - For every $k \geq 0$: $xy^kz \in L_1$

* The shortest string $w \in L_1$ has length of at least $3 \Rightarrow \forall w \in L_1$: $|w| \geq 3 \Rightarrow n \geq 3$

Take $w = a\,b^n\,c\,a^n\,a \in L_1$. We can break w into 3 parts such that the first two constraints written above are satisfied as follows:

$x = ab^{n-2}$, $y = b$, $z = bca^na$ $\longrightarrow$ $|xy| = n \leq n$ ✓, $y = b \neq \varepsilon$

If $k = 0$: $ab^{n-2}b^0bca^na = ab^{n-1}ca^na \notin L_1 \Rightarrow L_1$ is NOT regular.

- If L_3 were regular, then there would be a constant n such that $|w| \geq n$ we could break w into 3 parts, $w = xyz$, such that:
  - $|xy| \leq n$
  - $y \neq \varepsilon$
  - For every $k \geq 0$: $xy^kz \in L_3$

Take $a\,b^n\,a\,c^n\,a = w$. We can break w into 3 parts such that the first two constraints written above are satisfied as follows: $x = ab^{n-2}$ $y = b$ $z = bac^na$

If $k = 0$: $ab^{n-2}b^0bac^na = ab^{n-1}ac^na \notin L_3 \Rightarrow$ NOT reg.