Lezione 8 (2023-05-19)

Sistemi lineari: Metodi iterativi

RICHIAMI

Metodi iterativi: dato sistema $\underline{\underline{A}}\,\underline{x} = \underline{b}$

calcoliamo soluzione $\underline{x}$ come limite della successione $\{\underline{x}_k\}$ convergente. Ovvero

$$\lim_{k \to \infty} \underline{x}_k = \underline{x}$$

Metodi stazionari: iterazioni assumono forma del tipo

$$\underline{x}_{k+1} = \underline{\underline{B}}\,\underline{x}_k + \underline{q}$$

matrice di iterazione $\underline{\underline{B}}$ e vettore $\underline{q}$ non dipendono dall'iterazione k

Metodi di rilassamento:

- classe di metodi iterativi di tipo stazionario
- matrice $\underline{\underline{A}}$ viene decomposta nella forma $\underline{\underline{A}} = \underline{\underline{M}} - \underline{\underline{N}}$

$\underline{\underline{M}}$ NON SINGOLARE

- $\underline{\underline{B}} = \underline{\underline{M}}^{-1}\underline{\underline{N}}$ e $\underline{q} = \underline{\underline{M}}^{-1}\underline{b}$

Metodi classici

- quando elementi diagonali matrice $\underline{\underline{A}}$ NON NULLI
- matrici $\underline{\underline{M}}$ e $\underline{\underline{N}}$ costruite a partire da $\underline{\underline{D}}$, $\underline{\underline{E}}$ e $\underline{\underline{F}}$

Sia $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$

$$\underline{\underline{D}} = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}$$

$$-\underline{\underline{E}} = \begin{pmatrix} 0 & & & \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & a_{n\,n-1} & 0 \end{pmatrix} \qquad -\underline{\underline{F}} = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & \ddots & \vdots \\ & & 0 & a_{n-1\,n} \\ & & & 0 \end{pmatrix}$$

Metodo di Jacobi

$\underline{\underline{A}} = \underline{\underline{M}} - \underline{\underline{N}} = \underline{\underline{D}} - (\underline{\underline{E}} + \underline{\underline{F}})$

$$\underline{\underline{M}} = \underline{\underline{D}} \qquad \underline{\underline{N}} = \underline{\underline{E}} + \underline{\underline{F}}$$

$$\underline{\underline{B}}_J = \underline{\underline{D}}^{-1}(\underline{\underline{E}} + \underline{\underline{F}}) \qquad \underline{q} = \underline{\underline{D}}^{-1}\underline{b}$$

Metodo di Gauss-Seidel

$\underline{\underline{A}} = \underline{\underline{M}} - \underline{\underline{N}} = (\underline{\underline{D}} - \underline{\underline{E}}) - \underline{\underline{F}}$

$$\underline{\underline{M}} = \underline{\underline{D}} - \underline{\underline{E}} \qquad \underline{\underline{N}} = \underline{\underline{F}}$$

$$\underline{\underline{B}}_G = (\underline{\underline{D}} - \underline{\underline{E}})^{-1}\underline{\underline{F}} \qquad \underline{q} = (\underline{\underline{D}} - \underline{\underline{E}})^{-1}\underline{b}$$

CONVERGENZA

di un metodo iterativo $\underline{x}_{k+1} = \underline{\underline{B}}\,\underline{x}_k + \underline{q}$ (*)

TEOREMA

Metodo iterativo (*) converge $\forall \underline{x}_0 \in \mathbb{R}^n \iff \rho(\underline{\underline{B}}) < 1$

TEOREMA (condizione SUFFICIENTE, ma NON NECESSARIA)

Se $\|\underline{\underline{B}}\| < 1 \implies$ metodo iterativo (*) converge $\forall \underline{x}_0 \in \mathbb{R}^n$

dove:

$\rho(\underline{\underline{A}})$: raggio spettrale

$$\rho(\underline{\underline{A}}) = \max\{|\lambda_i| : \lambda_i \text{ autovalore di } \underline{\underline{A}}\} = \max_{1 \le i \le n} |\lambda_i|$$

$\|\cdot\|$: norma indotta qualsiasi

Esempi di norme indotte:

$\|\underline{\underline{A}}\|_\infty \longrightarrow$ del MASSIMO tra le SOMME per RIGHE

$\|\underline{\underline{A}}\|_1 \longrightarrow$ del MASSIMO tra le SOMME per COLONNE

Esercizio 4.3

Sia dato il seguente sistema lineare $A\mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

1. Si determini la matrice di iterazione B_J di Jacobi.
2. Si calcoli il raggio spettrale della matrice di iterazione B_J.
3. Si dica se il metodo di Jacobi converge.
4. Fissato $\mathbf{x}^{(0)} = (0, 0, 0)^T$, se il metodo di Jacobi converge, trovare la prima e la seconda soluzione approssimata.

1) Costruzione matrice $B_J = D^{-1}(E+F)$

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix} = I \qquad D^{-1} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$E = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ -2 & -2 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & -2 & 2 \\ & 0 & -1 \\ & & 0 \end{pmatrix} \quad E+F = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

$$B_J = \underbrace{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}}_{D^{-1} = I} \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}}_{E+F}$$

Saltiamo a pto 4 (pti 2 e 3 consentono di verificare che metodo di Jacobi CONVERGE!)

4) Trovare $\underline{x}_1$ e $\underline{x}_2$ noto $\underline{x}_0 = (0,0,0)^T$

$$\underline{x}_{k+1} = \underline{\underline{B}}_J \, \underline{x}_k + \underline{q}$$

$$\underline{q} = \underline{\underline{D}}^{-1} \underline{b}$$

$$\underline{\underline{D}} = \underbrace{\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}}_{\underline{\underline{I}}} \qquad \underline{b} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\underline{\underline{D}} = \underline{\underline{I}} = \underline{\underline{D}}^{-1} \implies \underline{q} = \underline{\underline{D}}^{-1}\underline{b} = \underline{\underline{I}}\,\underline{b} = \underline{b}$$

Prima soluzione

$$\underline{x}_1 = \underline{\underline{B}}_J \, \underline{x}_0 + \underline{q} = \underline{\underline{B}}_J \underbrace{\underline{x}_0}_{\text{vettore nullo}} + \underline{b} = \underline{b}$$

$\underline{x}_0 = (0,0,0)^T$

$$\implies \underline{x}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

Seconda soluzione

$\underline{\underline{B_J}}$ e $\underline{q}$ non vanno aggiornati

$$\underline{x}_2 = \underline{\underline{B_J}}\, \underline{x}_1 + \underline{q}$$

$$\underline{x}_2 = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} -2\cdot 3 + 2\cdot 5 \\ -1\cdot 1 - 1\cdot 5 \\ -2\cdot 1 - 2\cdot 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -6 \\ -8 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -3 \end{pmatrix}$$

2) Raggio spettrale $\underline{\underline{B}}_J$

raggio spettrale $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$: $\rho(\underline{\underline{A}}) = \max_{1 \le i \le n} |\lambda_i|$

calcolo autovalori λ_i : $\det(\underline{\underline{B}}_J - \lambda \underline{\underline{I}}) = 0$

$$\underline{\underline{B}}_J - \lambda \underline{\underline{I}} = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & -2 & 2 \\ -1 & -\lambda & -1 \\ -2 & -2 & -\lambda \end{pmatrix}$$

$\det(\underline{\underline{B}}_J - \lambda \underline{\underline{I}})$:

$$\begin{pmatrix} -\lambda & -2 & 2 \\ -1 & -\lambda & -1 \\ -2 & -2 & -\lambda \end{pmatrix} \begin{matrix} -\lambda & -2 \\ -1 & -\lambda \\ -2 & -2 \end{matrix}$$

$$-\lambda^3 - 4 + 4 - (4\lambda - 2\lambda - 2\lambda) = -\lambda^3$$

$$\det(\underline{\underline{B}}_J - \lambda \underline{\underline{I}}) = 0 \quad \Rightarrow \quad \lambda^3 = 0$$

$$\Rightarrow \quad \lambda_i = 0 \qquad i = 1, 2, 3$$

$$\rho(\underline{\underline{B}}_J) = \max_{1 \le i \le 3} |\lambda_i| = 0$$

3) convergenza metodo Jacobi?

$\rho(\underline{\underline{B}}_J) = 0 < 1 \Rightarrow$ converge $\forall \underline{x}_0 \in \mathbb{R}^3$

Esercizio 4.5

Sia dato il seguente sistema lineare $A\mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}.$$

1. Si determini la matrice di iterazione B_G.
2. Si dica perchè il metodo di Gauss Seidel converge sempre, utilizzando il teorema relativo a $\rho(B_G)$.
3. Fissato $\mathbf{x}^{(0)} = (0, 0, 0)^T$, trovare $\mathbf{x}^{(1)}$ e $\mathbf{x}^{(2)}$, prima e seconda soluzione approssimata, ottenuta con il metodo di Gauss Seidel (in aritmetica esatta).

1) Costruzione matrice $\underline{\underline{B}}_G = (\underline{\underline{D}} - \underline{\underline{E}})^{-1} \underline{\underline{F}}$

$$\underline{\underline{D}} = \begin{pmatrix} 2 & & \\ 0 & 3 & \\ 0 & 0 & 2 \end{pmatrix}$$

$$\underline{\underline{E}} = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 0 & -1 & 0 \end{pmatrix} \qquad \underline{\underline{F}} = \begin{pmatrix} 0 & -1 & 0 \\ & 0 & -1 \\ & & 0 \end{pmatrix}$$

$$\underline{\underline{D}} - \underline{\underline{E}} = \begin{pmatrix} 2 & & \\ 1 & 3 & \\ 0 & 1 & 2 \end{pmatrix} \longrightarrow \overbrace{(\underline{\underline{D}} - \underline{\underline{E}})^{-1} = \begin{pmatrix} 1/2 & & \\ -1/6 & 1/3 & \\ 1/12 & -1/6 & 1/2 \end{pmatrix}}^{\text{dettagli di calcolo a seguire}}$$

$$\underline{\underline{B}}_G = (\underline{\underline{D}} - \underline{\underline{E}})^{-1} \underline{\underline{F}}$$

$$\underline{\underline{B}}_G = \begin{pmatrix} 1/2 & & \\ -1/6 & 1/3 & \\ 1/12 & -1/6 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ & 0 & -1 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 & 0 \\ 0 & 1/6 & -1/3 \\ 0 & -1/12 & 1/6 \end{pmatrix}$$

Metodo 1 per calcolo $(\underline{\underline{D}} - \underline{\underline{E}})^{-1}$

Formule per matrice triangolare inferiore!

Sia $\underline{\underline{L}} = (\ell_{ij})$ e matrice inversa $\underline{\underline{L}}^{-1} = (y_{ij})$:

$$\left.\begin{aligned} y_{ii} &= \ell_{ii}^{-1} \\ y_{ji} &= -\ell_{jj}^{-1} \sum_{k=1}^{j-1} \ell_{jk}\, y_{ki} \qquad j = i+1, i+2, \dots, n \end{aligned}\right\} \quad i = 1, 2, \dots, n$$

$\underline{i = 1}$ (colonna 1)

$$y_{ii} = \ell_{ii}^{-1}$$

$$y_{ji} = -\ell_{jj}^{-1} \sum_{k=1}^{j-1} \ell_{jk}\, y_{ki}$$

$$\underline{\underline{L}} = \underline{\underline{D}} - \underline{\underline{E}} = \begin{pmatrix} 2 & & \\ 1 & 3 & \\ 0 & 1 & 2 \end{pmatrix}$$

$$y_{11} = 1/\ell_{11} = 1/2$$

$$y_{21} = -1/\ell_{22} \sum_{k=1}^{1} \ell_{2k}\, y_{k1} = -\frac{1}{\ell_{22}} (\ell_{21}\, y_{11})$$

$$= -1/3\, (1 \cdot 1/2) = -1/6$$

$$y_{31} = -\frac{1}{\ell_{33}} \sum_{k=1}^{2} \ell_{3k}\, y_{k1} = -\frac{1}{\ell_{33}} (\ell_{31}\, y_{11} + \ell_{32}\, y_{21})$$

$$= -\frac{1}{2} \left[0 \cdot 1/2 + 1 \cdot (-1/6) \right] = \frac{1}{12}$$

$\underline{i=2}$ (colonna 2)

$$\underline{\underline{L}} = \begin{pmatrix} 2 & & \\ 1 & 3 & \\ 0 & 1 & 2 \end{pmatrix}$$

$$y_{ii} = \ell_{ii}^{-1}$$

$$y_{ji} = -\ell_{jj}^{-1} \sum_{k=1}^{j-1} \ell_{jk}\, y_{ki}$$

$$\underline{\underline{L}}^{-1} = \begin{pmatrix} 1/2 & & \\ -1/6 & y_{22} & \\ 1/12 & y_{32} & y_{33} \end{pmatrix}$$

$$y_{22} = 1/\ell_{22} = 1/3$$

$$y_{32} = -\frac{1}{\ell_{33}} \sum_{k=1}^{2} \ell_{3k}\, y_{k2} = -\frac{1}{\ell_{33}} \left(\ell_{31}\, y_{12} + \ell_{32}\, y_{22} \right)$$

$$= -\frac{1}{2} \left(0 \cdot 0 + 1 \cdot 1/3 \right) = -\frac{1}{6}$$

$\underline{i=3}$ (colonna 3)

$$\underline{\underline{L}} = \begin{pmatrix} 2 & & \\ 1 & 3 & \\ 0 & 1 & 2 \end{pmatrix}$$

$$y_{ii} = \ell_{ii}^{-1}$$

$$y_{33} = \frac{1}{\ell_{33}} = 1/2$$

$$\underline{\underline{L}}^{-1} = \begin{pmatrix} 1/2 & & \\ -1/6 & 1/3 & \\ 1/12 & -1/6 & y_{33} \end{pmatrix}$$

Risultato Metodo 1

$$(\underline{\underline{D}} - \underline{\underline{E}})^{-1} = \underline{\underline{L}}^{-1} = \begin{pmatrix} 1/2 & & \\ -1/6 & 1/3 & \\ 1/12 & -1/6 & 1/2 \end{pmatrix}$$

Metodo 2 per calcolo $(\underline{\underline{D}}-\underline{\underline{E}})^{-1}$

Sfruttiamo definizione MATRICE INVERSA

$$(\underline{\underline{D}}-\underline{\underline{E}})^{-1}(\underline{\underline{D}}-\underline{\underline{E}}) = \underline{\underline{I}} \qquad \underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\underline{\underline{D}}-\underline{\underline{E}})^{-1} = \begin{pmatrix} 1/2 & & \\ a & 1/3 & \\ b & c & 1/2 \end{pmatrix} \qquad (\underline{\underline{D}}-\underline{\underline{E}}) = \begin{pmatrix} 2 & & \\ 1 & 3 & \\ 0 & 1 & 2 \end{pmatrix}$$

$2a + 1/3 = 0$ $\quad a = -1/6$ $\quad$ (2° riga × 1° colonna)

$2b + c = 0$ $\quad b = 1/12$ $\quad$ (3° riga × 1° colonna)

$3c + 1/2 = 0$ $\quad c = -1/6$ $\quad$ (3° riga × 2° colonna)

Risultato Metodo 2

$$(\underline{\underline{D}}-\underline{\underline{E}})^{-1} = \begin{pmatrix} 1/2 & & \\ -1/6 & 1/3 & \\ 1/12 & -1/6 & 1/2 \end{pmatrix}$$

Pto 2 consente di verificare che metodo Gauss-Seidel converge!

3) Determinare $\underline{x}_1$ e $\underline{x}_2$, partendo da $\underline{x}_0 = (0,0,0)^T$

$$\underline{x}_{k+1} = \underline{\underline{B}}_G \, \underline{x}_k + \underline{g}$$

$$\underline{g} = (\underline{\underline{D}} - \underline{\underline{E}})^{-1} \underline{b}$$

$$(\underline{\underline{D}} - \underline{\underline{E}})^{-1} = \begin{pmatrix} 1/2 & & \\ -1/6 & 1/3 & \\ 1/12 & -1/6 & 1/2 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$$

$$\underline{g} = \begin{pmatrix} 1/2 & & \\ -1/6 & 1/3 & \\ 1/12 & -1/6 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \cdot 3 \\ -1/6 \cdot 3 + 1/3 \cdot 5 \\ 1/12 \cdot 3 - 1/6 \cdot 5 + 1/2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 7/6 \\ 11/12 \end{pmatrix}$$

$$\underline{x}_1 = \underline{\underline{B}}_G \, \underline{x}_0 + \underline{g} = \underline{g} = (3/2, \; 7/6, \; 11/12)^T$$

$\underline{x}_0$ vettore nullo

$$\underline{x}_2 = \underline{\underline{B}}_G \, \underline{x}_1 + \underline{g} = \begin{pmatrix} 0 & -1/2 & 0 \\ 0 & 1/6 & -1/3 \\ 0 & -1/12 & 1/6 \end{pmatrix} \begin{pmatrix} 3/2 \\ 7/6 \\ 11/12 \end{pmatrix} + \begin{pmatrix} 3/2 \\ 7/6 \\ 11/12 \end{pmatrix}$$

$$= \begin{pmatrix} -7/12 \\ -1/9 \\ 1/18 \end{pmatrix} + \begin{pmatrix} 3/2 \\ 7/6 \\ 11/12 \end{pmatrix} = \begin{pmatrix} 11/12 \\ 19/18 \\ 35/36 \end{pmatrix}$$

2) Discutere convergenza metodo Gauss Seidel utilizzando teorema relativo a $\rho(\underline{\underline{B}}_G)$

$$\underline{\underline{B}}_G = \begin{pmatrix} 0 & -1/2 & 0 \\ 0 & 1/6 & -1/3 \\ 0 & -1/12 & 1/6 \end{pmatrix}$$

Calcolo AUTOVALORI

$$\det(\underline{\underline{B}}_G - \lambda \underline{\underline{I}}) = 0$$

$$\begin{pmatrix} -\lambda & -1/2 & 0 \\ 0 & 1/6-\lambda & -1/3 \\ 0 & -1/12 & 1/6-\lambda \end{pmatrix} \begin{matrix} -\lambda & -1/2 \\ 0 & 1/6-\lambda \\ 0 & -1/12 \end{matrix}$$

$$-\lambda \cdot (1/6-\lambda)^2 - (-\lambda \cdot 1/36) = 0$$

$$\lambda^2 \cdot (\lambda - 1/3) = 0 \quad \Longrightarrow \quad \lambda_1 = 1/3 \quad \lambda_{2,3} = 0$$

Raggio spettrale:

$$\rho(\underline{\underline{B}}_G) = \max_{1 \le i \le 3} |\lambda_i| = 1/3 < 1 \quad \Longrightarrow$$

Gauss Seidel CONVERGE sempre

Esercizio 4.13 ✓

Sia dato il seguente sistema lineare $A\mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} -10 & 1 & 1/2 \\ 1 & 5 & 2 \\ 3/5 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

1. Si calcoli (in aritmetica esatta) la matrice di iterazione B_J del metodo iterativo di Jacobi.
2. Si dica se il metodo di Jacobi applicato a tale sistema, converge.
3. Fissato $\mathbf{x}^{(0)} = (0, 0, 0)^T$, se il metodo converge, si trovino la prima e la seconda soluzione approssimata $\mathbf{x}^{(1)}$ e $\mathbf{x}^{(2)}$ (facendo prima i calcoli in aritmetica esatta e poi restituendo i due risultati con almeno 8-9 cifre decimali).
4. Sapendo che la soluzione esatta del sistema è $\mathbf{x} = (-5/67, 4/67, 26/67)^T$, facendo i calcoli con almeno 8-9 cifre decimali, si valuti il vettore errore rispetto alla seconda iterata $\mathbf{x}^{(2)}$.

1) Calcolare matrice $\underline{\underline{B_J}}$

$$\underline{\underline{B_J}} = \underline{\underline{D}}^{-1} (\underline{\underline{E}} + \underline{\underline{F}})$$

$$\underline{\underline{A}} = \begin{pmatrix} -10 & 1 & 1/2 \\ 1 & 5 & 2 \\ 3/5 & -2 & 3 \end{pmatrix}$$

$$\underline{\underline{B_J}} = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} \\ -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{10} & \frac{1}{20} \\ -\frac{1}{5} & 0 & -\frac{2}{5} \\ -\frac{1}{5} & \frac{2}{3} & 0 \end{pmatrix}$$

Verificare $\underline{\underline{B_J}} = \underline{\underline{D}}^{-1} (\underline{\underline{E}} + \underline{\underline{F}}) \ldots$ (procedimento Es 4.3)

Pto 2 consente di verificare che metodo Jacobi converge!

3) Valutare soluzioni approssimate $\underline{x}_1$ e $\underline{x}_2$ dato $\underline{x}_0 = (0,0,0)^T$

$$\underline{x}_{k+1} = \underline{\underline{B}}_J \, \underline{x}_k + \underline{q}$$

$$\underline{\underline{B}}_J = \underline{\underline{D}}^{-1}(\underline{\underline{E}} + \underline{\underline{F}})$$

$$\underline{q} = \underline{\underline{D}}^{-1} \underline{b}$$

$$\underline{q} = \begin{pmatrix} -1/10 & & \\ 0 & 1/5 & \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/10 \\ 1/5 \\ 1/3 \end{pmatrix}$$

$$\underline{x}_1 = \underline{\underline{B}}_J \, \underline{x}_0 + \underline{q} = \underline{q} = (-1/10,\ 1/5,\ 1/3)^T$$

$\underline{x}_0$ vettore nullo

$$\underline{x}_1 = \begin{pmatrix} -1/10 \\ 1/5 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -0,1 \\ 0,2 \\ 0,333\,333\,333 \end{pmatrix}$$

$$\underline{x}_2 = \underline{\underline{B}}_J \, \underline{x}_1 + \underline{q} = \begin{pmatrix} 0 & 1/10 & 1/20 \\ -1/5 & 0 & -2/5 \\ -1/5 & 2/3 & 0 \end{pmatrix} \begin{pmatrix} -1/10 \\ 1/5 \\ 1/3 \end{pmatrix} + \begin{pmatrix} -1/10 \\ 1/5 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 11/300 \\ -17/150 \\ 23/150 \end{pmatrix} + \begin{pmatrix} -1/10 \\ 1/5 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -19/300 \\ 13/150 \\ 73/150 \end{pmatrix}$$

$$\underline{x}_2 = \begin{pmatrix} -19/300 \\ 13/150 \\ 73/150 \end{pmatrix} = \begin{pmatrix} -0,633\ 333\ 333 \cdot 10^{-1} \\ 0,866\ 666\ 667 \cdot 10^{-1} \\ 0,486\ 666\ 667 \end{pmatrix}$$

4) Vettore errore seconda iterata

$$\underline{e}_2 = \underline{x} - \underline{x}_2 = \begin{pmatrix} -5/67 \\ 4/67 \\ 26/67 \end{pmatrix} - \begin{pmatrix} -19/300 \\ 13/150 \\ 73/150 \end{pmatrix}$$

$$= \begin{pmatrix} -227/20\,100 \\ -271/10\,050 \\ -991/10\,050 \end{pmatrix} = \begin{pmatrix} -0,112\ 935\ 323 \cdot 10^{-1} \\ -0,269\ 651\ 741 \cdot 10^{-1} \\ -0,986\ 069\ 652 \cdot 10^{-1} \end{pmatrix}$$

2) Si dica se metodo Jacobi converge

Approccio 1: valuto $\rho(\underline{\underline{B}}_J)$

Calcolo AUTOVALORI λ_i imponendo $\det(\underline{\underline{B}}_J - \lambda \underline{\underline{I}}) = 0$

$$\underline{\underline{B}}_J = \begin{pmatrix} 0 & 1/10 & 1/20 \\ -1/5 & 0 & -2/5 \\ -1/5 & 2/3 & 0 \end{pmatrix} \qquad \underline{\underline{B}}_J - \lambda \underline{\underline{I}} = \begin{pmatrix} -\lambda & 1/10 & 1/20 \\ -1/5 & -\lambda & -2/5 \\ -1/5 & 2/3 & -\lambda \end{pmatrix}$$

$$\det(\underline{\underline{B}}_J - \lambda \underline{\underline{I}}) = 0 \implies \lambda^3 + \frac{89}{300}\lambda - \frac{1}{750} = 0$$

$$\rho(\underline{\underline{B}}_J) = \max_{1 \le i \le 3} |\lambda_i| \simeq 0,545 < 1 \implies$$ Jacobi converge

Approccio 2: valutare norma $\|\underline{\underline{B}}_J\|$

2.1) Norma del massimo tra le somme per righe

$$\|\underline{\underline{A}}\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\underline{\underline{B}}_J = \begin{pmatrix} 0 & 1/10 & 1/20 \\ -1/5 & 0 & -2/5 \\ -1/5 & 2/3 & 0 \end{pmatrix}$$

riga 1 $\quad |a_{11}| + |a_{12}| + |a_{13}| = 1/10 + 1/20 = 3/20$

riga 2 $\quad |a_{21}| + |a_{22}| + |a_{23}| = 1/5 + 2/5 = 3/5$

riga 3 $\quad |a_{31}| + |a_{32}| + |a_{33}| = 1/5 + 2/3 = 13/15$

$$\|\underline{\underline{B_J}}\|_\infty = \max\{3/20,\ 3/5,\ 13/15\} = 13/15$$

$\|\underline{\underline{B_J}}\|_\infty = 13/15 < 1 \implies$ Jacobi converge

2.2) Norma del massimo tra le somme per colonne

$$\|\underline{\underline{A}}\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

$$\underline{\underline{B_J}} = \begin{pmatrix} 0 & 1/10 & 1/20 \\ -1/5 & 0 & -2/5 \\ -1/5 & 2/3 & 0 \end{pmatrix}$$

colonna 1 $\quad |a_{11}| + |a_{21}| + |a_{31}| = 1/5 + 1/5 = 2/5$

col. 2 $\quad |a_{12}| + |a_{22}| + |a_{32}| = 1/10 + 2/3 = 23/30$

col. 3 $\quad |a_{13}| + |a_{23}| + |a_{33}| = 1/20 + 2/5 = 9/20$

$$\|\underline{\underline{B_J}}\|_1 = \max\{2/5,\ 23/30,\ 9/20\} = 23/30$$

$\|\underline{\underline{B_J}}\|_1 = 23/30 < 1 \implies$ Jacobi converge

Approccio 3 : esaminare matrice $\underline{\underline{A}}$

3.1) STRETTAMENTE DIAGONALE DOMINANTE per RIGHE

$$\underline{\underline{A}} = \begin{pmatrix} -10 & 1 & 1/2 \\ 1 & 5 & 2 \\ 3/5 & -2 & 3 \end{pmatrix}$$

riga 1 $\quad |a_{11}| = 10 \quad > \quad |a_{12}| + |a_{13}| = 1 + 1/2 = 3/2$

riga 2 $\quad |a_{22}| = 5 \quad > \quad |a_{21}| + |a_{23}| = 1 + 2 = 3$

riga 3 $\quad |a_{33}| = 3 \quad > \quad |a_{31}| + |a_{32}| = 3/5 + 2 = 13/5$

Verificata $\Rightarrow$ Jacobi converge

3.2) STRETTAMENTE DIAGONALE DOMINANTE per COLONNE

$$\underline{\underline{A}} = \begin{pmatrix} -10 & 1 & 1/2 \\ 1 & 5 & 2 \\ 3/5 & -2 & 3 \end{pmatrix}$$

colonna 1 $\quad |a_{11}| = 10 \quad > \quad |a_{21}| + |a_{31}| = 1 + 3/5 = 8/5$

col. 2 $\quad |a_{22}| = 5 \quad > \quad |a_{12}| + |a_{32}| = 1 + 2 = 3$

col. 3 $\quad |a_{33}| = 3 \quad > \quad |a_{13}| + |a_{23}| = 1/2 + 2 = 5/2$

Verificata $\Rightarrow$ Jacobi converge

Esercizio 4.4

Si consideri il seguente sistema di equazioni lineari, che ha per soluzione esatta $\mathbf{x} = (1, 1, 1)^T$

$$\begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 6 \end{pmatrix}.$$

1. Si dica per quale ragione, guardando il sistema, si deduce che il metodo di Gauss–Seidel converge.
2. Si determini (in aritmetica esatta) la matrice di iterazione B_G del metodo di Gauss Seidel.
3. Si verifichi che per tale matrice è verificato il teorema di convergenza basato sulla norma.
4. Fissato $\mathbf{x}^{(0)} = (0, 0, 0)^T$, trovare, con il metodo di Gauss-Seidel, la prima e la seconda soluzione approssimata $\mathbf{x}^{(1)}$ e $\mathbf{x}^{(2)}$ (in aritmetica esatta).

1) Analisi matrice $\underline{\underline{A}}$ --- tralasciamo

2) Determinare matrice $\underline{\underline{B_G}}$

$$\underline{\underline{B_G}} = (\underline{\underline{D}} - \underline{\underline{E}})^{-1} \underline{\underline{F}}$$

$$\underline{\underline{A}} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\underline{\underline{D}} = \begin{pmatrix} 5 & & \\ 0 & 5 & \\ 0 & 0 & 5 \end{pmatrix} \qquad \underline{\underline{E}} = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 0 & -1 & 0 \end{pmatrix} \qquad \underline{\underline{F}} = \begin{pmatrix} 0 & -1 & 0 \\ & 0 & -1 \\ & & 0 \end{pmatrix}$$

$$\underline{\underline{D}} - \underline{\underline{E}} = \begin{pmatrix} 5 & & \\ 1 & 5 & \\ 0 & 1 & 5 \end{pmatrix} \qquad (\underline{\underline{D}} - \underline{\underline{E}})^{-1} = \begin{pmatrix} 1/5 & & \\ -1/25 & 1/5 & \\ 1/125 & -1/25 & 1/5 \end{pmatrix}$$

dettagli calcolo a seguire

calcolo inversa $(\underline{\underline{D}}-\underline{\underline{E}})^{-1}$

$$(\underline{\underline{D}}-\underline{\underline{E}})^{-1}(\underline{\underline{D}}-\underline{\underline{E}}) = \underline{\underline{I}}$$

$$\begin{pmatrix} 1/5 & & \\ a & 1/5 & \\ b & c & 1/5 \end{pmatrix} \begin{pmatrix} 5 & & \\ 1 & 5 & \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

riga 2 × col 1 $\qquad 5a + 1/5 = 0 \quad \Rightarrow \quad a = -1/25$

riga 3 × col. 1 $\qquad 5b + c = 0 \qquad\qquad b = -c/5 = 1/125$

riga 3 × col 2 $\qquad 5c + 1/5 = 0 \quad \Rightarrow \quad c = -1/25$

$$(\underline{\underline{D}}-\underline{\underline{E}})^{-1} = \begin{pmatrix} 1/5 & & \\ -1/25 & 1/5 & \\ 1/125 & -1/25 & 1/5 \end{pmatrix}$$

completo calcolo matrice $\underline{\underline{B_G}}$ Gauss-Seidel

$$\underline{\underline{B_G}} = (\underline{\underline{D}}-\underline{\underline{E}})^{-1}\,\underline{\underline{F}}$$

$$\underline{\underline{B_G}} = \begin{pmatrix} 1/5 & & \\ -1/25 & 1/5 & \\ 1/125 & -1/25 & 1/5 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ & 0 & -1 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1/5 & 0 \\ 0 & 1/25 & -1/5 \\ 0 & -1/125 & 1/25 \end{pmatrix}$$

3) Verifica TEOREMA CONVERGENZA basato su NORMA

$\|\underline{\underline{B_G}}\| < 1 \Rightarrow$ Gauss-Seidel CONVERGE

$\|\cdot\|$: norma indotta qualsiasi

Norma del massimo tra le somme per righe

$$\|\underline{\underline{A}}\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\underline{\underline{B_G}} = \begin{pmatrix} 0 & -1/5 & 0 \\ 0 & 1/25 & -1/5 \\ 0 & -1/125 & 1/25 \end{pmatrix}$$

riga 1 $(i=1)$ $\sum_{j=1}^{3} |a_{1j}| = 0 + 1/5 + 0 = 1/5$

riga 2 $(i=2)$ $\sum_{j=1}^{3} |a_{2j}| = 0 + 1/25 + 1/5 = 6/25$

riga 3 $(i=3)$ $\sum_{j=1}^{3} |a_{3j}| = 0 + 1/125 + 1/25 = 6/125$

$$\|\underline{\underline{B_G}}\|_\infty = \max\left\{ 1/5,\ 6/25,\ 6/125 \right\} = 6/25$$

$\|\underline{\underline{B_G}}\|_\infty = 6/25 < 1 \Rightarrow$ converge $\forall\, \underline{x_0} \in \mathbb{R}^3$

Norma del massimo tra le somme per colonne

$$\|\underline{\underline{A}}\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

$$\underline{\underline{B}}_G = \begin{pmatrix} 0 & -1/5 & 0 \\ 0 & 1/25 & -1/5 \\ 0 & -1/125 & 1/25 \end{pmatrix}$$

col. 1 $\quad (j=1) \quad \sum_{i=1}^{3} |a_{i1}| = 0 + 0 + 0 = 0$

col. 2 $\quad (j=2) \quad \sum_{i=1}^{3} |a_{i2}| = 1/5 + 1/25 + 1/125 = 31/125$

col. 3 $\quad (j=3) \quad \sum_{i=1}^{3} |a_{i3}| = 0 + 1/5 + 1/25 = 6/25$

$\|\underline{\underline{B}}_G\|_1 = 31/125 < 1 \quad \Rightarrow \quad$ converge $\forall \underline{x}_0 \in \mathbb{R}^3$

4) Valutare soluzioni approssimate $\underline{x}_1$ e $\underline{x}_2$ a partire da $\underline{x}_0 = (0,0,0)^T$

$$\underline{x}_{k+1} = \underline{\underline{B}}_G \, \underline{x}_k + \underline{q}$$

$$\underline{\underline{B}}_G = (\underline{\underline{D}} - \underline{\underline{E}})^{-1} \underline{\underline{F}}$$

$$\underline{q} = (\underline{\underline{D}} - \underline{\underline{E}})^{-1} \underline{b}$$

$$\underline{q} = \begin{pmatrix} 1/5 & & \\ -1/25 & 1/5 & \\ 1/125 & -1/25 & 1/5 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 6/5 \\ 29/25 \\ 121/125 \end{pmatrix}$$

$$\underline{x}_1 = \underline{\underline{B}}_G \underbrace{\underline{x}_0}_{\text{vettor nullo}} + \underline{q} = \underline{q} = \left(6/5 ,\ 29/25 ,\ 121/125\right)^T$$

$$\underline{x}_2 = \underline{\underline{B}}_G \underline{x}_1 + \underline{q}$$

$$= \begin{pmatrix} 0 & -1/5 & 0 \\ 0 & 1/25 & -1/5 \\ 0 & -1/125 & 1/25 \end{pmatrix} \begin{pmatrix} 6/5 \\ 29/25 \\ 121/125 \end{pmatrix} + \begin{pmatrix} 6/5 \\ 29/25 \\ 121/125 \end{pmatrix}$$

$$= \begin{pmatrix} -29/125 \\ -92/625 \\ 92/3125 \end{pmatrix} + \begin{pmatrix} 6/5 \\ 29/25 \\ 121/125 \end{pmatrix} = \begin{pmatrix} 121/125 \\ 633/625 \\ 3117/3125 \end{pmatrix}$$