Lezione 7 (2023-05-12)

Sistemi lineari: Metodi diretti

Esercizio 3.22 ✓

(Da svolgere in aritmetica esatta). Si consideri la seguente matrice non singolare

$$A = \begin{pmatrix} 3 & -1 & 2 \\ -3 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix},$$

ed i due sistemi $A\mathbf{x} = \mathbf{b}$ e $A\overline{\mathbf{x}} = \overline{\mathbf{b}}$, aventi la stessa matrice A e due diversi termini noti $\mathbf{b} = (4, -1, 6)^T$ e $\overline{\mathbf{b}} = (7, 2, 10)^T$.

1. Utilizzando il metodo di Gauss con pivoting e la matrice aumentata $(A|\mathbf{b}\overline{\mathbf{b}})$ si costruiscano simultaneamente i due sistemi triangolari superiori $U\mathbf{x} = \mathbf{y}$ e $U\overline{\mathbf{x}} = \overline{\mathbf{y}}$, indicando chiaramente quanto valgono tutte le matrici di permutazione elementari determinate ad ogni passo.
2. Si determinino le soluzioni $\mathbf{x}$ e $\overline{\mathbf{x}}$ dei due sistemi di partenza (non si deve utilizzare la fattorizzazione $PA = LU$ per la risoluzione di tali sistemi).
3. Si dica quanto valgono L e P (la matrice di permutazione ottenuta come prodotto delle matrici di permutazione elementari determinate ad ogni passo).
4. Si verifichi che il prodotto $L \times U$ restituisca il prodotto $P \times A$.
5. Si calcoli il determinante di A.

Matrice $\underline{A}$ NON SINGOLARE

$\det \underline{\underline{A}} \neq 0$

$$\begin{pmatrix} 3 & -1 & 2 \\ -3 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \begin{matrix} 3 & -1 \\ -3 & 1 \\ 3 & 2 \end{matrix}$$

$$3 - 3 - 12 - (6 + 6 + 3) = -27$$

1) Gauss + pivoting su matrice aumentata

$$(\underline{\underline{A}} \mid \underline{b}\ \underline{\bar{b}}) = \left(\begin{array}{ccc|cc} 3 & -1 & 2 & 4 & 7 \\ -3 & 1 & 1 & -1 & 2 \\ 3 & 2 & 1 & 6 & 10 \end{array}\right)$$

$\underline{b}$ $\quad \underline{\bar{b}}$

Passo 1

cerco r t.c. $|a_{r1}^{(0)}| = \max_{1 \le i \le 3} |a_{i1}^{(0)}|$

$r = 1 \quad a_{11}^{(0)} = 3 \quad \Rightarrow$ non serve scambio righe

matrice permutazione elementare

$$\underline{\underline{P}}_1 = \underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{P}}_1 (\underline{\underline{A}} \mid \underline{b}\ \underline{\bar{b}})^{(0)} = (\underline{\underline{A}} \mid \underline{b}\ \underline{\bar{b}})^{(0)} = \left(\begin{array}{ccc|cc} \boxed{3} & -1 & 2 & 4 & 7 \\ -3 & 1 & 1 & -1 & 2 \\ 3 & 2 & 1 & 6 & 10 \end{array}\right)$$

moltiplicatori	matrice aggiornata
$\ell_{21} = -3/3 = -1$ $\ell_{31} = 3/3 = 1$	$(\underline{\underline{A}} \mid \underline{b}\ \underline{\bar{b}})^{(1)} = \left(\begin{array}{ccc\|cc} 3 & -1 & 2 & 4 & 7 \\ 0 & 0 & 3 & 3 & 9 \\ 0 & 3 & -1 & 2 & 3 \end{array}\right)$

$(2^\circ \text{ riga}) - \ell_{21} \times (1^\circ \text{ riga})$

$(3^\circ \text{ riga}) - \ell_{31} \times (1^\circ \text{ riga})$

Passo 2

cerco r t.c. $|a_{r2}^{(1)}| = \max_{2 \le i \le 3} |a_{i2}^{(1)}|$

$r = 3 \quad a_{32}^{(1)} = 3 \quad \Rightarrow$ scambio righe 2 e 3
matrice permutazione elementare

$$\underline{\underline{P_2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underline{\underline{P_2}} (\underline{\underline{A}} | \underline{b}\, \underline{\bar{b}})^{(1)} = \left(\begin{array}{ccc|cc} 3 & -1 & 2 & 4 & 7 \\ 0 & \boxed{3} & -1 & 2 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{array}\right)$$

SCAMBIARE MOLTIPLICATORI già DETERMINATI!
$l_{21} = 1 \qquad l_{31} = -1$

moltiplicatore	matrice aggiornata
$l_{32} = 0/3 = 0$	$\underline{\underline{A}}^{(2)} = \left(\begin{array}{ccc\|cc} 3 & -1 & 2 & 4 & 7 \\ 0 & 3 & -1 & 2 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{array}\right)$

(3° riga) − l_{32} × (2° riga)

Sistemi triangolari superiori:

$\underline{\underline{U}}\, \underline{x} = \underline{y}$

$$\underbrace{\begin{pmatrix} 3 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix}}_{\underline{\underline{U}}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}}_{\underline{y}}$$

$\underline{\underline{U}}\,\bar{\underline{x}} = \bar{\underline{y}}$

$$\underbrace{\begin{pmatrix} 3 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix}}_{\underline{\underline{U}}} \underbrace{\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}}_{\bar{\underline{x}}} = \underbrace{\begin{pmatrix} 7 \\ 3 \\ 9 \end{pmatrix}}_{\bar{\underline{y}}}$$

Matrici di permutazione elementari

$$\underline{\underline{P_1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{P_2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2) Determinare soluzioni $\underline{x}$ e $\bar{\underline{x}}$

$\underline{\underline{U}}\,\underline{x} = \underline{y}$

$$\begin{pmatrix} 3 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \overbrace{\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}}^{\underline{x}}$$

Risoluzione <u>SOSTITUZIONI ALL'INDIETRO</u>

$3x_3 = 3 \quad \Rightarrow \quad x_3 = 1$

$3x_2 - x_3 = 2 \quad \Rightarrow \quad x_2 = \frac{1}{3}(2 + x_3) = 1$

$3x_1 - x_2 + 2x_3 = 4 \quad \Rightarrow \quad x_1 = \frac{1}{3}(4 + x_2 - 2x_3)$

$\qquad = \frac{1}{3}(4 + 1 - 2) = 1$

$\underline{x} = (1, 1, 1)^T$

$$\underline{\underline{U}}\,\underline{\bar{x}} = \underline{\bar{y}} \qquad \begin{pmatrix} 3 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 7 \\ 3 \\ 9 \end{pmatrix}}_{\bar{x}}$$

$$3\bar{x}_3 = 9 \implies \bar{x}_3 = 3$$

$$3\bar{x}_2 - \bar{x}_3 = 3 \implies \bar{x}_2 = \frac{1}{3}(3+\bar{x}_3) = \frac{1}{3}(3+3) = 2$$

$$3\bar{x}_1 - \bar{x}_2 + 2\bar{x}_3 = 7 \implies \bar{x}_1 = \frac{1}{3}(7+\bar{x}_2 - 2\bar{x}_3)$$

$$= \frac{1}{3}(7+2-2\cdot 3)$$

$$= 1$$

$$\bar{x} = (1, 2, 3)^T$$

3) Esplicitare $\underline{\underline{L}}$ e $\underline{\underline{P}}$

$$\underline{\underline{P}} = \underline{\underline{P_2}}\,\underline{\underline{P_1}} = \underline{\underline{P_2}}\,\underline{\underline{I}} = \underline{\underline{P_2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\underline{\underline{L}} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{\ell_{21}} & 1 & 0 \\ \boxed{\ell_{31}} & \ell_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{1} & 1 & 0 \\ \boxed{-1} & 0 & 1 \end{pmatrix}$$

abbiamo eseguito uno scambio di righe

4) Verificare $\underline{\underline{L}}\,\underline{\underline{U}} = \underline{\underline{P}}\,\underline{\underline{A}}$

$$\underline{\underline{P}}\,\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 \\ -3 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 3 & 2 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

$$\underline{\underline{L}}\,\underline{\underline{U}} = \underline{\underline{A}} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 2 \\ 3 & 2 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

5) Determinante di $\underline{\underline{A}}$

$$\det(\underline{\underline{P}}) \cdot \det(\underline{\underline{A}}) = \det(\underline{\underline{L}}) \cdot \det(\underline{\underline{U}})$$

$\det(\underline{\underline{P}}) = (-1)^s = -1$ $\qquad$ $s = 1$, uno scambio di righe

$\det(\underline{\underline{L}}) = 1$

$\det(\underline{\underline{U}}) = 3 \times 3 \times 3 = 27$

$$\Rightarrow \quad \det(\underline{\underline{A}}) = -27$$

Esercizio 3.15
Sia dato il seguente sistema $A\mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -15 \\ -24 \\ 5 \end{pmatrix}.$$

1. Utilizzando l'aritmetica esatta, si utilizzino le formule di Gauss Crout per fattorizzare la matrice nella forma $A = L \times U$.
2. Si verifichi che $L \times U = A$.
3. Sempre utilizzando l'aritmetica esatta, si utilizzi il metodo di Gauss (Doolittle) per fattorizzare la stessa matrice nella forma $A = L_1 \times U_1$, verificando che la fattorizzazione ottenuta restituisca la matrice iniziale.
4. Utilizzando una qualsiasi delle due precedenti fattorizzazioni, si risolva il sistema lineare.
5. Utilizzando una qualsiasi delle due precedenti fattorizzazioni si determini $\det(A)$.

Interessati solo a fattorizzare $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$?

Ricorso a FORMULE COMPATTE:

- Gauss
- Gauss - Doolittle
- Gauss Crout

$\Big\}$ in questo esercizio

Risoluzione SISTEMA LINEARE:

$$\begin{cases} \underline{\underline{L}}\,\underline{y} = \underline{b} & \text{risolvo per } \underline{y} \\ \underline{\underline{U}}\,\underline{x} = \underline{y} & \text{risolvo per } \underline{x} \text{ (SOLUZIONE cercata)} \end{cases}$$

1) $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$ con <u>Gauss Crout</u>

$\underline{\underline{L}}$: triangolare inferiore qualsiasi
$\underline{\underline{U}}$: triangolare superiore a DIAGONALE UNITARIA

NO PIVOTING!

calcolo alternativamente COLONNA di $\underline{\underline{L}}$

RIGA di $\underline{\underline{U}}$

$$l_{i1} = a_{i1} \qquad u_{1i} = \frac{a_{1i}}{l_{11}} \qquad i = 1, \dots, n$$

$$\left.\begin{aligned}
&l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad i = j, \dots, n \\
&u_{jj} = 1 \\
&u_{ji} = \frac{1}{l_{jj}} \left(a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki} \right) \qquad i = j+1, \dots, n
\end{aligned}\right\} \; j = 2, \dots, n$$

STRUTTURA MATRICI FATTORIZZAZIONE

$\underline{\underline{A}}$ 3×3

$$\underline{\underline{L}} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \qquad \underline{\underline{U}} = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix}$$

PRIMA colonna $\underline{\underline{L}}$ / riga $\underline{\underline{U}}$ $(j=1)$

$l_{i1} = a_{i1}$

$i = 1, \dots, n$

$$\underline{\underline{L}} = \begin{pmatrix} -4 & 0 & 0 \\ -3 & l_{22} & 0 \\ 1 & l_{32} & l_{33} \end{pmatrix}$$

$u_{1i} = \dfrac{a_{1i}}{l_{11}}$

$i = 1, \dots, n$

$$\underline{\underline{U}} = \begin{pmatrix} -4/-4 & -11/-4 & 19/-4 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 11/4 & -19/4 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

SECONDA colonna $\underline{\underline{L}}$ / riga $\underline{\underline{U}}$ $(j=2)$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{L}} = \begin{pmatrix} -4 & 0 & 0 \\ -3 & \ell_{22} & 0 \\ 1 & \ell_{32} & \ell_{33} \end{pmatrix} \quad \underline{\underline{U}} = \begin{pmatrix} 1 & 11/4 & -19/4 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\ell_{ij} = a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj} \qquad i = j, \dots, n$$

$i=2$, $j=2$
$$\ell_{22} = a_{22} - \sum_{k=1}^{1} \ell_{2k} u_{k2} = a_{22} - \ell_{21} u_{12}$$

$i=3$, $j=2$
$$\ell_{32} = a_{32} - \sum_{k=1}^{1} \ell_{3k} u_{k2} = a_{32} - \ell_{31} u_{12}$$

$$\ell_{22} = -11 - (-3 \cdot 11/4) = -11 + \frac{33}{4} = -\frac{11}{4}$$

$$\ell_{32} = 3 - (1 \cdot 11/4) = 3 - \frac{11}{4} = \frac{1}{4}$$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{L}} = \begin{pmatrix} -4 & 0 & 0 \\ -3 & \ell_{22} & 0 \\ 1 & \ell_{32} & \ell_{33} \end{pmatrix} \quad \underline{\underline{U}} = \begin{pmatrix} 1 & 11/4 & -19/4 \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$u_{ji} = \frac{1}{\ell_{jj}} \left(a_{ji} - \sum_{k=1}^{j-1} \ell_{jk} u_{ki} \right) \qquad i = j+1, \dots, n$$

$i=3$, $j=2$
$$u_{23} = \frac{1}{\ell_{22}} \left(a_{23} - \sum_{k=1}^{1} \ell_{2k} u_{k3} \right) = \frac{1}{\ell_{22}} \left(a_{23} - \ell_{21} u_{13} \right)$$

$$u_{23} = \frac{1}{-11/4} \left[7 - (-3)\left(-19/4\right) \right] = -\frac{4}{11} \left(-\frac{29}{4} \right) = 29/11$$

TERZA colonna $\underline{\underline{L}}$ / riga $\underline{\underline{U}}$ $(j=3)$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{L}} = \begin{pmatrix} -4 & 0 & 0 \\ -3 & -11/4 & 0 \\ 1 & 1/4 & \ell_{33} \end{pmatrix} \quad \underline{\underline{U}} = \begin{pmatrix} 1 & 11/4 & -19/4 \\ 0 & 1 & 29/11 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\ell_{ij} = a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj} \qquad i = j, \dots, n$$

$i=3$
$j=3$

$$\ell_{33} = a_{33} - \sum_{k=1}^{2} \ell_{3k} u_{k3} = a_{33} - (\ell_{31} u_{13} + \ell_{32} u_{23})$$

$$\ell_{33} = -4 - \left[1 \cdot \left(-19/4\right) + 1/4 \cdot 29/11\right]$$

$$= -4 - \left(-\frac{19}{4} + \frac{29}{44}\right) = -4 + \frac{45}{11} = \frac{1}{11}$$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{L}} = \begin{pmatrix} -4 & 0 & 0 \\ -3 & -11/4 & 0 \\ 1 & 1/4 & 1/11 \end{pmatrix} \quad \underline{\underline{U}} = \begin{pmatrix} 1 & 11/4 & -19/4 \\ 0 & 1 & 29/11 \\ 0 & 0 & 1 \end{pmatrix}$$

2) Verificare $\underline{\underline{L}}\,\underline{\underline{U}} = \underline{\underline{A}}$

$lu_{11} = -4 \cdot 1 = -4 \qquad lu_{12} = -4 \cdot 11/4 = -11 \qquad lu_{13} = -4 \cdot (-19/4) = 19$

... completare conti per verificare che $\underline{\underline{L}}\,\underline{\underline{U}} = \underline{\underline{A}}$

1) $\underline{\underline{A}} = \underline{\underline{L_1}}\ \underline{\underline{U_1}}$ con Gauss-Doolittle

$\underline{\underline{L_1}}$: triangolare inferiore a DIAGONALE UNITARIA

$\underline{\underline{U_1}}$: triangolare superiore qualsiasi

NO PIVOTING!

calcolo alternativamente RIGA di $\underline{\underline{U}}$

COLONNA di $\underline{\underline{L}}$

$$\left.\begin{aligned} u_{ji} &= a_{ji} - \sum_{k=1}^{j-1} \ell_{jk} u_{ki} \qquad i = j, \dots, n \\ \ell_{jj} &= 1 \\ \ell_{ij} &= \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj} \right) \qquad i = j+1, \dots, n \end{aligned}\right\} j = 1, \dots, n$$

STRUTTURA MATRICI FATTORIZZAZIONE

$\underline{\underline{A}}$

3×3

$$\underline{\underline{L_1}} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix} \qquad \underline{\underline{U_1}} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{21} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix}$$

PRIMA riga $\underline{\underline{U}}$ / colonna $\underline{\underline{L}}$ $(j=1)$

$$u_{ji} = a_{ji} - \cancel{\sum_{k=1}^{j-1} \ell_{jk}} u_{ki} \qquad i = j, \dots, n$$

$j=1 \quad u_{1i} = a_{1i}$ $\qquad \underline{\underline{U_1}} = \begin{pmatrix} -4 & -11 & 19 \\ 0 & u_{21} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

$l_{jj} = 1 \implies l_{11} = 1$

$$l_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \cancel{\sum_{k=1}^{j-1} l_{ik} u_{kj}} \right) \quad i = j+1, \dots, n$$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix}$$

$j=1 \quad l_{i1} = \frac{a_{i1}}{u_{11}} = \frac{a_{i1}}{-4}$ $\qquad \underline{\underline{L_1}} = \begin{pmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ -1/4 & l_{32} & 1 \end{pmatrix}$

SECONDA riga $\underline{\underline{U}}$ / colonna $\underline{\underline{L}}$ $(j=2)$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{U_1}} = \begin{pmatrix} -4 & -11 & 19 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \quad \underline{\underline{L_1}} = \begin{pmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ -1/4 & l_{32} & 1 \end{pmatrix}$$

$$u_{ji} = a_{ji} - \sum_{k=1}^{j-1} l_{jk} u_{ki} \qquad i = j, \dots, n$$

$j=2$, $i=2$: $\quad u_{22} = a_{22} - \sum_{k=1}^{1} l_{2k} u_{k2} = a_{22} - l_{21} u_{12}$

$j=2$, $i=3$: $\quad u_{23} = a_{23} - \sum_{k=1}^{1} l_{2k} u_{k3} = a_{23} - l_{21} u_{13}$

$$u_{22} = -11 - \left[\frac{3}{4} \cdot (-11) \right] = -\frac{11}{4}$$

$$u_{23} = 7 - \left(\frac{3}{4} \cdot 19 \right) = -\frac{29}{4}$$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{U_1}} = \begin{pmatrix} -4 & -11 & 19 \\ 0 & -11/4 & -29/4 \\ 0 & 0 & u_{33} \end{pmatrix} \quad \underline{\underline{L_1}} = \begin{pmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ -1/4 & \ell_{32} & 1 \end{pmatrix}$$

$$\ell_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} u_{kj} \right) \qquad i = j+1, \ldots, n$$

$i=3$
$j=2$

$$\ell_{32} = \frac{1}{u_{22}} \left(a_{32} - \sum_{k=1}^{1} \ell_{3k} u_{k2} \right) = \frac{1}{u_{22}} \left(a_{32} - \ell_{31} u_{12} \right)$$

$$\ell_{32} = 1/_{-11/4} \left[3 - (-1/4)(-11) \right]$$

$$= -\frac{4}{11} \left[3 - \frac{11}{4} \right] = -\frac{1}{11}$$

TERZA riga $\underline{\underline{U}}$ / colonna $\underline{\underline{L}}$ $(j=3)$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{U}}_1 = \begin{pmatrix} -4 & -11 & 19 \\ 0 & -11/4 & -29/4 \\ 0 & 0 & u_{33} \end{pmatrix} \quad \underline{\underline{L}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ -1/4 & -1/11 & 1 \end{pmatrix}$$

$$u_{ji} = a_{ji} - \sum_{k=1}^{j-1} \ell_{jk} u_{ki} \qquad i = j, \dots, n$$

$j=3$
$i=3$

$$u_{33} = a_{33} - \sum_{k=1}^{2} \ell_{3k} u_{k3} = a_{33} - (\ell_{31} u_{13} + \ell_{32} u_{23})$$

$$u_{33} = -4 - \left[\left(-1/4\right) \cdot 19 + \left(-1/11\right) \cdot \left(-29/4\right)\right] = 1/11$$

verifichiamo $\underline{\underline{L}}_1 \underline{\underline{U}}_1 = \underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{pmatrix} -4 & -11 & 19 \\ -3 & -11 & 7 \\ 1 & 3 & -4 \end{pmatrix} \quad \underline{\underline{L}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ -1/4 & -1/11 & 1 \end{pmatrix} \quad \underline{\underline{U}}_1 = \begin{pmatrix} -4 & -11 & 19 \\ 0 & -11/4 & -29/4 \\ 0 & 0 & 1/11 \end{pmatrix}$$

$$\left.\begin{aligned} a_{11} &= (1)\cdot(-4) = -4 \\ a_{12} &= (1)\cdot(-11) = -11 \\ a_{13} &= (1)\cdot(19) = 19 \end{aligned}\right\}$$

completare verifica $\underline{\underline{L}}_1 \underline{\underline{U}}_1 = \underline{\underline{A}}$

confrontare con Gauss NO PIVOTING

3) Risolvere sistema lineare $\underline{\underline{A}}\,\underline{x} = \underline{b}$
(con una delle fattorizzazioni)

Uso Gauss Crout (provare altra fattorizzazione)

$$\underline{\underline{L}} = \begin{pmatrix} -4 & 0 & 0 \\ -3 & -11/4 & 0 \\ 1 & 1/4 & 1/11 \end{pmatrix} \quad \underline{\underline{U}} = \begin{pmatrix} 1 & 11/4 & -19/4 \\ 0 & 1 & 29/11 \\ 0 & 0 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} -15 \\ -24 \\ 5 \end{pmatrix}$$

$$\begin{cases} \underline{\underline{L}}\,\underline{y} = \underline{b} \\ \underline{\underline{U}}\,\underline{x} = \underline{y} \end{cases}$$

Risolvo TRIANGOLARE INFERIORE (sostituzioni in avanti)

$\underline{\underline{L}}\,\underline{y} = \underline{b}$

$$\begin{pmatrix} -4 & 0 & 0 \\ -3 & -11/4 & 0 \\ 1 & 1/4 & 1/11 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -15 \\ -24 \\ 5 \end{pmatrix}$$

$$-4\,y_1 = -15 \quad \Rightarrow \quad y_1 = 15/4$$

$$-3y_1 - 11/4\, y_2 = -24 \quad \Rightarrow \quad y_2 = -\frac{4}{11}\left(3\,\frac{15}{4} - 24\right) = \frac{51}{11}$$

$$1 \cdot y_1 + 1/4\, y_2 + 1/11\, y_3 = 5$$

$$\Rightarrow \quad y_3 = 11\left(-\frac{15}{4} - \frac{1}{4}\,\frac{51}{11} + 5\right) = 1$$

Risolvo TRIANGOLARE SUPERIORE (sostituzioni all'indietro)

$\underline{U}\,\underline{x} = \underline{y}$

$$\begin{pmatrix} 1 & 11/4 & -19/4 \\ 0 & 1 & 29/11 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 15/4 \\ 51/11 \\ 1 \end{pmatrix}$$

$1 \cdot x_3 = 1 \quad \Rightarrow \quad x_3 = 1$

$1 \cdot x_2 + \frac{29}{11} x_3 = \frac{51}{11} \quad \Rightarrow \quad x_2 = \frac{51}{11} - \frac{29}{11} = 2$

$1 \cdot x_1 + 11/4 \cdot x_2 - 19/4 \cdot x_3 = \frac{15}{4}$

$\Rightarrow \quad x_1 = -\frac{11}{4} \cdot 2 + \frac{19}{4} \cdot 1 + \frac{15}{4} = 3$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

5) Determinante matrice $\underline{\underline{A}}$

Uso Gauss Crout (provare alla fattorizzazione)

$\det \underline{\underline{A}} = \det \underline{\underline{L}} \cdot \det \underline{\underline{U}}$

$\det \underline{\underline{L}} = (-4)\left(-\frac{11}{4}\right)\frac{1}{11} = 1$ (prodotto termini diagonali)

$\det \underline{\underline{U}} = 1$ (prodotto termini diagonali)

$\det \underline{\underline{A}} = 1$

Esercizio 3.24 ✓

(Da svolgere in aritmetica esatta). Si consideri la seguente matrice simmetrica definita positiva

$$A = \begin{pmatrix} \frac{1}{4} & 1 & -\frac{1}{2} \\ 1 & 5 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{29}{4} \end{pmatrix}.$$

1. Applicando il metodo di fattorizzazione di Cholesky, si calcoli la matrice triangolare inferiore L.
2. Si verifichi che risulta $A = L \times L^T$.
3. Si calcoli il determinante di A.

Fattorizzazione Cholesky $\underline{\underline{L}}\,\underline{\underline{L}}^T = \underline{\underline{A}}$

SIMMETRICA

con

$$\underline{\underline{L}} = \begin{pmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}$$

non si impone che sia DIAGONALE UNITARIA

$$\ell_{11} = \sqrt{a_{11}} \qquad \ell_{i1} = \frac{a_{i1}}{\ell_{11}} \qquad i = 2, \dots, n$$

$$\left.\begin{aligned} \ell_{jj} &= \left(a_{jj} - \sum_{k=1}^{j-1} \ell_{jk}^2 \right)^{1/2} \\ \ell_{ij} &= \frac{1}{\ell_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} \ell_{ik}\,\ell_{jk} \right) \quad i = j+1, \dots, n \end{aligned}\right] \; j = 2, \dots, n$$

PRIMA COLONNA $(j=1)$

$$A = \begin{pmatrix} 1/4 & 1 & -1/2 \\ 1 & 5 & -1/2 \\ -1/2 & -1/2 & 29/4 \end{pmatrix}$$

$$\ell_{11} = \sqrt{a_{11}} = \sqrt{1/4} = 1/2$$

$$\ell_{i1} = \frac{a_{i1}}{\ell_{11}}$$

$$\underline{\underline{L}} = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/1/2 & \ell_{22} & 0 \\ -1/2/1/2 & \ell_{32} & \ell_{33} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 2 & \ell_{22} & 0 \\ -1 & \ell_{32} & \ell_{33} \end{pmatrix}$$

SECONDA COLONNA $(j=2)$

$$A = \begin{pmatrix} 1/4 & 1 & -1/2 \\ 1 & 5 & -1/2 \\ -1/2 & -1/2 & 29/4 \end{pmatrix} \qquad \underline{\underline{L}} = \begin{pmatrix} 1/2 & 0 & 0 \\ 2 & \ell_{22} & 0 \\ -1 & \ell_{32} & \ell_{33} \end{pmatrix}$$

$$\ell_{jj} = \left(a_{jj} - \sum_{k=1}^{j-1} \ell_{jk}^2 \right)^{1/2}$$

$$\ell_{ij} = \frac{1}{\ell_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} \ell_{ik} \ell_{jk} \right) \qquad i = j+1, \dots, n$$

$i=2$
$j=2$

$$\ell_{22} = \left(a_{22} - \sum_{k=1}^{1} \ell_{2k}^2 \right)^{1/2} = \left(a_{22} - \ell_{21}^2 \right)^{1/2}$$

$$= (5 - 2^2)^{1/2} = 1$$

$i=3$
$j=2$

$$\ell_{32} = \frac{1}{\ell_{22}}\left(a_{32} - \sum_{k=1}^{1} \ell_{3k}\,\ell_{2k}\right) = \frac{1}{\ell_{22}}\left(a_{32} - \ell_{31}\ell_{21}\right)$$

$$= 1/1\left[-1/2 - (-1)\cdot 2\right] = 3/2$$

$$\underline{\underline{L}} = \begin{pmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3/2 & \ell_{33} \end{pmatrix}$$

TERZA COLONNA ($j=3$)

$$A = \begin{pmatrix} 1/4 & 1 & -1/2 \\ 1 & 5 & -1/2 \\ -1/2 & -1/2 & 29/4 \end{pmatrix} \qquad \underline{\underline{L}} = \begin{pmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3/2 & \ell_{33} \end{pmatrix}$$

$$\ell_{jj} = \left(a_{jj} - \sum_{k=1}^{j-1} \ell_{jk}^2\right)^{1/2}$$

$j=3$

$$\ell_{33} = \left(a_{33} - \sum_{k=1}^{2} \ell_{3k}^2\right)^{1/2} = \left[a_{33} - \left(\ell_{31}^2 + \ell_{32}^2\right)\right]^{1/2}$$

$$= \left[29/4 - \left((-1)^2 + (3/2)^2\right)\right]^{1/2}$$

$$= \left[29/4 - \left(1 + 9/4\right)\right]^{1/2} = 2$$

$$\underline{\underline{A}} = \begin{pmatrix} 1/4 & 1 & -1/2 \\ 1 & 5 & -1/2 \\ -1/2 & -1/2 & 29/4 \end{pmatrix} \qquad \underline{\underline{L}} = \begin{pmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3/2 & 2 \end{pmatrix}$$

2) Verificare $\underline{\underline{L}}\,\underline{\underline{L}}^T = \underline{\underline{A}}$

$$\underline{\underline{L}}\,\underline{\underline{L}}^T = \begin{pmatrix} 1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3/2 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & 2 & -1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/4 & 1 & -1/2 \\ 1 & 5 & -1/2 \\ -1/2 & -1/2 & 29/5 \end{pmatrix}$$

3) Determinante di $\underline{\underline{A}}$

$$\det \underline{\underline{A}} = \det \underline{\underline{L}} \cdot \det \underline{\underline{L}}^T = \left(\det \underline{\underline{L}}\right)^2$$

$$= \left(1/2 \cdot 1 \cdot 2\right)^2 = 1$$

Esercizio 2.10

solo pti 1 e 5

Si consideri la funzione

$$f(x) = x^3 - 4x^2 + 5x - 2.$$

1. Si determinino graficamente gli intervalli contenenti le soluzioni reali dell'equazione $f(x) = 0$ e si dica quante sono (si ponga molta cura nel disegno e, nel dubbio, si disegni anche la curva $y = f(x)$ e si svolga il punto 5. di questo esercizio).

5. Si determinino analiticamente la/le radici reali di questa equazione.

1) $f(x) = x^3 - 4x^2 + 5x - 2$

$f'(x) = 3x^2 - 8x + 5$

$f''(x) = 6x - 8$

1.1) $3x^2 - 8x + 5 \geq 0$ $\qquad x \leq 1 \ \cup \ x \geq 5/3$

$3x^2 - 8x + 5 = 0$

$\Delta = 64 - 4 \cdot 3 \cdot 5 = 4$ $\qquad x_{1,2} = \dfrac{8 \pm 2}{6} \begin{cases} 1 \\ 5/3 \end{cases}$

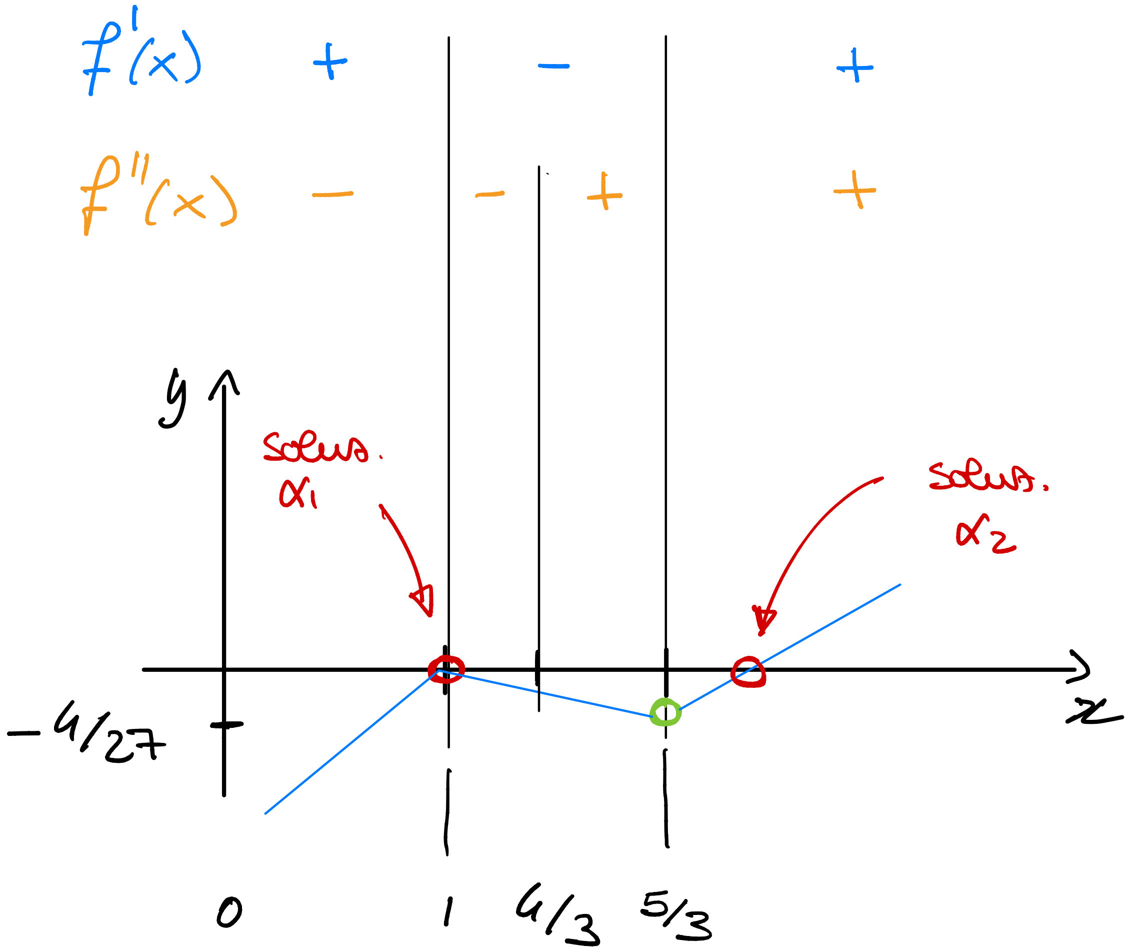
$f(x) = x^3 - 4x^2 + 5x - 2$

$f(1) = 0$

$f(5/3) = 125/27 - 4 \cdot 25/9 + 5 \cdot 5/3 - 2$

$= -4/27 \simeq -0,148$

1.2) $f''(x) \geq 0$ $\qquad x \geq 4/3$



x	$f(x) = x^3 - 4x^2 + 5x - 2$	
1	0	α_1
$4/3 \simeq 1,33$	$\simeq -0,074$	
$5/3 \simeq 1,67$	$\simeq -0,148$	
2	0	α_2
$5/2 = 2,5$	1,125	