Lezione 6 (2023-05-05)

Sistemi lineari: Metodi diretti

Esercizio 3.4 ✓

Si consideri la matrice

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

1. Si applichi (utilizzando l'aritmetica esatta) il metodo di Gauss senza pivoting, calcolando le due matrici L ed U della fattorizzazione.
2. Si verifichi che risulta $A = L \times U$.

5. Si calcolino poi con il metodo di Gauss con pivoting (utilizzando l'aritmetica esatta), le due matrici $\bar{L}$ ed $\bar{U}$ della decomposizione e la matrice di permutazione P.
6. Verificare che il prodotto $\bar{L} \times \bar{U}$ restituisce la matrice $P \times A$.
7. Si calcoli (in aritmetica esatta), utilizzando una delle fattorizzazioni trovate, il determinante di A.

Fattorizzazione $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$

matrice 3×3 in questo caso

Gauss

TRIANGOLARE INFERIORE

$$\underline{\underline{L}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}}$$

- diagonale unitaria
- costruita implicitamente

TRIANGOLARE SUPERIORE

$$\underline{\underline{U}} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ \boxed{0} & u_{22} & u_{23} \\ \boxed{0} & \boxed{0} & u_{33} \end{pmatrix}$$

passo 1

passo 2

1) Fattorizzazione $\underline{\underline{L}}\,\underline{\underline{U}} = \underline{\underline{A}}$ Gauss NO PIVOTING

Passo 1

$$\underline{\underline{A}}^{(0)} = \begin{pmatrix} \boxed{1} & 1 & 1 \\ 2 & -3 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

moltiplicatori	righe della matrice aggiornata
$\ell_{21} = a_{21}^{(0)} / a_{11}^{(0)}$	$1 \quad 1 \quad 1 \quad \times(-2) +$
	$2 \quad -3 \quad -1$
$= 2/1 = 2$	$0 \quad -5 \quad -3$
$\ell_{31} = a_{31}^{(0)} / a_{11}^{(0)}$	$1 \quad 1 \quad 1 \quad \times(-1) +$
	$1 \quad 2 \quad -1$
$= 1/1 = 1$	$0 \quad 1 \quad -2$

$$\underline{\underline{A}}^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{\text{Passo 1}} \underline{\underline{A}}^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 1 & -2 \end{pmatrix}$$

Passo 2

$$\underline{\underline{A}}^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \boxed{-5} & -3 \\ 0 & 1 & -2 \end{pmatrix}$$

moltiplicatori	righe della matrice aggiornata
$\ell_{32} = a_{32}^{(1)} / a_{22}^{(1)}$	$0 \quad -5 \quad -3 \quad \times \frac{1}{5} \; +$
	$0 \quad 1 \quad -2$
$= \frac{1}{-5} = -\frac{1}{5}$	$0 \quad 0 \quad -\frac{13}{5}$

$$\underline{\underline{A}}^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 1 & -2 \end{pmatrix} \xrightarrow{\text{Passo 2}} \underline{\underline{A}}^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & -\frac{13}{5} \end{pmatrix}$$

Matrici $\underline{\underline{L}}$ e $\underline{\underline{U}}$ FATTORIZZAZIONE

$$\underline{\underline{U}} = \underline{\underline{A}}^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & -\frac{13}{5} \end{pmatrix}$$

$$\underline{\underline{L}} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{5} & 1 \end{pmatrix}$$

2) Verificare $\underline{\underline{L}}\,\underline{\underline{U}} = \underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & 2 & -1 \end{pmatrix} \quad \underline{\underline{L}} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/5 & 1 \end{pmatrix} \quad \underline{\underline{U}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & -13/5 \end{pmatrix}$$

$$LU_{11} = 1 \cdot 1 = 1$$

$$LU_{12} = 1 \cdot 1 = 1$$

$$LU_{13} = 1 \cdot 1 = 1$$

$$LU_{21} = 2 \cdot 1 = 2$$

$$LU_{22} = 2 \cdot 1 + 1 \cdot (-5) = -3$$

$$LU_{23} = 2 \cdot 1 + 1 \cdot (-3) = -1$$

$$LU_{31} = 1 \cdot 1 = 1$$

$$LU_{32} = 1 \cdot 1 + (-1/5) \cdot (-5) = 2$$

$$LU_{33} = 1 \cdot 1 + (-1/5) \cdot (-3) + 1 \cdot (-13/5) = -1$$

✓ verificata

Tralasciamo pti 3-4

ESERCIZIO AGGIUNTIVO:

Determinare soluzione $\underline{x}$ del sistema $\underline{\underline{A}}\,\underline{x} = \underline{b}$
con $\underline{b} = (1/2 \quad 1 \quad 0)^T$

Sfruttiamo fattorizzazione e troviamo soluzione in due passaggi, ovvero

$$\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}} \quad \Rightarrow \quad \underline{\underline{A}}\,\underline{x} = \underline{\underline{L}}\,\underbrace{\underline{\underline{U}}\,\underline{x}}_{\underline{y}} = \underline{b}$$

$$\begin{cases} \underline{\underline{L}}\,\underline{y} = \underline{b} \\ \underline{\underline{U}}\,\underline{x} = \underline{y} \end{cases}$$

risolviamo in sequenza questi due sistemi

Determino $\underline{y}$

$\underline{\underline{L}}$ matrice triangolare inferiore
$\Rightarrow$ algoritmo risolutivo delle "SOSTITUZIONI in AVANTI"

$$\begin{cases} x_1 = b_1 / \ell_{11} \\ x_i = \dfrac{1}{\ell_{ii}} \left(b_i - \sum_{k=1}^{i-1} \ell_{ik} x_k \right) \qquad i = 2, \ldots, n \end{cases}$$

$$\underline{\underline{L}}\,\underline{y} = \underline{b} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$$

$1 \cdot y_1 = 1/2 \quad \Rightarrow \quad y_1 = 1/2$

$$y_2 = 1/\ell_{22} \, (b_2 - \ell_{21} \cdot y_1)$$

$2 \cdot y_1 + 1 \cdot y_2 = 1 \quad \Rightarrow \quad y_2 = \overbrace{1/1 \, (1 - 2 \cdot y_1)} = 0$

$1 \cdot y_1 + (-1/5) \cdot y_2 + 1 \cdot y_3 = 0 \quad \Rightarrow \quad y_3 = 1/1 \left[0 - 1 \cdot y_1 - (-1/5) y_2 \right] = -1/2$

$\underline{y} = (1/2 \quad 0 \quad -1/2)^T$

Determino $\underline{x}$

$\underline{U}$ matrice triangolare superiore

$\Rightarrow$ algoritmo risolutivo delle "SOSTITUZIONI all' INDIETRO"

$$\begin{cases} x_n = b_n / u_{nn} \\ x_i = \dfrac{1}{u_{ii}} \left(b_i - \sum_{k=i+1}^{n} u_{ik} \, x_k \right) \qquad i = n-1, \dots, 1 \end{cases}$$

$\underline{U} \; \underline{x} = \underline{y}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & -13/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$-13/5 \, x_3 = -1/2 \quad \Rightarrow \quad x_3 = 5/26$

$-5x_2 - 3x_3 = 0 \quad \Rightarrow \quad x_2 = -3/5 \, x_3 = -3/26$

$$x_1 + x_2 + x_3 = 1/2 \quad \Rightarrow \quad x_1 = 1/2 + 3/26 - 5/26$$

$$= 11/26$$

$$\underline{x} = \left(11/26,\ -3/26,\ 5/26\right)^T$$

5) Applicare Gauss con pivoting. Valutare $\underline{\underline{L}}, \underline{\underline{U}}, \underline{\underline{P}}$

$$\underline{\underline{A}}^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ \boxed{2} & -3 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

Passo 1

determinare r t.c. $|a_{r1}^{(0)}| = \max\limits_{1 \le i \le 3} |a_{i1}^{(0)}|$

$r = 2 \qquad a_{21}^{(0)} = 2 \qquad$ scambio righe 1 e 2

$$\underline{\underline{P_1}}\ \underline{\underline{A}}^{(0)} = \begin{pmatrix} 2 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

matrice di permutazione elementare

$$\underline{\underline{P_1}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

moltiplicatori (ci riferiamo a matrice $\underline{\underline{P_1}}\ \underline{\underline{A}}^{(0)}$)	righe della matrice aggiornata
$\ell_{21} = a_{21}/a_{11} = 1/2$	$\begin{array}{ccc} 2 & -3 & -1 \\ 1 & 1 & 1 \\ \hline 0 & 5/2 & 3/2 \end{array}$ $\times (-1/2) +$
$\ell_{31} = a_{31}/a_{11} = 1/2$	$\begin{array}{ccc} 2 & -3 & -1 \\ 1 & 2 & -1 \\ \hline 0 & 7/2 & -1/2 \end{array}$ $\times (-1/2) +$

$$\underline{\underline{A}}^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & 2 & -1 \end{pmatrix} \qquad \underline{\underline{P_1}}\,\underline{\underline{A}}^{(0)} = \begin{pmatrix} 2 & -3 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \qquad \underline{\underline{A}}^{(1)} = \begin{pmatrix} 2 & -3 & -1 \\ 0 & 5/2 & 3/2 \\ 0 & \boxed{7/2} & -1/2 \end{pmatrix}$$

Passo 2

determinare r t.c. $|a_{r2}^{(1)}| = \max\limits_{2 \le i \le 3} |a_{i2}^{(1)}|$

$r = 3 \qquad a_{32}^{(1)} = 7/2$ scambio righe 2 e 3

$$\underline{\underline{P_2}}\,\underline{\underline{A}}^{(1)} = \begin{pmatrix} 2 & -3 & -1 \\ 0 & 7/2 & -1/2 \\ 0 & 5/2 & 3/2 \end{pmatrix}$$

matrice di permutazione elementare

$$\underline{\underline{P_2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

SCAMBIARE MOLTIPLICATORI GIÀ DETERMINATI!

$\ell_{21} = 1/2 \qquad \ell_{31} = 1/2$

moltiplicatori (ci riferiamo a matrice $\underline{\underline{P_2}}\,\underline{\underline{A}}^{(1)}$)	righe della matrice aggiornata
$\ell_{32} = a_{32}/a_{22} = 5/7$	$\begin{array}{ccc} 0 & 7/2 & -1/2 \\ 0 & 5/2 & 3/2 \\ \hline 0 & 0 & 13/7 \end{array} \quad \times(-5/7)+$

$$\underline{\underline{A}}^{(1)} = \begin{pmatrix} 2 & -3 & -1 \\ 0 & 5/2 & 3/2 \\ 0 & 7/2 & -1/2 \end{pmatrix} \longrightarrow \underline{\underline{A}}^{(2)} = \begin{pmatrix} 2 & -3 & -1 \\ 0 & 7/2 & -1/2 \\ 0 & 0 & 13/7 \end{pmatrix}$$

Fattorizzazione $\underline{\underline{A}} = \underline{\underline{\bar{L}}}\,\underline{\underline{\bar{U}}}$ (Gauss + pivoting)

$$\underline{\underline{\bar{L}}} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{e_{21}} & 1 & 0 \\ \boxed{e_{31}} & e_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{1/2} & 1 & 0 \\ \boxed{1/2} & 5/7 & 1 \end{pmatrix}$$

$$\underline{\underline{\bar{U}}} = \begin{pmatrix} 2 & -3 & -1 \\ 0 & 7/2 & -1/2 \\ 0 & 0 & 13/7 \end{pmatrix}$$

$$\underline{\underline{P}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{P}} = \underline{\underline{P}}_2\,\underline{\underline{P}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6) Verificare $\underline{\underline{L}} \times \underline{\underline{U}} = \underline{\underline{P}} \times \underline{\underline{A}}$

$$\underline{\underline{P}} \times \underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\underline{\underline{L}} \times \underline{\underline{U}} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 5/7 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & -1 \\ 0 & 7/2 & -1/2 \\ 0 & 0 & 13/7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

7) Calcolare determinante di $\underline{\underline{A}}$

7.1) Gauss NO PIVOTING : $\det \underline{\underline{A}} = \det \underline{\underline{L}} \cdot \det \underline{\underline{U}}$

$$\underline{\underline{L}} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/5 & 1 \end{pmatrix} \qquad \underline{\underline{U}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & -13/5 \end{pmatrix}$$

$$\left.\begin{aligned} \det \underline{\underline{L}} &= 1 \cdot 1 \cdot 1 = 1 \\ \det \underline{\underline{U}} &= 1 \cdot (-5) \cdot (-13/5) = 13 \end{aligned}\right\} \Rightarrow \det \underline{\underline{A}} = 13$$

7.2) Gauss PIVOTING : $\det \underline{\underline{P}} \cdot \det \underline{\underline{A}} = \det \underline{\underline{\bar{L}}} \cdot \det \underline{\underline{\bar{U}}}$ equivalente a

$$\det \underline{\underline{A}} = \det \underline{\underline{P}} \cdot \det \underline{\underline{\bar{L}}} \cdot \det \underline{\underline{\bar{U}}}$$

$$\det \underline{\underline{P}} = (-1)^s = (-1)^2 = 1$$

s : numero scambi righe

$$\underline{\underline{\bar{L}}} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 5/7 & 1 \end{pmatrix} \qquad \underline{\underline{\bar{U}}} = \begin{pmatrix} 2 & -3 & -1 \\ 0 & 7/2 & -1/2 \\ 0 & 0 & 13/7 \end{pmatrix}$$

$$\det \underline{\underline{\bar{L}}} = 1 \cdot 1 \cdot 1 = 1$$

$$\det \underline{\underline{\bar{U}}} = 2 \cdot 7/2 \cdot 13/7 = 13$$

$$\det \underline{\underline{A}} = \det \underline{\underline{P}} \cdot \det \underline{\underline{\bar{L}}} \cdot \det \underline{\underline{\bar{U}}} = 1 \cdot 1 \cdot 13 = 13$$

Esercizio 3.18 ✓

(Da svolgere in aritmetica esatta). Si consideri la seguente matrice non singolare

$$A = \begin{pmatrix} -1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \end{pmatrix}.$$

1. Utilizzando il metodo di Gauss con pivoting si fattorizzi la matrice A nel prodotto $L \times U$, indicando ad ogni passo la matrice di permutazione elementare corrispondente.
2. Si dica quanto valgono L, U e P (la matrice di permutazione ottenuta come prodotto delle matrici di permutazione elementari determinate ad ogni passo).
3. Si verifichi che il prodotto $L \times U$ restituisca il prodotto $P \times A$.
4. Si calcoli il determinante di A.
5. Sapendo che $\mathbf{b} = (0, 4, 6, 10)^T$, si determini la soluzione del sistema $A\mathbf{x} = \mathbf{b}$ (ovvero $PA\mathbf{x} = P\mathbf{b}$), utilizzando le due matrici L ed U.

$$\underline{\underline{A}}^{(0)} = \begin{pmatrix} -1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ \boxed{2} & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

<u>Passo 1</u>

determinare r t.c. $|a_{r1}^{(0)}| = \max_{1 \le i \le n} |a_{i1}^{(0)}|$

$r = 3 \quad a_{31}^{(0)} = 2$ scambio righe 1 e 3

matrice di permutazione elementare

$$\underline{\underline{P_1}}\,\underline{\underline{A}}^{(0)} = \begin{pmatrix} \boxed{2} & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ -1 & 0 & 1 & -1/2 \\ 0 & 2 & 2 & 0 \end{pmatrix} \qquad \underline{\underline{P_1}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

moltiplicatori	righe della matrice aggiornata
$\ell_{21} = 0/2 = 0$	$\begin{array}{cccc} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ \hline 0 & 1 & 0 & 1/2 \end{array}$ $\times\ 0 \quad +$
$\ell_{31} = -1/2$	$\begin{array}{cccc} 2 & 0 & 0 & 1 \\ -1 & 0 & 1 & -1/2 \\ \hline 0 & 0 & 1 & 0 \end{array}$ $\times\ 1/2 \quad +$

$$\ell_{41} = 0/2 = 0$$

2	0	0	1	×	0	+
0	2	2	0			
0	2	2	0			

$$\underline{\underline{A}}^{(1)} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & \boxed{2} & 2 & 0 \end{pmatrix}$$

Passo 2

determinare r t.c. $|a_{r2}^{(1)}| = \max_{2 \le i \le 4} |a_{i2}^{(1)}|$

$r = 4$ $\quad a_{42}^{(1)} = 2$ $\quad$ scambio righe 2 e 4

matrice di permutazione elementare

$$\underline{\underline{P_2}}\,\underline{\underline{A}}^{(1)} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & \boxed{2} & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 \end{pmatrix} \qquad \underline{\underline{P_2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

SCAMBIARE MOLTIPLICATORI GIÀ DETERMINATI

$\ell_{21} = 0 \qquad \ell_{41} = 0$

moltiplicatori	righe della matrice aggiornata
$\ell_{32} = 0/2 = 0$	0 2 2 0 × 0 +
	0 0 1 0
	0 0 1 0

$\ell_{42} = 1/2$

$$\begin{array}{cccc} 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1/2 \\ \hline 0 & 0 & -1 & 1/2 \end{array} \quad \times(-1/2) +$$

$$\underline{\underline{A}}^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$$

Passo 3

$$a_{33}^{(2)} = \max_{2 \le i \le 4} |a_{i3}^{(2)}| \Rightarrow$$ non necessario scambiare righe

$\Rightarrow$ matrice permutazione elementare

$$\underline{\underline{P}}_3 = \underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{P}}_3 \underline{\underline{A}}^{(2)} = \underline{\underline{A}}^{(2)} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & -1 & 1/2 \end{pmatrix}$$

moltiplicatore	riga della matrice aggiornata
$\ell_{43} = -1/1 = -1$	$0 \quad 0 \quad 1 \quad 0 \quad \times 1 \quad +$
	$0 \quad 0 \quad -1 \quad 1/2$
	$0 \quad 0 \quad 0 \quad 1/2$

$$\underline{\underline{A}}^{(3)} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

2) Fattorizzazione $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$ (Gauss + pivoting)

$$\underline{\underline{L}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{\ell_{21}} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \boxed{\ell_{41}} & \ell_{42} & \ell_{43} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ \boxed{0} & 1/2 & -1 & 1 \end{pmatrix}$$

$$\underline{\underline{U}} = \underline{\underline{A}}^{(3)} = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$\underline{\underline{P}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{P}} = \underline{\underline{P_3}}\,\underline{\underline{P_2}}\,\underline{\underline{P_1}} = \underline{\underline{I}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3) Verificare $\underline{\underline{L}} \times \underline{\underline{U}} = \underline{\underline{P}} \times \underline{\underline{A}}$

$$\underline{\underline{P}} \times \underline{\underline{A}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 \end{pmatrix}$$

$$\underline{\underline{L}} \times \underline{\underline{U}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ \boxed{0} & 1/2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 \end{pmatrix}$$

√ verificata

4) Determinante di $\underline{\underline{A}}$

$$\det(\underline{\underline{P}}\,\underline{\underline{A}}) = \det(\underline{\underline{L}}\,\underline{\underline{U}})$$

$$\Rightarrow \quad \det(\underline{\underline{P}}) \cdot \det(\underline{\underline{A}}) = \det(\underline{\underline{L}}) \cdot \det(\underline{\underline{U}})$$

$\det(\underline{\underline{P}}) = (-1)^s$ $\qquad$ s: numero di scambi

$\det(\underline{\underline{L}}) = 1$ $\qquad$ triangolare inferiore con termini diagonali unitari

$\det(\underline{\underline{U}}) = \prod_{i=1}^{4} u_{ii}$ $\qquad$ triangolare superiore

$$\det(\underline{\underline{A}}) = \underbrace{(-1)^s}_{=1} \times \underbrace{\det(\underline{\underline{U}})}_{=2} = 2$$

perché $s = 2$ $\qquad$ $2 = 2 \times 2 \times 1 \times 1/2$

5) Soluzione $\underline{x}$ per termine noto $\underline{b} = (0, 4, 6, 10)^T$

$$\underline{\underline{P}}\,\underline{\underline{A}}\,\underline{x} = \underline{\underline{P}}\,\underline{b} \quad \Leftrightarrow \quad \underline{\underline{L}}\,\underbrace{\underline{\underline{U}}\,\underline{x}}_{\underline{y}} = \underbrace{\underline{\underline{P}}\,\underline{b}}_{\text{termine noto}}$$

$$\underline{\underline{P}}\,\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$$

Risolviamo due sistemi triangolari:

$$\begin{cases} \underline{\underline{L}}\,\underline{y} = \underline{\underline{P}}\,\underline{b} \\ \underline{\underline{U}}\,\underline{x} = \underline{y} \end{cases}$$

Valuto $\underline{y}$

$\underline{\underline{L}}$ matrice triangolare inferiore

$\Rightarrow$ algoritmo risolutivo delle "SOSTITUZIONI in AVANTI"

$$\begin{cases} x_1 = b_1 / l_{11} \\ x_i = \dfrac{1}{l_{ii}} \left(b_i - \sum_{k=1}^{i-1} l_{ik} x_k \right) \qquad i = 2, \dots, n \end{cases}$$

$$\underline{\underline{P}}\,\underline{b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ 0 \\ 4 \end{pmatrix}$$

$$\underline{\underline{L}}\,\underline{y} = \underline{\underline{P}}\,\underline{b} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 0 & 1/2 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ 0 \\ 4 \end{pmatrix}$$

$1 \cdot y_1 = 6 \quad \Rightarrow \quad y_1 = 6/1 = 6$

$1 \cdot y_2 = 10 \quad \Rightarrow \quad y_2 = 10/1 = 10$

$-\frac{1}{2} y_1 + y_3 = 0 \quad \Rightarrow \quad y_3 = y_1/2 = 6/2 = 3$

$\frac{1}{2} y_2 - y_3 + y_4 = 4 \quad \Rightarrow \quad y_4 = 4 - \frac{1}{2} y_2 + y_3$

$= 4 - 10/2 + 3$

$= 2$

$$\boxed{\underline{y} = (6, 10, 3, 2)^T}$$

Vettore $\underline{x}$

$\underline{U}$ matrice diagonale superiore

$\Rightarrow$ algoritmo risolutivo delle "SOSTITUZIONI all' INDIETRO"

$$\begin{cases} x_n = b_n / u_{nn} \\ x_i = \dfrac{1}{u_{ii}} \left(b_i - \sum_{k=i+1}^{n} u_{ik} x_k \right) \qquad i = n-1, \dots, 1 \end{cases}$$

$$\underline{\underline{U}} \, \underline{x} = \underline{y} \qquad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ 3 \\ 2 \end{pmatrix}$$

$$x_4 = 2 / {1/2} = 4$$

$$x_3 = 3/1 = 3$$

$$2x_2 + 2x_3 = 10 \quad \Rightarrow \quad x_2 = \frac{1}{2}(10 - 2x_3)$$

$$= \frac{1}{2}(10-6) = 2$$

$$2 \cdot x_1 + 1 \cdot x_4 = 6 \quad \Rightarrow \quad x_1 = \frac{1}{2}(6 - x_4) = 1$$

$$\boxed{\underline{x} = (1, 2, 3, 4)^T}$$

ESERCIZIO AGGIUNTIVO: Risolvo con Gauss NO PIVOTING

$$(\underline{A}|\underline{b})^{(0)} = \left(\begin{array}{cccc|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 4 \\ 2 & 0 & 0 & 1 & 6 \\ 0 & 2 & 2 & 0 & 10 \end{array}\right)$$

Lavoro su matrice aumentata

moltiplicatori	righe della matrice aggiornata
$\ell_{21} = a_{21}^{(0)} / a_{11}^{(0)} = 0/-1 = 0$	$\begin{array}{cccc\|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 4 \\ \hline 0 & 1 & 0 & 1/2 & 4 \end{array} \quad \times 0 \; +$
$\ell_{31} = a_{31}^{(0)} / a_{11}^{(0)} = 2/-1 = -2$	$\begin{array}{cccc\|c} -1 & 0 & 1 & -1/2 & 0 \\ 2 & 0 & 0 & 1 & 6 \\ \hline 0 & 0 & 2 & 0 & 6 \end{array} \quad \times 2 \; +$
$\ell_{41} = a_{41}^{(0)} / a_{11}^{(0)} = 0/-1 = 0$	$\begin{array}{cccc\|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & 2 & 2 & 0 & 10 \\ \hline 0 & 2 & 2 & 0 & 10 \end{array} \quad \times 0 \; +$

$$(\underline{A}|\underline{b})^{(1)} = \left(\begin{array}{cccc|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 4 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 2 & 2 & 0 & 10 \end{array}\right)$$

$$(\underline{\underline{A}}\,|\,\underline{b})^{(1)} = \left(\begin{array}{cccc|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & \boxed{1} & 0 & 1/2 & 4 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 2 & 2 & 0 & 10 \end{array}\right)$$

moltiplicatori	righe della matrice aggiornata
$l_{32} = a^{(1)}_{32} / a^{(1)}_{22} = 0/1 = 0$	$\begin{array}{cccc\|c} 0 & 1 & 0 & 1/2 & 4 \\ 0 & 0 & 2 & 0 & 6 \\ \hline 0 & 0 & 2 & 0 & 6 \end{array}$ $\times 0 \; +$
$l_{42} = a^{(1)}_{42} / a^{(1)}_{22} = 2/1 = 2$	$\begin{array}{cccc\|c} 0 & 1 & 0 & 1/2 & 4 \\ 0 & 2 & 2 & 0 & 10 \\ \hline 0 & 0 & 2 & -1 & 2 \end{array}$ $\times (-2) \; +$

$$(\underline{\underline{A}}\,|\,\underline{b})^{(2)} = \left(\begin{array}{cccc|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 4 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & -1 & 2 \end{array}\right)$$

$$(\underline{\underline{A}} \mid \underline{b})^{(2)} = \left(\begin{array}{cccc|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 4 \\ 0 & 0 & \boxed{2} & 0 & 6 \\ 0 & 0 & 2 & -1 & 2 \end{array}\right)$$

moltiplicatori	righe della matrice aggiornata
$\ell_{43} = a_{43}^{(2)} / a_{33}^{(2)} = 2/2 = 1$	$\begin{array}{cccc\|c} 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 0 & 0 & 0 & -1 & -4 \end{array}$ $\times(-1)$ +

$$(\underline{\underline{A}} \mid \underline{b})^{(3)} = \left(\begin{array}{cccc|c} -1 & 0 & 1 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2 & 4 \\ 0 & 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right) = (\bar{\underline{\underline{U}}} \mid \bar{\underline{y}})$$

$$\bar{\underline{\underline{U}}} = \begin{pmatrix} -1 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \bar{\underline{y}} = \begin{pmatrix} 0 \\ 4 \\ 6 \\ -4 \end{pmatrix}$$

$$\underline{\underline{\bar{L}}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

1) verificare $\underline{\underline{\bar{L}}} \times \underline{\underline{\bar{U}}} = \underline{\underline{A}}$

2) verificare: $\underline{\underline{\bar{L}}}\ \underline{\bar{y}} = \underline{b}$ con $\underline{\bar{y}} = (0, 4, 6, -4)^T$

3) Determinare: $\underline{\bar{x}}$ come $\underline{\underline{\bar{U}}}\ \underline{\bar{x}} = \underline{\bar{y}}$

4) verificare: $\underline{\bar{x}} = \underline{x}$ con $\underline{x} = (1, 2, 3, 4)^T$

5) Osservare: $\underline{\underline{\bar{L}}} \neq \underline{\underline{L}}$, $\underline{\underline{\bar{U}}} \neq \underline{\underline{U}}$