SOLUZIONI TUTORATO 7 - 24/04/2024

(1) VEDI ES. 6 TUTORATO 6 (17/04/2024) $\boxed{(\sim 40)}$

(2)

$$p_1(x) = x + x^2 = \frac{1}{3} q_3 + q_2 - 2 q_1$$

INFATTI

$$\frac{q_3}{3} + q_2 - 2q_1 = \frac{\cancel{3}x^2}{\cancel{3}} + x + 2 - 2\cdot 1 = x^2 + x$$

PIU' IN GENERALE UNO LO PUO' VEDERE USANDO:

$$p_1(x) = x + x^2 = \underbrace{\alpha q_1 + \beta q_2 + \gamma q_3}_{\parallel}$$

$$= \underbrace{\alpha \cdot 1 + \beta(x+2) + \gamma(3x^2)}_{\parallel}$$

$$\Updownarrow \qquad \alpha + \beta x + 2\beta + 3\gamma x^2$$

QUINDI

$$\begin{cases} \alpha + 2\beta = 0 \rightarrow \alpha = -2 \\ \beta = 1 \\ 3\gamma = 1 \rightarrow \gamma = 1/3 \end{cases}$$

COME SCRITTO SOPRA!

$$p_2(x) = 1 + 3x + 2x^2 = \underbrace{\alpha q_1(x) + \beta q_2(x) + \gamma q_3(x)}$$

$$\| \quad \alpha \cdot 1 + \beta(x+2) + \gamma(3x^2)$$

$$\| \quad (\alpha + 2\beta) + \beta x + 3\gamma x^2$$

$$\Leftrightarrow \begin{cases} \alpha + 2\beta = 1 \longrightarrow \alpha = -5 \\ \beta = 3 \\ 3\gamma = 2 \rightarrow \gamma = 2/3 \end{cases}$$

i.e. $p_2(x) = 1 + 3x + 2x = -5 q_1 + 3 q_2 + \frac{2}{3}(q_3)$

$$p_3(x) = 2 - x^2 = 2 q_1 - \frac{1}{3} q_3 + 0 q_2$$

Da cui:

$$M_{B \to B'} = \begin{pmatrix} -2 & -5 & 2 \\ 1 & 3 & 0 \\ 1/3 & 2/3 & -\frac{1}{3} \end{pmatrix}$$

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(a) A_1, A_2 B BASE POICHÉ

$$\begin{pmatrix} -1 & -2 & 1 & 3 \\ 5 & 0 & 1 & -7 \\ -2 & 0 & 3 & 0 \\ +1 & 0 & -4 & 0 \end{pmatrix} \xrightarrow[\text{GAUSS}]{} \dots$$

$$\dots \to \begin{pmatrix} -1 & -2 & 1 & 3 \\ 0 & -10 & 6 & 8 \\ 0 & 0 & 17/5 & -14/5 \\ 0 & 0 & 0 & -35/17 \end{pmatrix}$$

→ 4 VETTORI LIN. INDIP.

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B BASE DI $M_{2,2}$

(b) EQUIVALENTEMENTE:

$$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4 = 0$$

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$$\lambda_1 \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 5 & 0 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} -2\lambda_1 - \lambda_2 + \lambda_3 + 5\lambda_4 \\ -2\lambda_2 = 0 \\ 3\lambda_1 + \lambda_2 - 4\lambda_3 + \lambda_4 = 0 \\ 3\lambda_2 - 7\lambda_4 = 0 \end{cases} \xrightarrow[\text{RISOLVO}]{} \dots \to \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$$

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4 VETTORI LIN. INDIP.

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B BASE DI $M_{2,2}(\mathbb{R})$

(b)

$$\underset{A_1}{\begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}} = -2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{A_2}{\begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}} = -1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 4\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 \\ 1 & -7 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - 7\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

QUINDI:

$$M_{B\to C} = \begin{pmatrix} -2 & -1 & 1 & 5 \\ 0 & -2 & 0 & 0 \\ 3 & 1 & -4 & 1 \\ 0 & 3 & 0 & -7 \end{pmatrix}$$

DOVE C BASE CANONICA DI $M_{2,2}(\mathbb{R}) \cong \mathbb{R}^4$.

(4) $f: \mathbb{R}^2 \to \mathbb{R}^2$
$f(x,y) = (2x+3y,\ x-y)$

(a)
$C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$f(1,0) = (2,1) = 2(1,0) + 1(0,1)$

$f(0,1) = (3,-1) = 3(1,0) - 1(0,1)$

$\to M_{C,C}(f) = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

(b) $B' = \{(1,1), (1,2)\}$

IN GENERALE, SE $f: V \to W$ LINEARE CON B_V, B_V' BASI DI V, B_W, B_W' BASI DI W.

ALLORA

$$\underbrace{M^{B_W'}_{B_V'}(f)}_{} = \underbrace{\left(M_{B_W \to B_W'}\right)}_{} \cdot \underbrace{M^{B_W}_{B_V}(f)}_{} \cdot \underbrace{\left(M_{B_V \to B_V'}\right)^{-1}}_{}$$

MATRICE DI f RISPETTO A B_V' DI V (DOMINIO) E B_W' DI W (CODOMINIO)

CAMBO DI BASE IN W (CODOMINIO)

MATRICE DI f RISPETTO A B_V DI V (DOMINIO) E B_W DI W (CODOMINIO)

CAMBIO DI BASE INVERSO IN V (DOMINIO)

IN QUESTO CASO:

$$M_B^{B'}(f) = M_{C\to B'}\; M_C^C(f)\left(M_{C\to B'}\right)^{-1}$$

USO $B_V = B_W = C$

$B'_V = B'_W = B'$

DOVE $M_C^C(f) \overset{NOT}{=} M^f_{C,C} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

$M_{C\to B'} = ?$

$$id\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$id\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$M_{C\to B'} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

QUINDI

$$M_B^{B'}(f) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}^{-1}}_{\displaystyle = \frac{1}{1}\cdot\begin{pmatrix} 1 & +1 \\ +1 & 2 \end{pmatrix}} =$$

USO

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 8 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 10 & 17 \\ -5 & -9 \end{pmatrix}$$

(c) CALCOLIAMO $M^{B'}_{B'}(f)$:

$f(1,1) = (5, 0) = \alpha(1,1) + \beta(1,2)$

i.e.

~~$f(1,2) = (8,-1)$~~

$\begin{cases} \alpha+\beta = 5 \\ \alpha + 2\beta = 0 \end{cases} \rightarrow \begin{cases} \beta = -5 \\ \alpha = 10 \end{cases} \rightarrow f(1,1) = 10\cdot(1,1) + -5(1,2)$

$f(1,2) = (8,-1) = \tilde{\alpha}(1,1) + \tilde{\beta}(1,2)$

i.e.

$\begin{cases} \tilde{\alpha} + \tilde{\beta} = 8 \\ \tilde{\alpha} + 2\tilde{\beta} = -1 \end{cases} \rightarrow \begin{cases} \tilde{\alpha} = 17 \\ \tilde{\beta} = -9 \end{cases} \rightarrow f(1,2) = 17(1,1) + -9(1,2)$

$$M^{B'}_{B'}(f) = \begin{pmatrix} 10 & 17 \\ -5 & -9 \end{pmatrix}$$

CHE È UGUALE A QUELLA TROVATA IN (b)

(5) $f: \mathbb{R}^2 \to \mathbb{R}^3$ t.c. $f(1,0) = (1,2,1)$
$f(0,1) = (1,0,-1)$

(a) SIANO $C_{\mathbb{R}^2}$, $C_{\mathbb{R}^3}$ LE BASI CANONICHE.

$M^{C_{\mathbb{R}^3}}_{C_{\mathbb{R}^2}}(f) = ?$

$f(1,0) = (1,2,1) = 1(1,0,0) + 2(0,1,0) + 1(0,0,1)$

$f(0,1) = (1,0,-1) = 1(1,0,0) + 0(0,1,0) - 1(0,0,1)$

$\rightarrow M^{C_{\mathbb{R}^3}}_{C_{\mathbb{R}^2}}(f) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$

(b) USO $$M^{B'_W}_{B'_V}(f) = M_{B_W \to B'_W} \cdot M^{B_W}_{B_V}(f) \cdot \left(M_{B_V \to B'_V}\right)^{-1}$$

DOVE
$$B_V = C_{\mathbb{R}^2}$$
$$B_W = C_{\mathbb{R}^3}$$
$$B'_V = B = \{(1,0),(0,1)\} = C_{\mathbb{R}^2}$$
$$B'_W = B' = \{(0,1,0),(0,0,1),(1,1,1)\}$$

DA CUI,

$$M^{B'}_{B}(f) = \left(M_{C_{\mathbb{R}^3} \to B'}\right) \cdot M^{C_{\mathbb{R}^3}}_{C_{\mathbb{R}^2}}(f) \cdot \underbrace{\left(M_{C_{\mathbb{R}^2} \to C_{\mathbb{R}^2}}\right)^{-1}}_{= (\mathbb{I})^{-1} = \mathbb{I}}$$

POICHÉ NON CAMBIO BASE NEL DOMINIO $\mathbb{R}^2$ (i.e. $B_{V'} = B_V$)

CALCOLIAMOLA:

$(1,0,0) = (1,1,1) - (0,1,0) - (0,0,1)$

$(0,1,0) = (0,1,0)$

$(0,0,1) = (0,0,1)$

QUINDI, $$M_{C_{\mathbb{R}^3} \to B'} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

DA CUI

$$M^{B'}_{B}(f) = \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}} \cdot \underbrace{\underset{3\times 2}{\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}} \cdot \mathbb{I}}_{= \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -2 \end{pmatrix}$$

CHECK (FACOLTATIVO):

CALCOLO $M_B^{B'}(f)$ DA f:

$$f(1,0) = (1,2,1) = \alpha(0,1,0) + \beta(0,0,1) + \gamma(1,1,1)$$

i.e. $\begin{cases} 1 = \gamma \\ 2 = \alpha + \gamma \\ 1 = \beta + \gamma \end{cases} \rightarrow \begin{cases} \gamma = 1 \\ \alpha = 1 \\ \beta = 0 \end{cases}$

$$f(0,1) = (1,0,-1) = \alpha(0,1,0) + \beta(0,0,1) + \gamma(1,1,1) \rightarrow \begin{matrix} \alpha = -1 \\ \beta = -2 \\ \gamma = 1 \end{matrix}$$

DA CUI $M_B^{B'}(f) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -2 \end{pmatrix}$

N.B.: QUA HO USATO $B' = \{(1,1,1), (0,1,0), (0,0,1)\}$

$\uparrow$ 1° VETTORE, $\uparrow$ 2° VETTORE, $\uparrow$ 3° VETTORE

PER COSTRUIRE $M_B^{B'}(f)$ (SIA ORA CHE PRIMA)

MA NULLA MI VIETA DI USARE UN ALTRO ORDINAMENTO, AD ESEMPIO

$B' = \{(0,1,0), (0,0,1), (1,1,1)\}$

~~[illegible]~~ IN QUESTO CASO TROVO

$$M_B^{B'}(f) = \begin{pmatrix} 1 & -1 \\ 0 & -2 \\ 1 & 1 \end{pmatrix}$$

~~INFATTI ALTRO~~

CHE VA BENE!