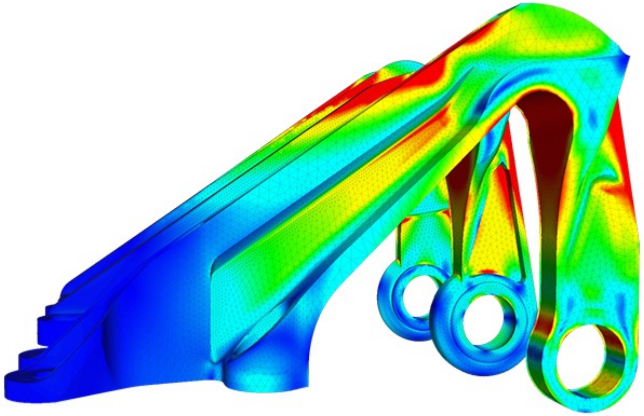
The finite element method

Computational Material Science
Lecture 9



Last time

- Discrete dislocation plasticity studies the micro-scale
- It can be used to study boundary value problems in microcomponents, such as the plasticity size effect, but also micro-scale phenomena as the competition between crack propagation and plasticity (e.g. brittle and ductile failure)
- The method is based on a continuum description of the body and a discrete description of dislocations, sources, slip planes, obstacles and grain boundaries
- The solution is given as the sum of the elastic solution of dislocations in an infinite medium and the complementary boundary value problem
- Solution to a LEBVP can be obtained by FEM, BEM, etc.
- FEM is the topic of today

Today

- What is the finite element method?
- What should be initially defined when using a commercial package?
- How to tackle a boundary value problem
- Discretization of a continuum problem: the mesh, the element, the isoparametric element
- Nodal displacements and interpolation

Coding: Initialization of a FEM code for the bending of a cantilever
Material properties and discretization

Learning goals:
The student can list the main quantities that need to be defined when using a FEM commercial package
Can describe how a continuum can be discretized for use in a FEM code
Can discretize a rectangular cantilever with rectangular elements

The finite element method

The finite element method (FEM) is …..

The finite element method

The finite element method (FEM) is …A COMPUTER TECHNIQUE TO SOLVE PARTIAL DIFFERENTIAL EQUATIONS.

One application is to predict the deformation and stress fields within solid bodies subject to external forces.

It is capable of solving large classes of PDE on almost any arbitrarily shaped region.

FEM can also be used to solve problems involving: **fluid flow, heat transfer, electromagnetic fields, diffusion, …**

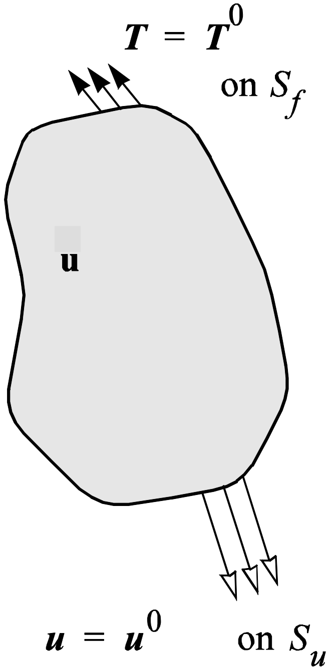
Elastic Boundary Value Problem

$$\begin{cases} \nabla \cdot \sigma = 0 \\ \varepsilon = \nabla u & \text{in } V \\ \sigma = L : \varepsilon \end{cases}$$

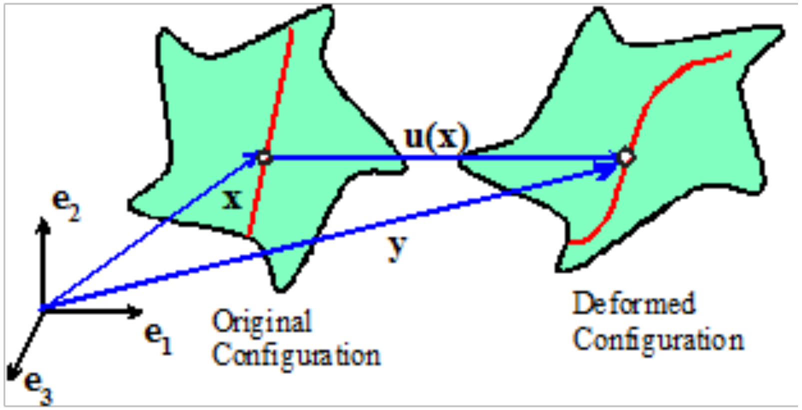
with the boundary conditions :

$$\sigma \cdot n = T = T_0 \qquad \text{on } S_f$$

$$u = u = u_0 \qquad \text{on } S_u$$



Objective



The principle objective of the displacement based finite element method is to compute the **displacement field** within a solid subjected to external forces.

The goal is therefore to determine the displacement vector **u**(**x**), which specifies the motion of the point at position **x** in the undeformed solid. Once **u(x)** is known, the strain and stress fields in the solid can be calculated.

How to set up a calculation

To set up a finite element calculation, you will need to specify:

1. The geometry of the solid as well as generate a finite element mesh to discretize the solid.

2. The material properties (by specifying a constitutive law for the solid).

3. The loading applied to the solid (by specifying the boundary conditions for the problem).

4. If your analysis involves contact between two or more solids, specify the surfaces that might come into contact, and the properties (e.g. friction coefficient) of the contact.

5. For problems involving additional fields, you may need to specify initial values for these field variables (e.g. initial temperature distribution in a thermal analysis).

Additionally:

You will also need to specify some additional aspects of the problem you are solving and the solution procedure to be used:

1. Whether the computation should take into account finite changes in the geometry of the solid.

2. You will need to specify the time period of the analysis and the time increment with which you want to perform the simulation

3. Whether the problem is linear, or nonlinear. Linear problems are very easy to solve. Nonlinear problems may need special procedures.

4. You will need to specify what the finite element method must compute.

WARNING!!!!

It is deceivingly easy to run a finite element simulation with a commercial package!

If you get a result $\not\Rightarrow$ the result is correct

Do all possible sanity checks:

1. Check against analytical solutions
2. Check for symmetry
3. Refining the mesh must give a more accurate solution until convergence is reached (approximation!) unless you are dealing with a singular field.

Boundary conditions

Boundary conditions are used to specify the loading applied to a solid. There are two main types of boundary conditions:

1. Displacement boundary conditions (Dirichlet)
2. Force boundary conditions (Neumann)

A finite element program can correctly solve a problem if a unique static equilibrium solution to the problem exists.

Some FEM package will not give you an error if you try to solve a problem which does have more than one solutions, and will simply give you one.

Elastic Boundary Value Problem

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with the boundary conditions :

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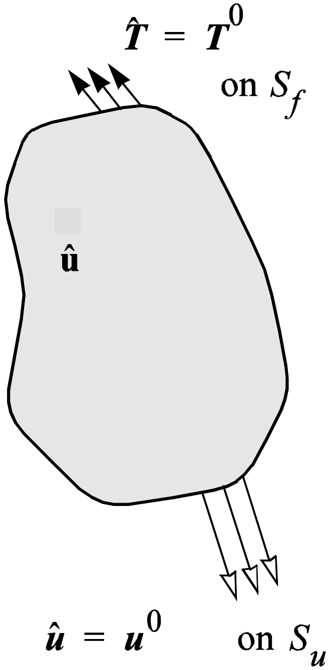
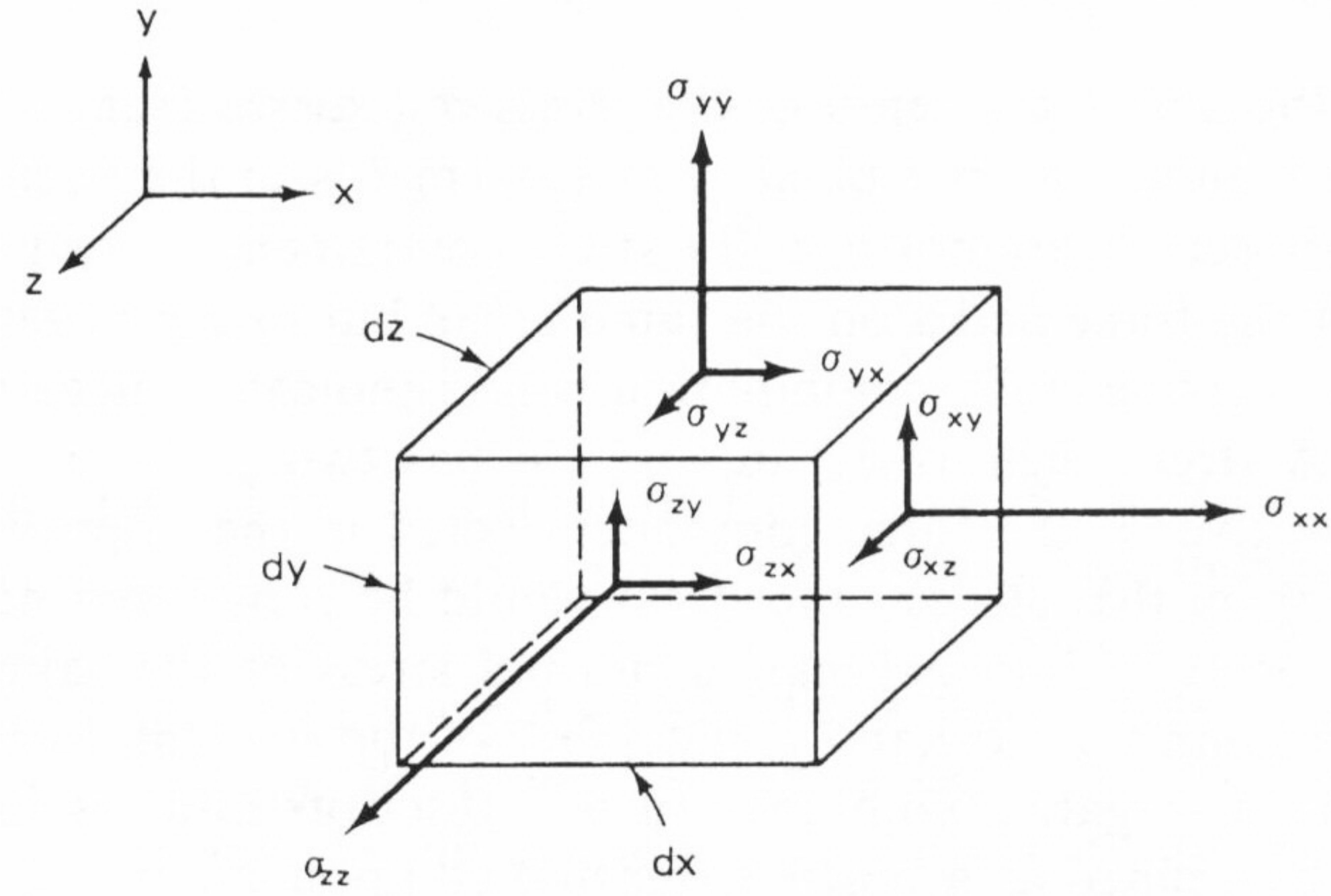
$$u = u = u_0 \qquad \text{on } S_u$$



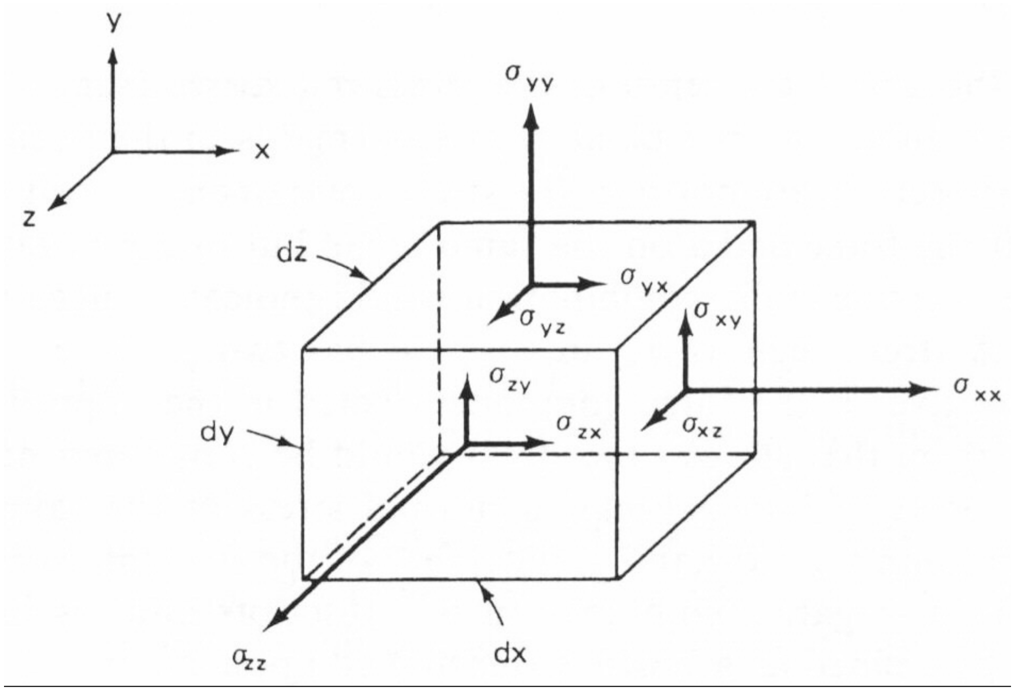
Stress tensor



each force component is divided by the area of the face upon which it acts ==> stress tensor (9 components)

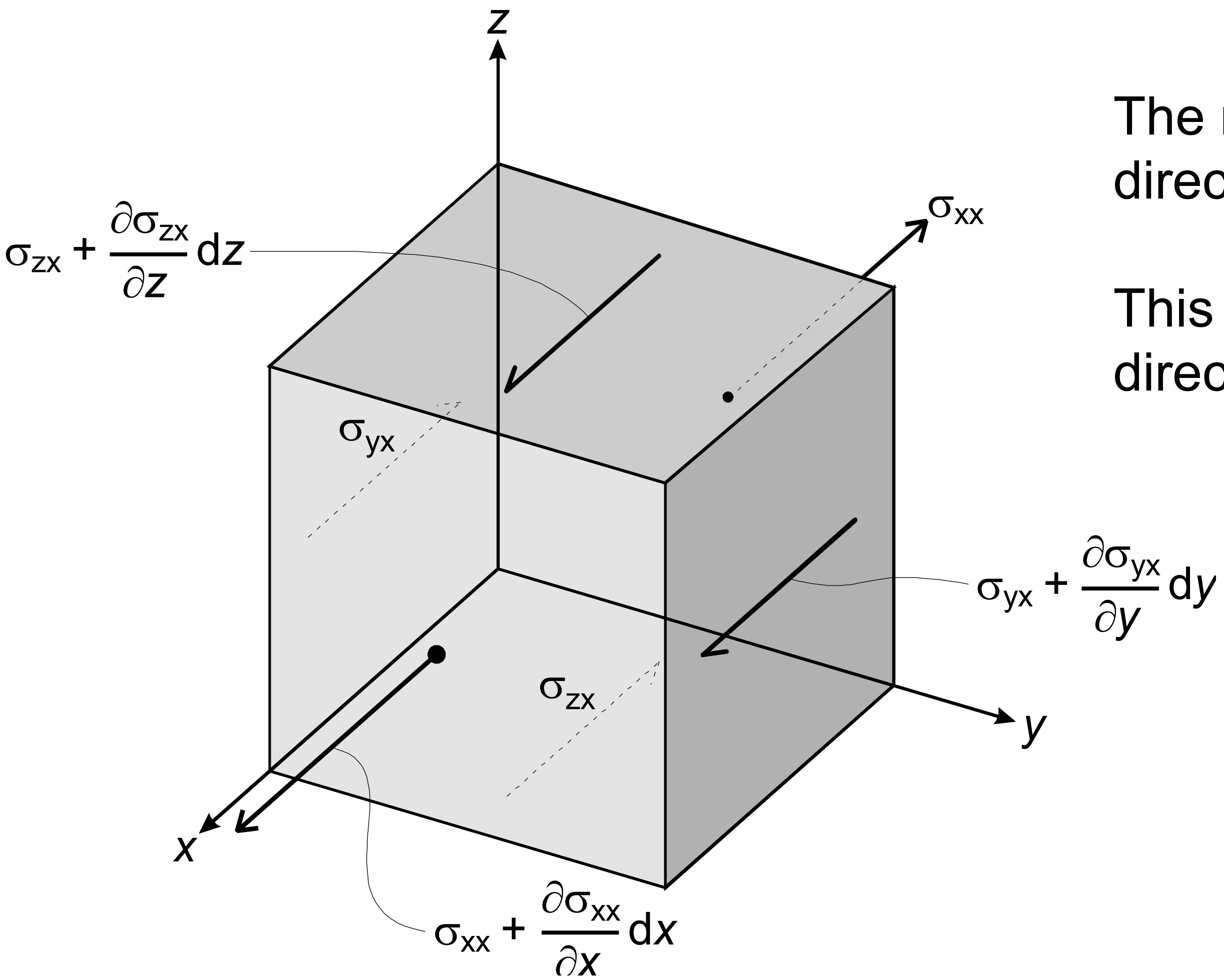
Rotational equilibrium

The sum of the moments about each of the axes should be zero



Rotational equilibrium ==> 6 independent components

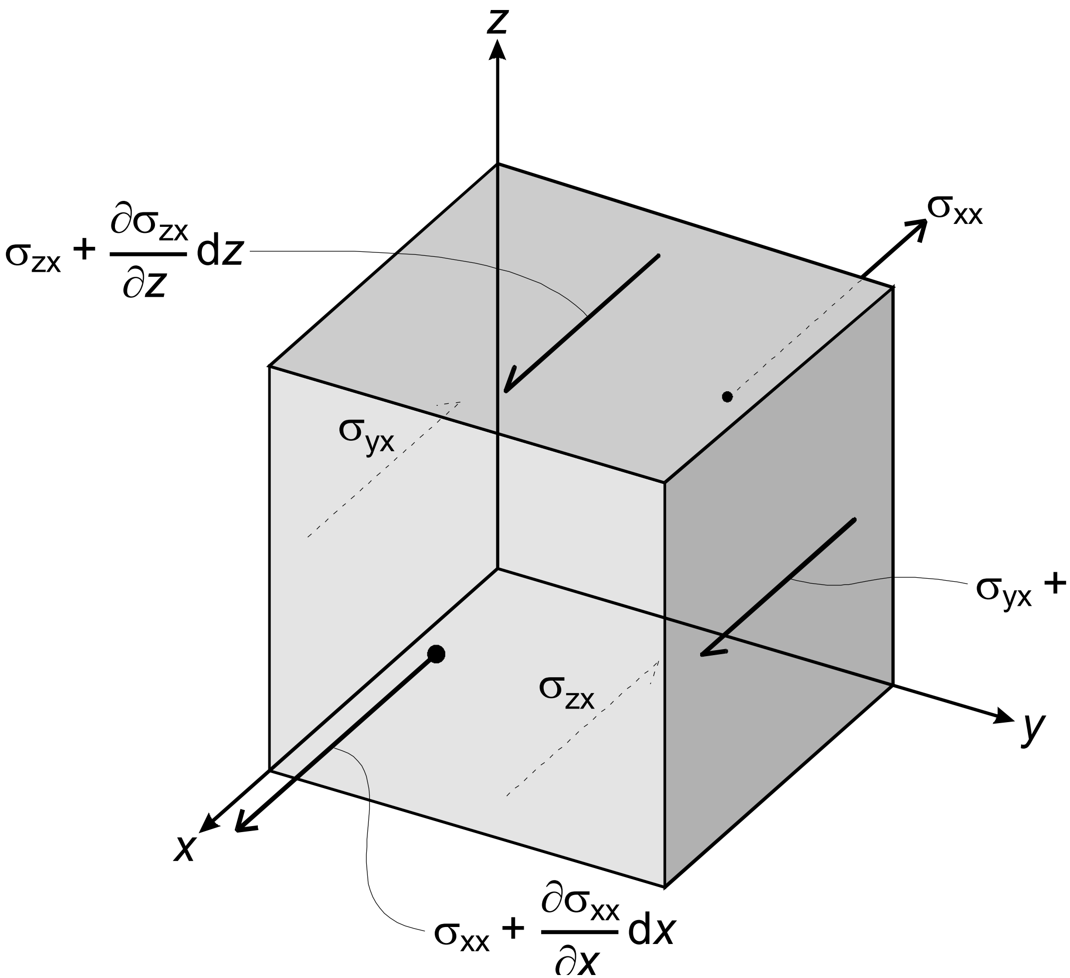
Translational equilibrium



The resultant of all forces in x direction must be zero.

This also holds for y and z direction.

Equilibrium equations



The resultant of all forces in x direction must be zero.

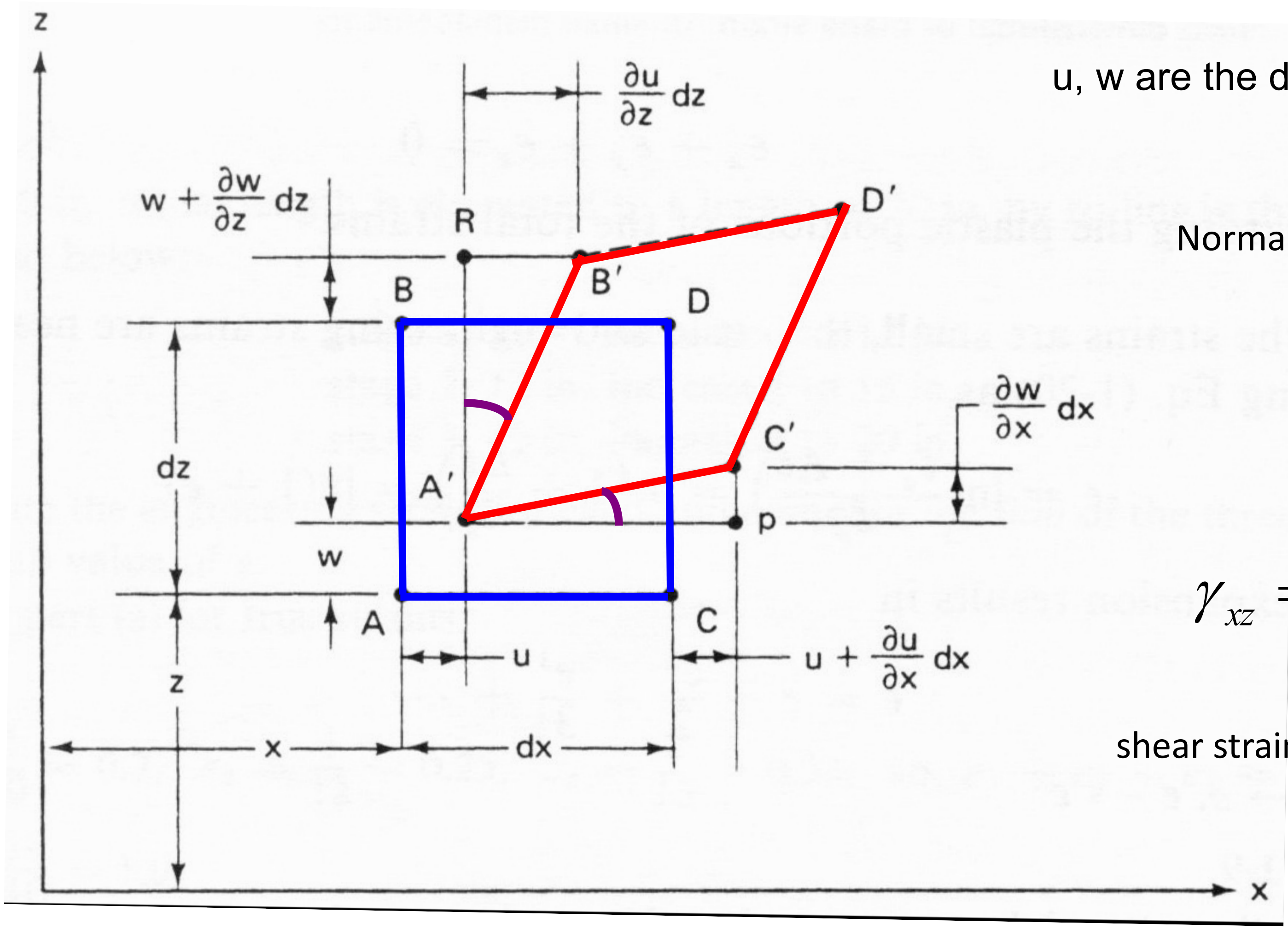
This also holds for y and z direction.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Strain compatibility (2D)



u, w are the displacements

Normal strain: extension

$$e_{xx} = \frac{\partial u}{\partial x}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

shear strain: angular distortion

Strain tensor

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} = \gamma_{xy}/2 & \varepsilon_{xz} = \gamma_{xz}/2 \\ \varepsilon_{yx} = \gamma_{yx}/2 & \varepsilon_{yy} & \varepsilon_{yz} = \gamma_{yz}/2 \\ \varepsilon_{zx} = \gamma_{zx}/2 & \varepsilon_{zy} = \gamma_{zy}/2 & \varepsilon_{zz} \end{bmatrix}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Elastic Boundary Value Problem

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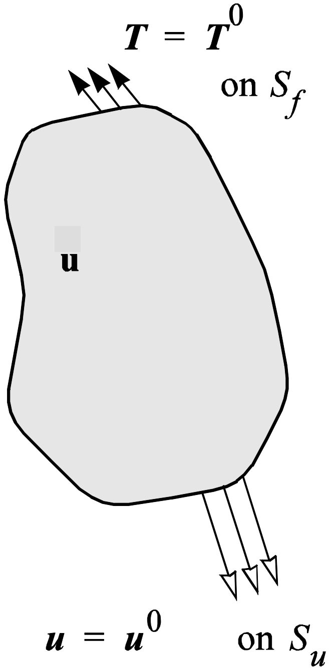


Robert Hooke

Ut tensio, sic vis

As is extension, so is force

1635-1703

Generic linear anisotropic material

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} C_{12} C_{13} C_{14} C_{15} C_{16} \\ C_{12} C_{22} C_{23} C_{24} C_{25} C_{26} \\ C_{13} C_{23} C_{33} C_{34} C_{35} C_{36} \\ C_{41} C_{42} C_{43} C_{44} C_{45} C_{46} \\ C_{51} C_{52} C_{53} C_{54} C_{55} C_{56} \\ C_{61} C_{62} C_{63} C_{64} C_{65} C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

$$C_{ij} = C_{ji}$$

The most general linear anisotropic material requires 21 independent material constants

Linear isotropic material

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) & \\ 0 & 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

$C_{11} = 1/E$, $C_{12} = -\nu/E$, and $2(C_{11}-C_{12}) = 1/G$

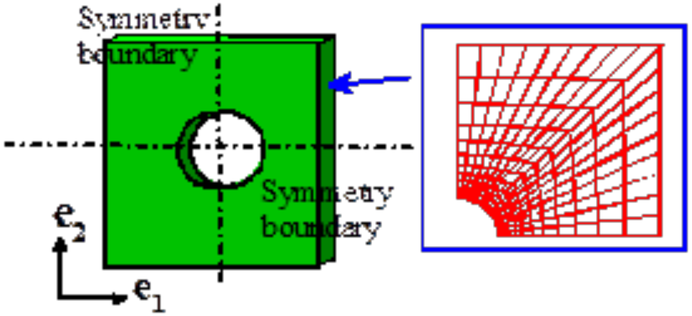
Only 2 independent material constants

Common assumption: material linear isotropic

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

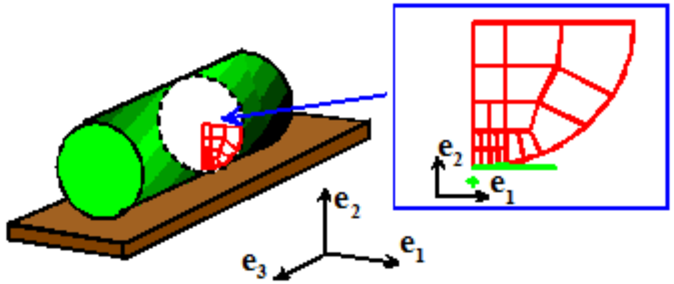
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}$$

Assumption 2: Plane stress or plane strain



$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

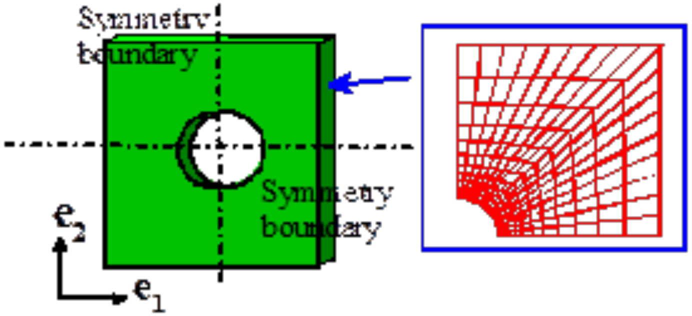
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$



$$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yx} \end{Bmatrix} = \frac{1}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

Plane stress



$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ \sigma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

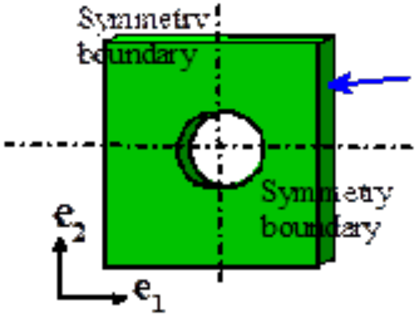
These components are zero

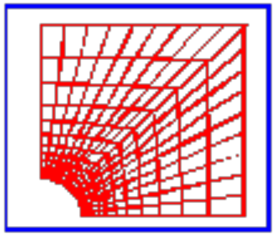
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{Bmatrix} \sigma_{xx} - \nu\sigma_{yy} \\ -\nu\sigma_{xx} + \sigma_{yy} \\ 2(1+\nu)\sigma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

Is eps_zz=0??

Plane stress





$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ \sigma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

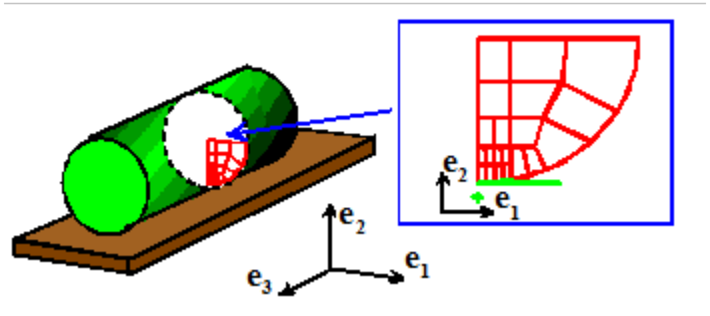
These components are zero

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{Bmatrix} \sigma_{xx} - \nu\sigma_{yy} \\ -\nu\sigma_{xx} + \sigma_{yy} \\ 2(1+\nu)\sigma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

Plane strain

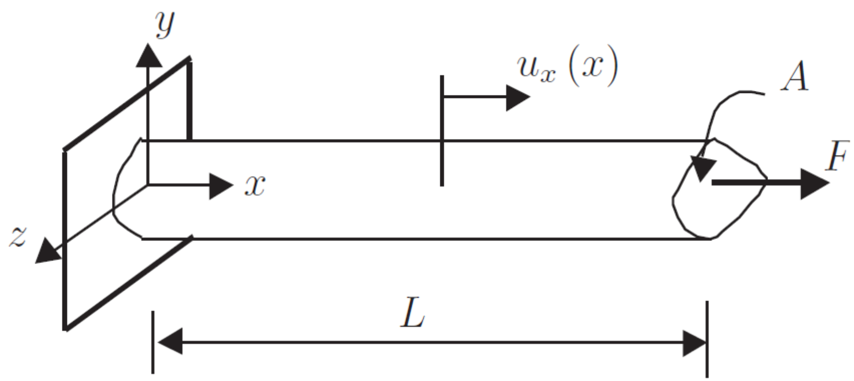


$$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yx} \end{Bmatrix} = \frac{1}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy})$$

Continuum system: bar in tension



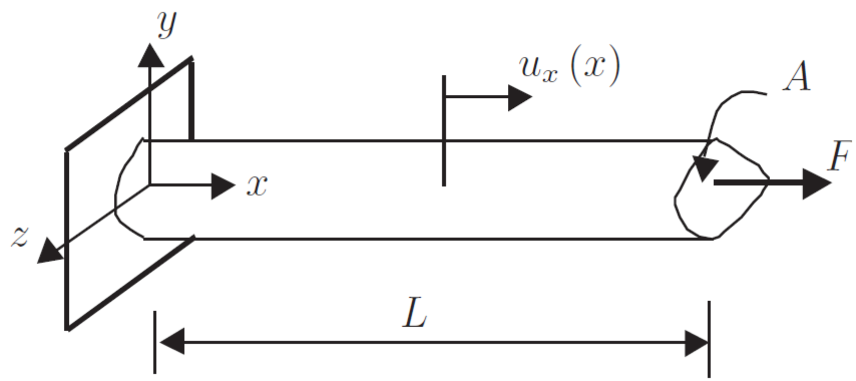
1) Equilibrium (conservation of linear momentum): $\frac{d\sigma_{xx}}{dx} = 0$

2) Constitutive Law: $\sigma_{xx} = E\varepsilon_{xx}$

3) Compatibility equation: $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$

What are the boundary conditions?

Continuum system: bar in tension



1) Equilibrium (conservation of linear momentum): $\frac{d\sigma_{xx}}{dx} = 0$

2) Constitutive Law: $\sigma_{xx} = E\varepsilon_{xx}$

3) Compatibility equation: $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$

4) Boundary Conditions: $u_x(x=0) = 0$, $\sigma_{xx}(x=L) = \frac{F}{A}$

$$\frac{d\sigma_{xx}}{dx} = 0 \implies \sigma_{xx} = C_1$$

$$\sigma_{xx}(x=L) = C_1 = \frac{F}{A} \implies \sigma_{xx}(x) = \frac{F}{A}$$

Continuum system: bar in tension

$$\frac{\partial u_x}{\partial x} = \varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{\frac{F}{A}}{E}$$

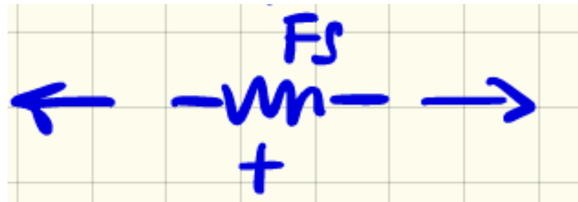
$$\frac{du_x}{dx} = \frac{F}{AE} \text{ or } \int_0^x du_x = \int_0^x \frac{F}{EA}\, dx$$

$$u_x(x) = \int_0^x \frac{F}{EA}\, dx = \frac{F}{EA}\int_0^x dx = \left(\frac{F}{EA}\right) x + C$$

$$u_x(x=0) = \frac{F}{EA}(0) + C \implies C = 0$$

$$u_x(x) = \left(\frac{F}{EA}\right) x \qquad \sigma_{xx}(x) = \frac{F}{A} \qquad \delta_{end} = u_x(x=L) = \frac{FL}{EA}$$

Discrete system: the spring



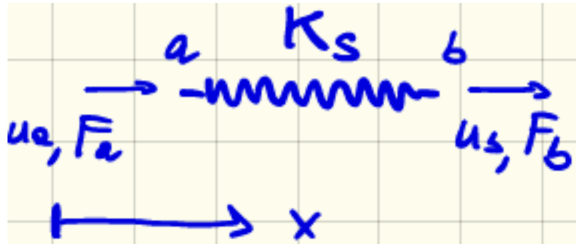
Assumption: the force is positive if the spring is elongated

$$F_s = k_s \delta \qquad \delta = u_2 - u_1$$

Constitutive equation

Displacements are discrete: they exist only at the end of the spring, while for the bar we have u(x).

Discrete system: the spring



Internal force:

$$F_s = k_s \delta = k_s (u_b - u_a)$$

$$\begin{cases} F_a + F_s = 0 \\ -F_s + F_b = 0 \end{cases}$$

$$\begin{cases} F_a + k_s (u_b - u_a) = 0 \\ -k_s (u_b - u_a) + F_b = 0 \end{cases}$$

$$\begin{cases} k_s u_b - k_s u_a = -F_a \\ -k_s u_b + k_s u_a = -F_b \end{cases}$$

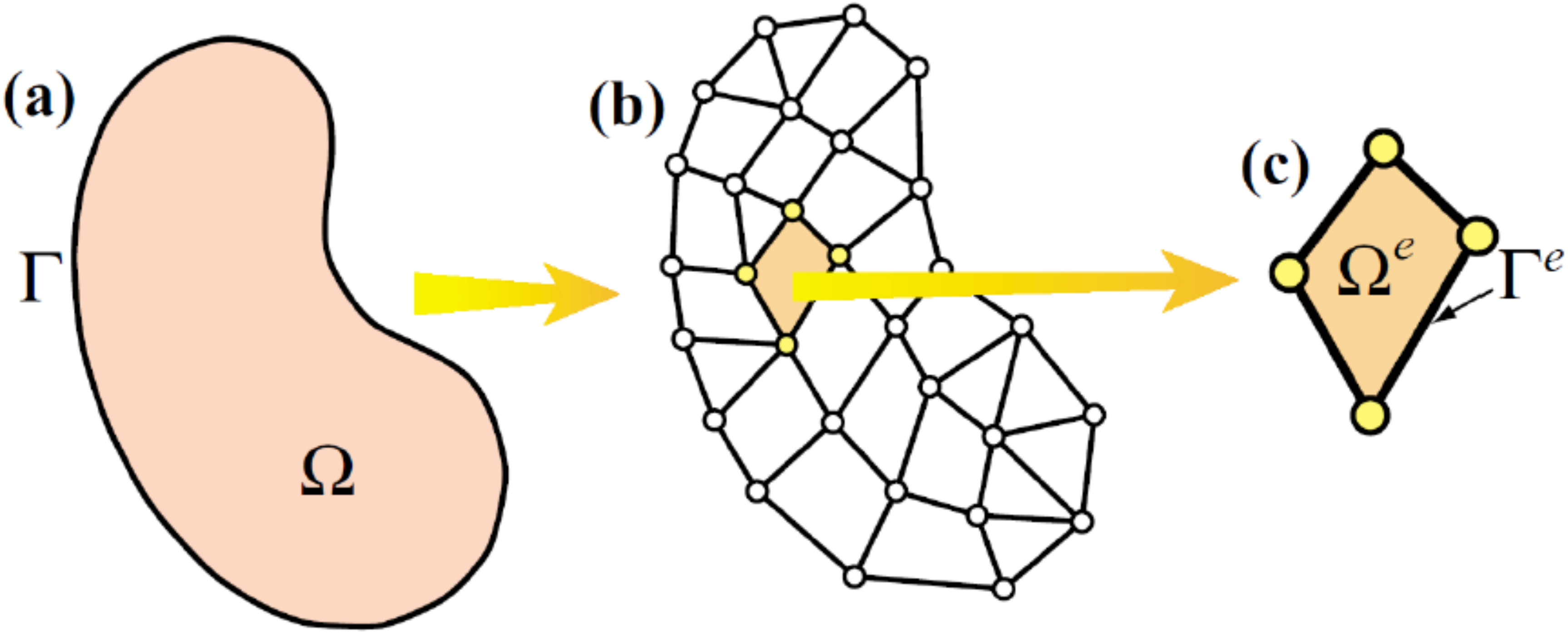
$$k_s \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \begin{bmatrix} F_a \\ F_b \end{bmatrix} \longrightarrow \underline{\underline{k}}\, \underline{u} = \underline{f}$$

Discretization:
bar=spring with
stiffness K=EA/L

$$\delta_{end} = u_x(x = L) = \frac{FL}{EA}$$

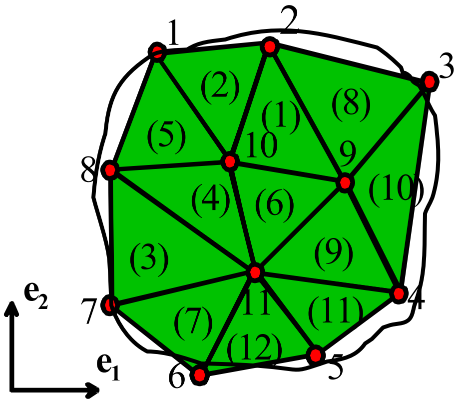
The finite element method

From continuum to discrete objects



Discretization: mesh and connectivity

- A finite element mesh is defined as a set of elements and nodes.
- In the example below the elements are 3 noded triangles
- The nodes are numbered 1,2,3…*N*, while the elements are numbered (1),(2)…(*L)*. Element numbers are shown in parentheses.
- The position of the *a*th node is specified by its coordinates x,y
- During deformation the nodes move. Their displacement (u1,u2) is calculated during the simulation
- The *element connectivity* specifies the node numbers attached to each element. The connectivity for element 1 is (10,9,2); for element 2 it is (10,2,1)



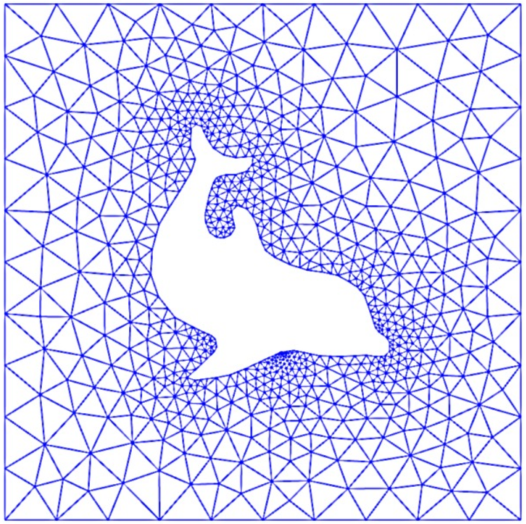
Discretization

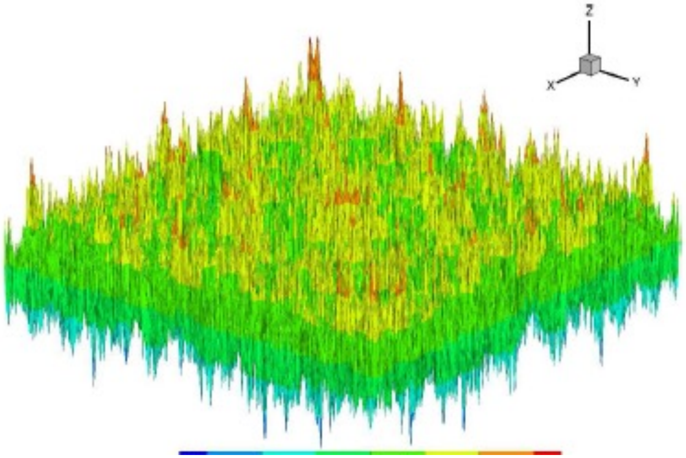
A finite element mesh can be coarse or fine depending on:

(1) The geometry of the body
(to model surface roughness accurately you would need about 10^8 discretization points)

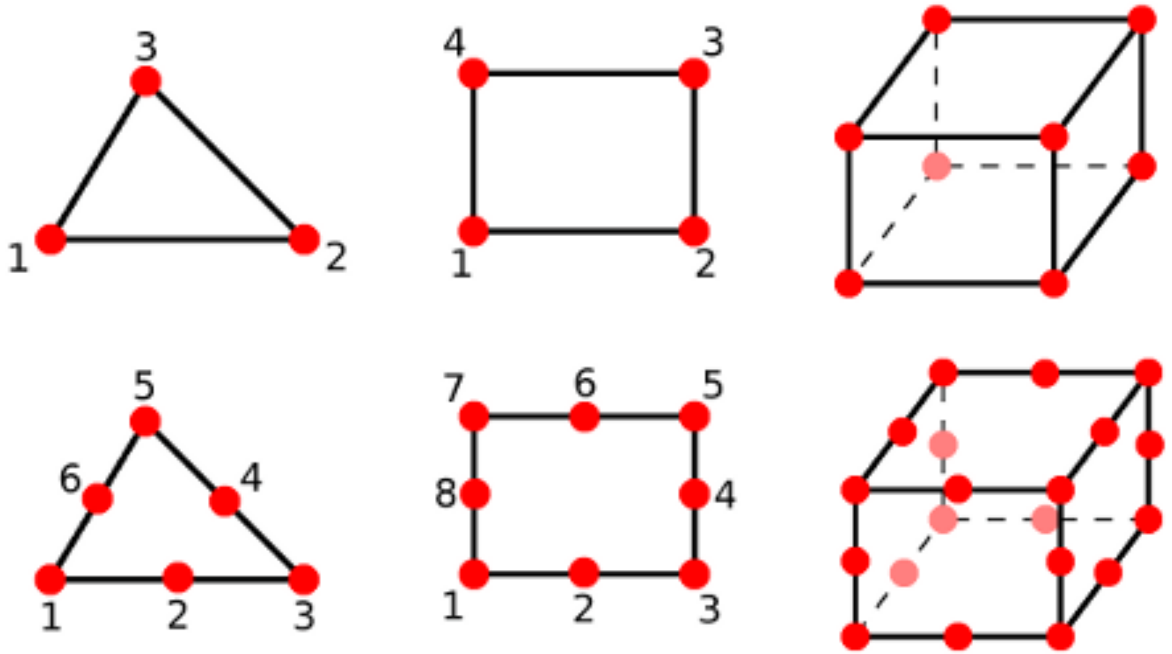
(2) The state of stress and deformation in the body, i.e. more or less homogeneous

(3) The level of accuracy required





Various types of elements



Triangle element with 3 and 6 nodes

Rectangle element with 4 and 8 nodes

Box element (for 3D) with 8 and 20 nodes

Sample of some simple element shapes and standard node placement. By convention nodes are numbered anti-clockwise.

The displacement vector

We will approximate the displacement field by interpolating between values at the nodes, as follows. Let $u_i^{(a)}$ denote the unknown displacement vector at nodes $a = 1, 2, \ldots N$. In a finite element code, the displacements are normally stored as a column vector like the one shown below:

$$\underline{u} = \begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} & \cdots & \end{bmatrix}^T$$

Note that the entries of the vector are 2*nnod=ndof for a 2D problem, they are 3*nnod=ndof for a 3D problem

The unknown displacement components will be determined by minimizing the potential energy of the solid.

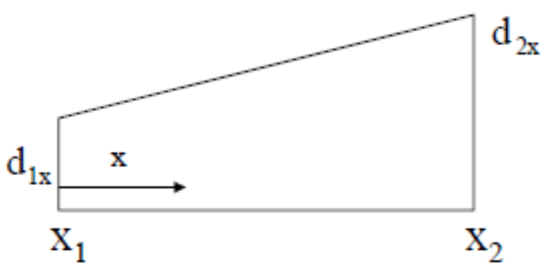
Shape functions (interpolation functions)

In order to know the displacement anywhere in the body we need to interpolate the node values through functions that are called SHAPE FUNCTIONS (low-order polynomial functions of position).

Example: uni-dimensional 2-noded element, linear interpolation

Simplest assumption: displacement varying linearly inside each bar

$$w(x) = a_0 + a_1 x$$



How to obtain a_0 and a_1?

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

X1 and x2 are the nodal positions,
d1 and d2 are the nodal displacements

2-noded element

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

Solve simultaneously

$$a_0 = \frac{x_2}{x_2 - x_1} d_{1x} - \frac{x_1}{x_2 - x_1} d_{2x}$$
$$a_1 = -\frac{1}{x_2 - x_1} d_{1x} + \frac{1}{x_2 - x_1} d_{2x}$$

Hence

$$w(x) = a_0 + a_1 x = \underbrace{\frac{x_2 - x}{x_2 - x_1}}_{N_1(x)} d_{1x} + \underbrace{\frac{x - x_1}{x_2 - x_1}}_{N_2(x)} d_{2x} = N_1(x) d_{1x} + N_2(x) d_{2x}$$

"**Shape functions**" $N_1(x)$ and $N_2(x)$

2-noded element

In matrix notation, we write

$$\boxed{w(x) = \underline{N}\,\underline{d}} \qquad (1)$$

Vector of nodal shape functions

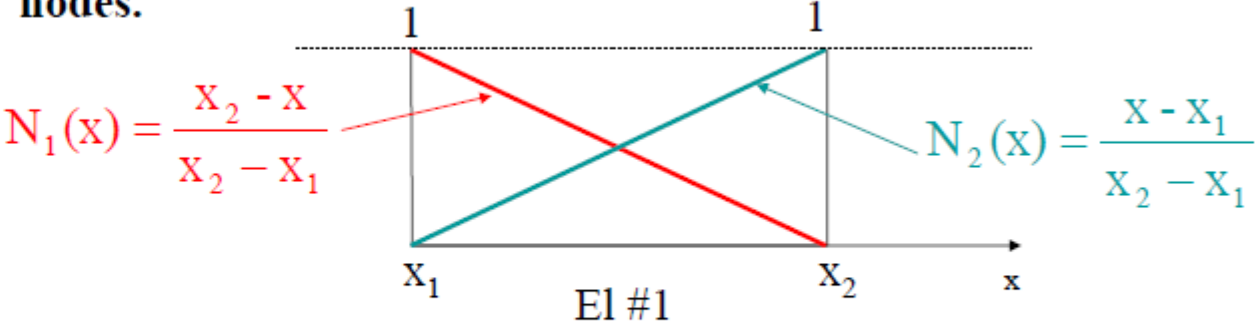
$$\underline{N} = \left[N_1(x) \quad N_2(x) \right] = \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

Vector of nodal displacements

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Properties of shape functions

1. <u>Kronecker delta property</u>: The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.



Check

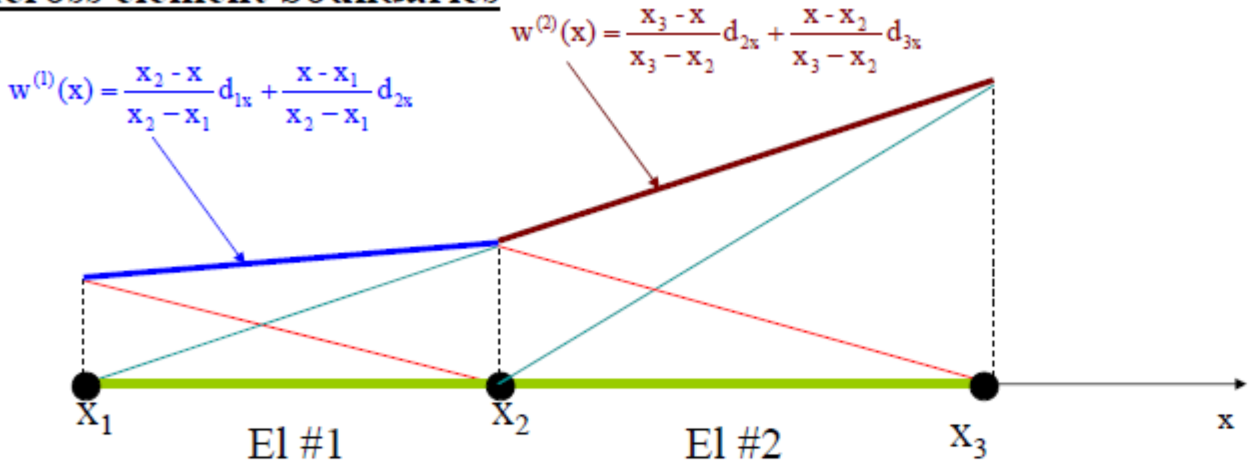
$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\Rightarrow N_1(x = x_1) = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} = 0$$

Properties of shape functions

2. <u>Compatibility</u>: The displacement approximation is **<u>continuous across element boundaries</u>**



At $x=x_2$

$$w^{(1)}(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} d_{1x} + \frac{x_2 - x_1}{x_2 - x_1} d_{2x} = d_{2x}$$

$$w^{(2)}(x = x_2) = \frac{x_3 - x_2}{x_3 - x_2} d_{2x} + \frac{x_2 - x_2}{x_3 - x_2} d_{3x} = d_{2x}$$

Hence the displacement approximation is continuous across elements

Strain in the bar

$$w(x) = \underline{N}\,\underline{d}$$

Recall that the **strain** in the bar $\varepsilon = \frac{dw}{dx}$

Hence

$$\boxed{\varepsilon = \left[\frac{d\underline{N}}{dx}\right]\underline{d} = \underline{B}\,\underline{d}} \qquad (2)$$

The matrix $\underline{B}$ is known as the "**strain-displacement matrix**"

$$\underline{B} = \left[\frac{d\underline{N}}{dx}\right]$$

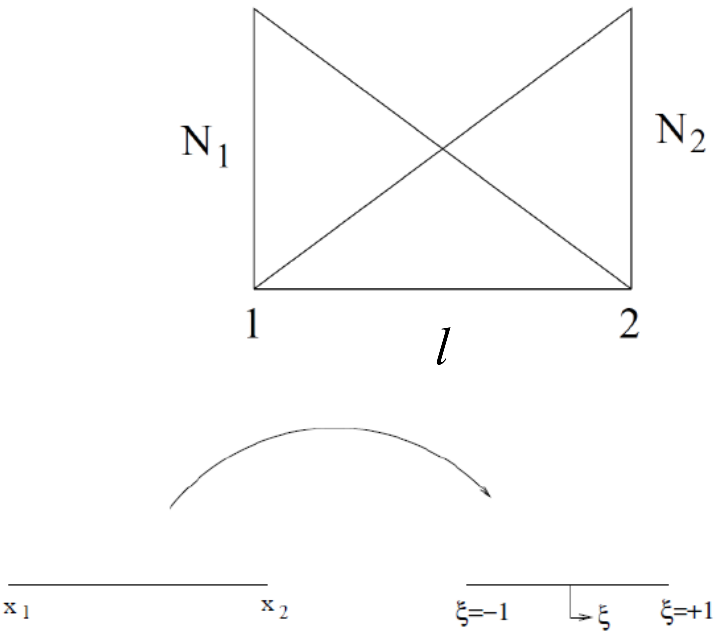
$$\underline{B} = \left[\begin{matrix} \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{matrix}\right] = \frac{1}{x_2 - x_1}\left[\begin{matrix} -1 & 1 \end{matrix}\right]$$

$$\varepsilon = \underline{B}\,\underline{d} = \left[\begin{matrix} \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{matrix}\right]\begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

$$= \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$

Hence, strain is a **constant** within each element (only for a linear element)!

Isoparametric elements



Mapping of an element to a master element

To map a generic element to one of length 1 we change variables and keep the requirement of linearity: the constants ai, bi are evaluated with the new boundary values.

Shape functions

$$N_i(\xi) = a_i + b_i\xi,$$

$$N_1(\xi) = \frac{1}{2}(1-\xi),\ N_2(\xi) = \frac{1}{2}(1+\xi),$$

$$x = N_1(\xi)x_1 + N_2(\xi)x_2,$$

$$u = N_1(\xi)u_1 + N_2(\xi)u_2$$

Isoparametric elements

Shape functions

$$N_i(\xi) = a_i + b_i\xi,$$

$$N_1(\xi) = \frac{1}{2}(1-\xi),\ N_2(\xi) = \frac{1}{2}(1+\xi),$$

$$x = N_1(\xi)x_1 + N_2(\xi)x_2,$$

$$u = N_1(\xi)u_1 + N_2(\xi)u_2$$

$$\text{Strain } \epsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{1}{l_e}(u_2 - u_1).$$

$$\epsilon = \mathbf{B}\hat{\mathbf{u}} \implies \mathbf{B} = \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}, \quad \hat{\mathbf{u}} = \begin{bmatrix}u_1 \\ u_2\end{bmatrix}$$

Isoparametric elements

Shape functions

$$N_i(\xi) = a_i + b_i\xi,$$

$$N_1(\xi) = \frac{1}{2}(1-\xi),\ N_2(\xi) = \frac{1}{2}(1+\xi),$$

$$x = N_1(\xi)x_1 + N_2(\xi)x_2,$$

$$u = N_1(\xi)u_1 + N_2(\xi)u_2$$

$$x = \frac{1}{2}x_1 - \frac{1}{2}\xi x_1 + \frac{1}{2}x_2 + \frac{1}{2}\xi x_2$$

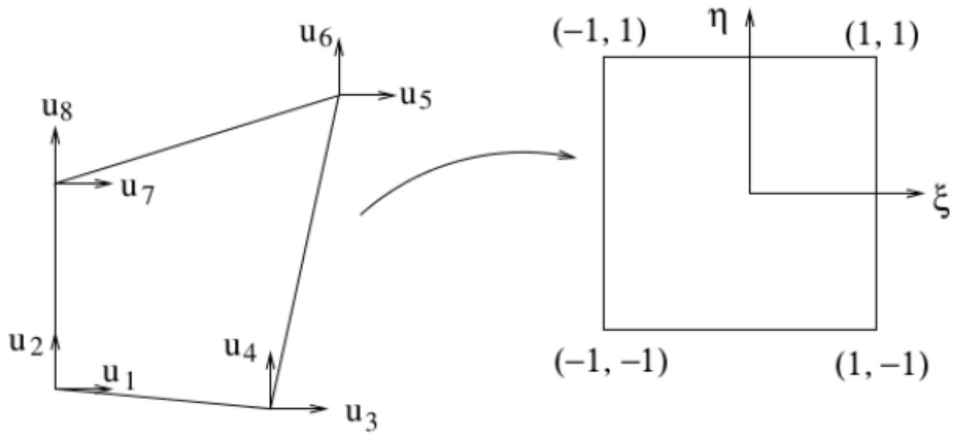
$$\xi = \frac{2x}{x_2-x_1} - \frac{x_1+x_2}{x_2-x_1} \qquad \frac{d\xi}{dx} = \frac{2}{x_2-x_1} = \frac{2}{l_e}$$

$$\frac{du}{d\xi} = -\frac{1}{2}u_1 + \frac{1}{2}u_2 = \frac{u_2-u_1}{2}$$

$$\text{Strain } \epsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{1}{l_e}(u_2-u_1).$$

$$\epsilon = \mathbf{B}\hat{\mathbf{u}} \implies \mathbf{B} = \frac{1}{l_e}\begin{bmatrix}-1 & 1\end{bmatrix}, \quad \hat{\mathbf{u}} = \begin{bmatrix}u_1 \\ u_2\end{bmatrix}$$

For a rectangular element



Shape functions: $N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$

The displacement fields can be then written as:

$u = N_1 u_1 + N_2 u_3 + N_3 u_5 + N_4 u_7,$

$v = N_1 u_2 + N_2 u_4 + N_3 u_6 + N_4 u_8$

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \qquad \mathbf{u} = \mathbf{N}\hat{\mathbf{u}}$$

The Jacobian

In 2D to find the strain:

By using chain rule, we get: $$\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix},$$

where $\mathbf{J}$ is the Jacobian such that $\det(\mathbf{J}) = |\mathbf{J}| > 0$

f =u

Thus, $$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \mathbf{\Gamma} \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix},$$

where $\mathbf{\Gamma} = \mathbf{J}^{-1}$, is the inverse Jacobian

with $\Gamma_{11} = \frac{J_{22}}{|\mathbf{J}|}; \quad \Gamma_{12} = -\frac{J_{12}}{|\mathbf{J}|}; \quad \Gamma_{21} = -\frac{J_{21}}{|\mathbf{J}|}; \quad \Gamma_{22} = \frac{J_{11}}{|\mathbf{J}|}$

Take home messages

- The FEM is used to solve partial differential equations with prescribed boundary conditions
- It is used in solid mechanics to calculate the deformation of bodies subjected to loading.
- The body is discretized using finite elements and nodes
- The displacements at the nodes are the unknown of the problem and obtained as a solution
- The fields inside the elements are approximated using shape functions, which can be linear, bi-linear, etc.
- It is common to map elements onto isoparametric elements (mostly useful when elements have curvilinear boundaries)
- The isoparametric elements require a change of variables and therefore a Jacobian to move from isoparametric variables to real variables.