5_Esercizi-fattorizzazione LU

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ESERCIZIO 1

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 3 \\ -1 & 4 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

(A) verificare le ipotesi del teorema per l'esistenza della fattorizzazione di Cholesky

(B) Trovare una matrice di permutazione per cui la matrice B (= A permutata) soddisfi le ipotesi

(C) Trovare la fattorizzazione $B = LL^T$

(D) Risolvere $By = c$ e ritrovare la soluzione x di $Ax = b$

(E) Calcolare il determinante : $\det(A^{-3})$

Soluzione

A) Considero i minori di A :

$$A_1 = 2 \,, \qquad A_2 = \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix} \rightsquigarrow \det A_2 = -4 < 0$$

Inoltre, A <u>NON</u> è simmetrica

B) Scambiando le colonne 2 e la colonna 3 ottengo :

$$B = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -2 \\ -1 & -2 & 4 \end{bmatrix} = AP = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Osserviamo :

- B è simmetrica
- $B_1 = 2$, $B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\det B_2 = 6 > 0$

  $B_3 = B$, $\det B = 13 > 0$

$\Rightarrow$ possiamo calcolare $B = LL^T$

c) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -2 \\ -1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$

$$\Rightarrow \quad l_{11}^2 = 2 \quad \longrightarrow \quad l_{11} = \sqrt{2}$$

$$l_{21}\, l_{11} = 0 \quad \longrightarrow \quad l_{21} = 0$$

$$l_{31}\, l_{11} = -1 \quad \longrightarrow \quad l_{31} = -1/\sqrt{2}$$

$$l_{21}^2 + l_{22}^2 = 3 \quad \rightarrow \quad l_{22} = \sqrt{3-0} = \sqrt{3}$$

$$l_{21}\, l_{31} + l_{22}\, l_{32} = -2 \quad \longrightarrow \quad l_{32} = -\frac{2}{l_{22}} = -2/\sqrt{3}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 4 \quad \rightarrow \quad l_{33} = \sqrt{4 - \frac{1}{2} - \frac{4}{3}} = \sqrt{13/6}$$

quindi : $\quad L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ -1/\sqrt{2} & -2/\sqrt{3} & \sqrt{13/6} \end{bmatrix}$

D) $\quad Ax = b$

$AP = B \qquad (\text{con } P^{-1} = P^T \iff PP^T = I)$

$\underbrace{AP}_{B}\,\underbrace{P^T x}_{y} = \underbrace{b}_{c}$ $\qquad$ quindi $\quad B = AP$

$\qquad y = P^T x \rightsquigarrow x = Py$

$\qquad c = b$

Risolvo $\quad By = c$:

$$L\underbrace{L^T y}_{z} = b \quad \iff \quad \begin{cases} Lz = b \\ L^T y = z \end{cases}$$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ -1/\sqrt{2} & -2/\sqrt{3} & \sqrt{13/6} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$\hookrightarrow \quad z_1 = 0$

$\qquad z_2 = 5/\sqrt{3}$

$\qquad z_3 = \sqrt{26/3}$

$\begin{bmatrix} \sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \end{bmatrix} \qquad \begin{bmatrix} 0 \end{bmatrix}$

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ 0 & \sqrt{3} & -2/\sqrt{3} \\ 0 & 0 & \sqrt{13/6} \end{bmatrix} \begin{bmatrix} \cdots \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cdots \\ 5/\sqrt{3} \\ \sqrt{26/3} \end{bmatrix}$$

$$\hookrightarrow \quad y_3 = 2$$

$$y_2 = 3$$

$$y_1 = 1$$

E per ottenere x:

$$x = Py = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_3 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(per verificare $b - Ax \stackrel{?}{=} 0$)

E) $$\det(A^{-3}) = \frac{1}{(\det A)^3} = \frac{1}{(\det(P^T B))^3} =$$

$B = PA$
$A = P^{-1}B = P^T B$

$$= \frac{1}{(\det P^T \cdot \det B)^3}$$

ho un solo scambio

$$\det P = \det P^T = (-1)^S = -1$$

$$\det B = \det L \cdot \det L^T = (\det L)^2 = \left(\prod l_{ii}\right)^2$$

$$\Rightarrow \det(A^{-3}) = \left[\frac{1}{(-1)\left(\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{13/6}\right)^2}\right]^3 = \left(\frac{-1}{2\cdot 3\cdot \frac{13}{6}}\right)^3 =$$

$$= -\frac{1}{(13)^3}$$

ESERCIZIO 2

$$A = \begin{bmatrix} 7 & 28 & 21 \\ 3 & 17 & 19 \\ 4 & 19 & \alpha \end{bmatrix} \quad \text{con } \alpha \in \mathbb{R}$$

A) Trovare fattorizzazione LU con $u_{ii} = 1$

B) Trovare il valore di α per cui $Tr(L) = 20$

C) Calcolare $\det(A^{-2})$

D) Fissato α del punto (B). Dato $b = \begin{bmatrix} 91 \\ 64 \\ 83 \end{bmatrix}$,

risolvere $Ax = b$

Soluzione:

A) $$A = \begin{bmatrix} 7 & 28 & 21 \\ 3 & 17 & 19 \\ 4 & 19 & \alpha \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = LU$$

quindi:

$l_{11} = 7$ $\quad l_{11} \cdot u_{12} = 28 \rightarrow u_{12} = 4$ $\quad l_{11} u_{13} = 21 \rightarrow u_{13} = 3$

$l_{21} = 3$ $\quad l_{21} u_{12} + l_{22} = 17 \rightarrow l_{22} = 5$ $\quad l_{21} u_{13} + l_{22} u_{23} = 19 \rightarrow u_{23} = 2$

$l_{31} = 4$ $\quad l_{31} u_{12} + l_{32} = 19 \rightarrow l_{32} = 3$ $\quad l_{31} u_{13} + l_{32} u_{23} + l_{33} = \alpha$

$\hookrightarrow l_{33} = \alpha - 18$

$$\Rightarrow A = LU = \begin{bmatrix} 7 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 3 & \alpha-18 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

B) $Tr(L) = 7 + 5 + \alpha - 18 \underbrace{= 20}_{\uparrow\ \text{impongo}} \iff \alpha = 26$

(da qui segue che $l_{33} = 26 - 18 = 8$)

C) $$\det(A^{-2}) = \frac{1}{(\det A)^2} = \frac{1}{(\det L \cdot \det U)^2} =$$

con $\det U = 1$

$\det L = \prod_{i=1}^{3} l_{ii}$

$$= \frac{1}{\quad}$$

$(7 \cdot 5 \cdot 8)^2$

D) $A = LU = \begin{bmatrix} 7 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 3 & 8 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 91 \\ 64 \\ 83 \end{bmatrix}$

$$LU\,x = b \iff \begin{cases} L\,y = b \\ U\,x = y \end{cases}$$

$$\begin{bmatrix} 7 & 0 & 0 \\ 3 & 5 & 0 \\ 4 & 3 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 91 \\ 64 \\ 83 \end{bmatrix}$$

$\hookrightarrow$

$$y_1 = 91/7 = 13$$
$$y_2 = \frac{1}{5}(64 - 3 \cdot 13) = 5$$
$$y_3 = \frac{1}{8}(83 - 4 \cdot 13 - 3 \cdot 5) = 2$$

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \\ 2 \end{bmatrix}$$

$\hookrightarrow$

$$x_3 = 2$$
$$x_2 = 5 - 2 \cdot x_3 = 1 \quad \Rightarrow \quad x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$
$$x_1 = 13 - 4 \cdot x_2 - 3 \cdot x_3 = 3$$

$\underline{\qquad\qquad} \cdot \underline{\qquad\qquad\qquad}$