Knowledge Representation and Learning
Final exam

Prof. Luciano Serafini

February 20, 2024

Exercise 1 (4 points) *Find a formula ϕ that has the following truth table, and explain the method you have followed to find it.*

p	q	r	ϕ
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Solution A possible way to proceed is based on three main steps (1) building is by associating a conjunction of literals that fully describes each interpretation that satisfies ϕ; (2) put them in a disjunction and (3) simplify the resulting disjunction as much as possible.

1. For every interpretation $\mathcal{I}$ on the set of proposition $\mathcal{P}$ we can define the conjunction of the literals that are satisfied by $\mathcal{I}$

   $$\psi_{\mathcal{I}} \triangleq \bigwedge_{\substack{l \in Lit \\ \mathcal{I} \models l}} l$$

   where Lit is the set of literals on the propositional variables in $\mathcal{P}$. Notice that $\psi_{\mathcal{I}}$ is satisfied by $\mathcal{I}$ and $\mathcal{I}$ is the only model that satisfies $\psi_{\mathcal{I}}$.

2. Let us put in disjunction all the $\psi_{\mathcal{I}}$ such that $\mathcal{I} \models \phi$.

   $$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \tag{1}$$

   Notice taht the above formula has exactly the same models than ϕ since is it the disjunction of the formulas that are true in each model of ϕ.

3. We can then simplify it.

   $$(p \wedge q) \vee (\neg p \wedge \neg q \wedge r) \tag{2}$$

   that can be reduced in the following set of clauses

   $$(p \vee \neg q) \wedge (p \vee r) \wedge (q \vee \neg p) \wedge (q \vee r) \tag{3}$$

4. that cam be simplified in

$$(p \leftrightarrow q) \wedge (\neg p \rightarrow r) \tag{4}$$

$\square$

Exercise 2 (5 points) *Consider a directed graph $G = (V, E)$ composed of a set of nodes $V = \{1, \ldots, n\}$ and a set of directed edges between them. Formulate the problem of finding the shortest path that visits all the nodes starting from a node s and ending in a node e without passing through the node p.*

Solution For every edge $(i, j) \in E$ and for every $1 \leq t \leq |V|$, we define a propositional variable p_{ij}^t that is true if at time t we traverse the edge (i, j), The problem asks for a path:

$$\begin{aligned}
\text{Startig at } i : & \bigvee_{h:(i,h)\in E} p_{ih}^1 \\
\text{Ending at } j : & \bigvee_{m=1}^{n} \left(\bigvee_{h:(h,j)\in E} p_{hj}^m \wedge \bigwedge_{\substack{m<t\leq n \\ e\in E}} \neg p_e^l \right) \\
\text{Passing through } k : & \bigvee_{\substack{h:(k,h)\in E \\ 1\leq t\leq n}} p_{k,h}^t
\end{aligned}$$

The fact that we want to find the minimal path can be formulated with a set of weighted clauses, that adds a cost of 1 every time we tracerse an edge. i.e., for every $e \in E$ and $1 \leq i \leq n$, we add the weighted clause

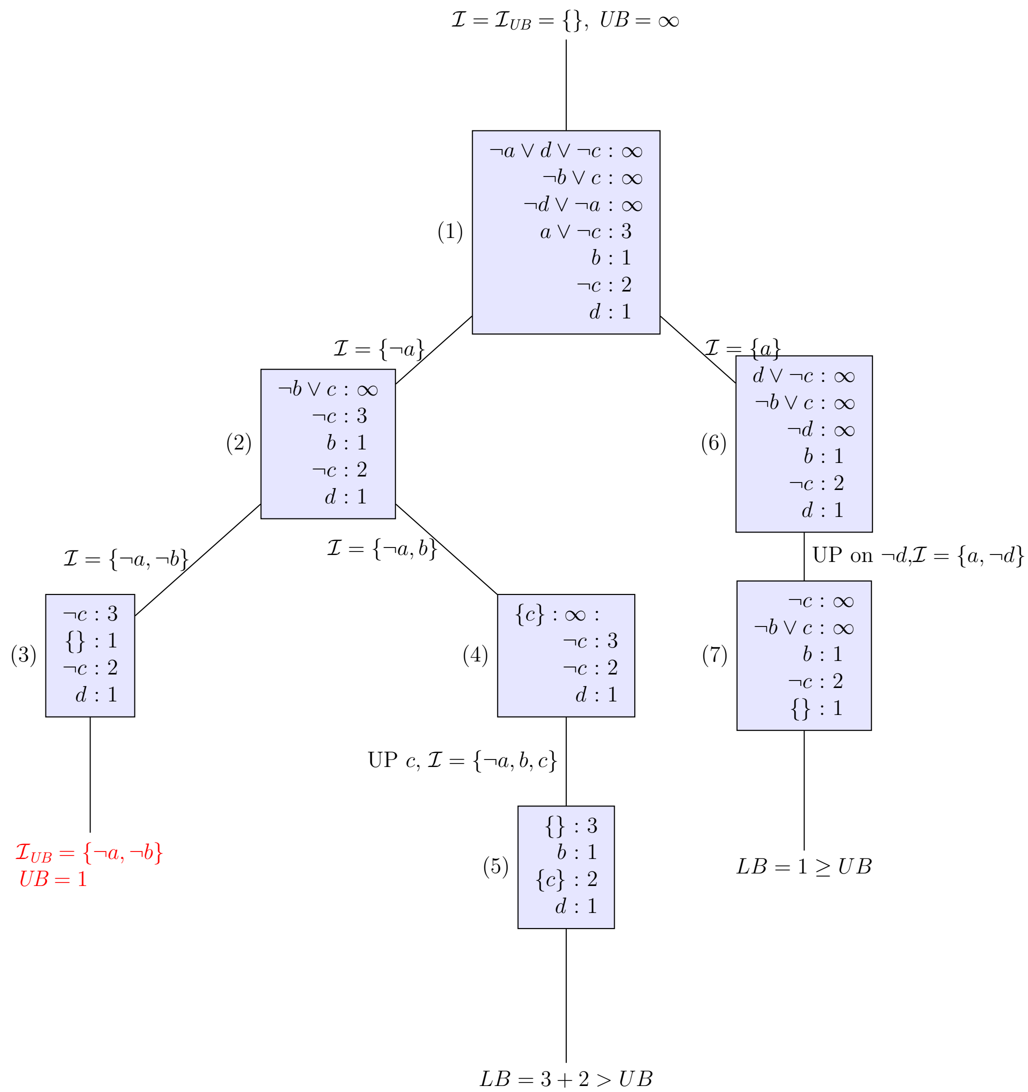
$$1 : p_e^t$$

$\square$

Exercise 3 (4 points) *Use B&B algorithm to solve the following maxSat problem*

$$\begin{array}{rrr}
(\neg a \vee d \vee \neg c : \infty) & (a \vee \neg c : 3) & (\neg c : 2) \\
(\neg b \vee c : \infty) & (b : 1) & (d : 1) \\
(\neg d \vee \neg a : \infty) & &
\end{array}$$

Solution



□

Exercise 4 (5 points) *Suppose you have three coins: the faces of the first coin are black and white, the faces of the second coin are yellow and green, and the faces of the third coin are red and green. In an experiment you toss the first coin; if you obtain a black you toss the second coin otherwise you toss the third coin.*

1. *Model this experiment in propositional logic by using the following propositinal variables*

$$
\begin{aligned}
B &: \textit{The result of tossing the first coing is black} \\
W &: \textit{The result of tossing the first coing is wite} \\
Y &: \textit{you toss the second coin and the result is yellow} \\
R &: \textit{you toss the third coin and the result is red} \\
G &: \textit{you toss either the second or the third coin and the result is green}
\end{aligned}
$$

2. *Let p, q and r be the probability of obtaining a black, yellow, and red faces when tossing the first, second and third coin respectively. Compute the probability of obtaining an outcome which is either red or green.*

Solution

$$
\begin{aligned}
B \leftrightarrow \neg W &: \text{The first coin is tossed and the result is either black or white} \\
B \rightarrow (Y \leftrightarrow \neg G) &: \text{If the result of the first toss is blach then the second coin is tossed} \\
&\quad \text{and either we have a yellow or a green} \\
B \rightarrow \neg R &: \text{If the result of the first toss is blach the third coing is not tossed and} \\
&\quad \text{therefore we cannot obtain a red} \\
W \rightarrow (R \leftrightarrow \neg G) &: \text{If the result of the first toss is white then the third coin is tossed and} \\
&\quad \text{either we have a red or a green} \\
W \rightarrow \neg Y &: \text{If the result of the first toss is white the second coing is not tossed} \\
&\quad \text{and therefore we cannot obtain a yellow}
\end{aligned}
$$

The set of assignments that satisfy the above formulas are the followin:

B	W	Y	G	R
1	0	1	0	0
1	0	0	1	0
0	1	0	1	0
0	1	0	0	1

The relative probabilities are the following

B	W	Y	G	R	Pr
1	0	1	0	0	pq
1	0	0	1	0	$p(1-q)$
0	1	0	1	0	$(1-p)(1-r)$
0	1	0	0	1	$(1-p)r$

□

Exercise 5 (4 points) *Describe all the models of the following set of formulas in the domain* $\{1, 2, 3\}$.

$$\forall x \exists y, \neg R(x, y) \tag{5}$$
$$\forall x \forall y (R(x, y) \rightarrow R(y, x)) \wedge \tag{6}$$
$$\forall x (A(x) \rightarrow (\forall y (R(x, y) \rightarrow A(y)))) \tag{7}$$

Solution R can be interpreted in any symmetric relations on $\{1, 2, 3\}$ (to satisfy the second formula) such that from every node can exit at most 2 arrows (to satisfy the first axioms) and such that if there is an R-arc between two points either they are both in A or both not in A (to satisfy the third formula). □

Exercise 6 (5 points) *Consider the following statements*

A grandparent of a person is a parent of a parent of the person

Translate the above facts in FOL using the following symbols:

$$P(x, y) = x \textit{ is a parent of } y,\ P \textit{ is a binary predicate}$$
$$G(x, y) = x \textit{ is a grandparent of } y,\ G \textit{ is a binary predicate}$$

Then use resolution to show that if x and y have the same parents they also have the same grandparents. Only formulate the problem in clausal form without doing the resolution proof.

Solution The definition of grandparent can be obtrained by formalizing the sentence: "x is a granparent of y if there is a z that has x as parent and is the parent of z. In FOL

$$\forall x \forall y (G(x, y) \leftrightarrow \exists z P(x, z) \land P(z, y))$$

in clausal form after Skolemization:

$$\{\neg G(x, y), P(x, f(x, y))\}$$
$$\{\neg G(x, y), P(f(x, y), y)\}$$
$$\{\neg P(x, f(x, y)), \neg P(f(x, y), y), G(x, y)\}$$

The goal states that if two people has the same parents, then they have the same grandparents. Let us first formalize the sentence x and y has the same parents. This can be formalized by $\forall z (P(z, x) \leftrightarrow P(z, y))$ Similarly "x, y having the same grapdparents" can be formalized with the formula $\forall z (G(z, x) \leftrightarrow G(z, y))$. The entire statement is the implication between the two formulas for every x and y. i.e.,

$$\forall x \forall y (\forall z (P(z, x) \leftrightarrow P(z, y)) \rightarrow \forall z (G(z, x) \leftrightarrow G(z, y)))$$

Negate the goal and transform in CNF

$$\neg \forall x \forall y (\forall z (P(z, x) \leftrightarrow P(z, y)) \rightarrow \forall z (G(z, x) \leftrightarrow G(z, y)))$$
$$\exists x \exists y (\forall z (P(z, x) \leftrightarrow P(z, y)) \land \exists z \neg (G(z, x) \leftrightarrow G(z, y)))$$
$$\forall z (P(z, a) \leftrightarrow P(z, b)) \land \neg (G(c, a) \leftrightarrow G(c, b)))$$
$$\{\neg P(z, a), P(z, b)\}, \{\neg P(z, b), P(z, a)\}, \{\neg G(c, a), \neg G(c, b)\}, \{G(c, a), G(c, b)\},$$

$\square$

Exercise 7 (5 points) *Find a formula that counts the number of models of the formula:*

$$\forall xy (A(x) \land A(y) \rightarrow \exists z R(x, z, y))$$

on a domain of n elements.

Solution Consider the case in which A is interpreted in a set $A^{\mathcal{I}}$ of a elements. Then for every pair (i, j) in $A^{\mathcal{I}} \times A^{\mathcal{I}}$ there should be at least an elements k such that $(i, k, j) \in R^{\mathcal{I}}$,

There sho $R(i, z, j)$ must not be empty so there are $2^n - 1$ possibilities. For all the other pairs (i, j) $R(i, z, j)$ can be freely interpreted This means that we have 2^n possibilities. In total we have

$$\sum_{a=0}^{n} \binom{n}{a} (2^n - 1)^{a^2} (2^n)^{n^2 - a^2}$$

$\square$