Exercise 1. Consider the Cauchy problem
\[
\begin{aligned}
    y' &= \frac{y^2 - 4}{t}, \\
y(1) &= 0.
\end{aligned}
\]

i) Determine the solution.
ii) Determine the domain of definition \([a, b]\) of the solution and the limits of \(y(t)\) when \(t \to a\) and \(t \to b\).

Exercise 2. Let
\[
D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 + xy\}.
\]

i) Show that \(D \neq \emptyset\) is the zero set of a submersion.
ii) Is \(D\) compact?
iii) Determine, if any, points of \(D\) at min/max distance to \(\mathbf{0}\).

Exercise 3. Let
\[
D := \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2)^{1/4} \leq z \leq 2 - x^2 - y^2\}.
\]

i) Draw \(D \cap \{x = 0\}\) and deduce a figure for \(D\).
ii) Compute the volume of \(D\).

Exercise 4. Let
\[
v(x, y) := e^{-y} (y \cos x + x \sin x), \quad (x, y) \in \mathbb{R}^2.
\]

i) Determine all possible \(u = u(x, y)\) in such a way that \(f(x + iy) := u(x, y) + iv(x, y)\) be \(\mathbb{C}\)-differentiable on \(\mathbb{R}^2\).
ii) Express the \(f\) found at i) as function of complex number \(z\), that is \(f = f(z)\).

Exercise 5. State the Green formula. Let \(f \in \mathcal{C}(\mathbb{R}^2)\) with \(\partial_i f, \partial_j (\partial_i f) \in \mathcal{C}(\mathbb{R}^2)\), for all \(i, j = 1, 2\).
Prove that
\[
\oint_{\partial D} f \nabla f = 0.
\]
Exercise 6. Consider the equation
\[ y' = \frac{e^y - 1}{t}, \quad t \neq 0. \]
i) Determine the constant solutions.
ii) Determine the solution of the Cauchy problem \( y(1) = -1 \).
iii) Determine in particular the domain of definition \( [a, b] \) of the solution and its limits when \( t \to a+ \) and \( t \to b- \).

Exercise 7. Let \( D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1, \ y^2 + z = 1 \} \).
i) Show that \( D \neq \emptyset \) is the zero set of a submersion \((g_1, g_2)\).
ii) Is \( D \) compact?
iii) Determine, if any, points of \( D \) at min/max distance to \( \mathbb{O} \).

Exercise 8. Let \( D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq 1 - y^2 \} \).
i) Draw \( D \cap \{x = 0\} \) and \( D \cap \{y = 0\} \). Is \( D \) invariant by some rotation? Justify your answer. Draw \( D \) as best as you can.
ii) Compute the volume of \( D \).

Exercise 9. Let \( \vec{F} := \left( \frac{ax^2 + by^2}{(x^2 + y^2)^2}, \frac{xy}{(x^2 + y^2)^2} \right) \) on \( D = \mathbb{R}^2 \setminus \{(0, 0)\} \). Here \( a, b \in \mathbb{R} \) are constants.
i) Determine all possible values for \( a, b \) in such a way \( \vec{F} \) be irrotational on \( D \).
ii) Determine values of \( a, b, c \) in such a way \( \vec{F} \) be conservative on \( D \), in this case determining also all the possible potentials.

Exercise 10. What are the Cauchy–Riemann equations (or conditions)? State precisely. Then, let \( f = u + iv \) (\( u = \text{Re} \ f \) and \( v = \text{Im} \ f \)) be a \( \mathbb{C} \) differentiable function on the entire plane \( \mathbb{C} \). Assume that also \( \overline{f} = u - iv = u + i(-v) \) is \( \mathbb{C} \) differentiable on \( \mathbb{C} \). What conclusion can you draw on \( f \)?
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Exercise 11. Consider the second order equation
\[ y'' - 2y' + y = e^{2t}. \]
   i) Determine the general integral.
   ii) Solve the Cauchy problem \( y(0) = 1, y'(0) = 0 \).
   iii) For which \( a \in \mathbb{R} \) there exists a solution such that \( y(0) = 0 \) and \( y(1) = a \)?

Exercise 12. Let
\[ f(x, y) := (x^2 + y^2)^3 - x^4 + y^4, \quad (x, y) \in \mathbb{R}^2. \]
   i) Compute, if it exists, \( \lim_{(x, y) \to \infty} f(x, y) \).
   ii) Discuss existence of min/max of \( f \) on \( \mathbb{R}^2 \) and find the eventual min/max points of \( f \). What about \( f(\mathbb{R}^2) \)?

Exercise 13. Let \( D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 \leq z \leq 4 - 3(x^2 + 2y^2)\} \).
   i) Draw the set \( D \). Someone says: ”\( D \) is a rotation volume with respect to the \( z \)-axis”. Is it true or false?
   ii) Compute the volume of \( D \).

Exercise 14. Let
\[ u(x, y) := x^2 + y^2. \]
   i) Determine, if any, \( v = v(x, y) \) in such a way that \( f(x + iy) := u(x, y) + iv(x, y) \) be \( \mathbb{C} \)-differentiable on \( \mathbb{C} \).
   ii) For the \( f \) you found at i), write \( f = f(z) \) as function of \( z \in \mathbb{C} \).

Exercise 15. State the Lagrange multipliers theorem. Then, consider a curve \( y = f(x) \) defined by a function \( f = f(x) : \mathbb{R} \to \mathbb{R}, f \in \mathcal{C}^1(\mathbb{R}) \). Let \( P = (a, b) \) a point in the cartesian plane not belonging to the curve \( y = f(x) \). Prove that if \( Q \) is a point of the curve \( y = f(x) \) where the distance to \( P \) is minimum, then the segment \( P - Q \) is perpendicular to the tangent to \( f \).
Exercise 16. Consider the differential equation
\[ y' = \frac{t - ty^2}{y + t^2y}. \]

i) Show that it is a separable variables equation and determine all possible constant solutions.
ii) Determine the solution of the Cauchy Problem with passage condition \( y(0) = 2 \).

Exercise 17. Let \( \Gamma \subset \mathbb{R}^3 \) the set described by equations
\[ \Gamma : \begin{cases} x^2 + y^2 = 1, \\ x^2 + z^2 = xz + 1. \end{cases} \]

i) Show that \( \Gamma \neq \emptyset \) is the zero set of a submersion on \( \Gamma \).
ii) Is \( \Gamma \) compact? Justify your answer.
iii) Determine points of \( \Gamma \) at minimum/maximum distance to \((0, 0, 0)\) (if any).

Exercise 18. Let \( D := \{(x, y, z) \in \mathbb{R}^3 : 1 - (x^2 + y^2) \leq z \leq \sqrt{1 - (x^2 + y^2)}\} \).

i) Draw \( D \cap \{y = 0\} \) and deduce a figure for \( D \).
ii) Compute the volume of \( D \).

Exercise 19. Let \( f = u + iv \) where
\[ u(x, y) := ax^2 + bxy + cy^2, \quad v(x, y) := xy, \quad x + iy \in \mathbb{C}. \]
\( (a, b, c \text{ are real constant}) \)

i) Determine all possible \( a, b, c \) such that \( f \) be holomorphic on \( \mathbb{C} \).
ii) For values found at i), determine the analytical expression for \( f = f(z) \) in terms of variable \( z \in \mathbb{C} \).

Exercise 20. Let \( \vec{a}_1, \ldots, \vec{a}_N \in \mathbb{R}^d \) be \( N \) fixed vectors, \( \vec{a}_i \neq \vec{a}_j \) for \( i \neq j \). Define
\[ f(\vec{x}) := \sum_{j=1}^{N} \|\vec{x} - a_j\|^2. \]

Discuss the problem of determining, if any, points of min/max for \( f \) on \( \mathbb{R}^d \). Justify carefully, state all general facts you use.
Exercise 21. Consider the equation \( y' = y \log y \).  

i) Determine, if any, all constant solutions.

ii) Solve the Cauchy problem with \( y(0) = a \).

iii) Determine, if any, values of \( a \) such that \( \lim_{t \to +\infty} y(t) = 0 \).

Exercise 22. Let \( D := \{(x,y,z) \in \mathbb{R}^3 : x^2 = y^2 + z^2, \quad x^2 + y^2 = xy + 1 \} \).

i) Show that \( D \) is the zero set of a submersion on \( D \) itself.

ii) Is \( D \) compact? Justify your answer.

iii) Determine, if any, the points of \( D \) at the min / max distance to the origin.

Exercise 23. Consider the vector field \( \mathbf{F}(x,y) := \left( \frac{ax + by}{\sqrt{x^2 + y^2}}, \frac{cx + dy}{\sqrt{x^2 + y^2}} \right) \), \( (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \).

i) Find all possible values of \( a, b, c, d \in \mathbb{R} \) such that \( \mathbf{F} \) is irrotational.

ii) Find all possible values for \( a, b, c, d \) such that \( \mathbf{F} \) is conservative. For such values, determine the potentials of \( \mathbf{F} \).

Exercise 24. Let \( D := \{(x,y,z) \in \mathbb{R}^3 : x^2 + 4y^2 - z^2 \leq 1, \quad 0 \leq z \leq 1 \} \). Draw \( D \) and calculate its volume.

Exercise 25. Let \( f = u + iv \) be holomorphic on \( D \subset \mathbb{C} \). Define \( g(z) := \overline{f(z)} \), \( z \in \overline{D} := \{w \in \mathbb{C} : \overline{w} \in D \} \).

i) Express real and imaginary part of \( g \) in terms of real and imaginary parts \( u \) and \( v \) of \( f \).

ii) Use i) to discuss whether \( g \) is holomorphic on \( \overline{D} \) or not.
Exercise 26. Consider the differential equation
\[ y'' + 2y' + y = t + 1. \]

i) Determine the general integral of the equation.
ii) Solve the Cauchy problem \( y(0) = 0, y'(0) = 1. \)
iii) Discuss the boundary value problem \( y(0) = 0, y(1) = 0. \)

Exercise 27. Let
\[ D := \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, y^2 + (z - 2)^2 = 1 \}. \]

i) Show that \( D \neq \emptyset \) and it is the zero set of a submersion.
ii) Is \( D \) compact? Prove or disprove.
iii) Find points of \( D \) at min/max distance to \( \bar{0} \).

Exercise 28. Let
\[ D := \{ (x, y) \in \mathbb{R}^2 : x \geq 1, x^3 \leq y \leq 3 \}. \]

i) Draw \( D \).
ii) By using the change of variables \( u = y - x^3, v = y + x^3 \), compute the integral
\[ \int_D x^2(y - x^3)e^{y+x^3} \, dx \, dy. \]

Exercise 29. Let \( v(x, y) := y^3 - 3x^2y + 4xy - x, (x, y) \in \mathbb{R}^2. \) Determine all possible \( u = u(x, y) \) such that
\[ f(x + iy) := u(x, y) + iv(x, y), \]
be holomorphic on \( \mathbb{C} \). What is \( f(z) \) as a function of \( z \)?

Exercise 30. What does it mean that a set \( C \subset \mathbb{R}^d \) is closed? What is the Cantor characterization of closed sets?

Given a generic set \( S \subset \mathbb{R}^d \), we define the frontier of \( S \) as the set
\[ \partial S := \{ \tilde{x} \in \mathbb{R}^d : \forall r > 0, B(\tilde{x}, r) \cap S \neq \emptyset, B(\tilde{x}, r) \cap S^c \neq \emptyset \}. \]
Is \( \partial S \) always closed? Justify your answer providing a proof if yes, a counterexample if no.
Exam Simulation

Exercise 31. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\sinh \frac{1}{z} = 0.$$ 

Exercise 32. Consider the set (surface)

$$D := \{(x, y, z) \in \mathbb{R}^3 : x^2 - 2xy + y^2 - x + y = 0\}.$$ 

Determine, if any, points of $D$ at min/max distance to the point $(1, 2, -3)$. Justify carefully the method you use.

Exercise 33. Let

$$D := \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq \frac{1}{\cosh(x^2 + y^2)}\}.$$ 


ii) Determine the volume of $D$.

iii) Determine for which values of $\alpha$ the following integral has a finite value:

$$\int_D e^{\alpha(x^2+y^2)} \, dx dy dz.$$ 

Exercise 34. Let

$$u(x, y) := x^3 + axy^2, \quad v(x, y) := bx^2 y - y^3, \quad (x, y) \in \mathbb{R}^2.$$ 

i) Determine $a, b \in \mathbb{R}$ in such a way that $f(x + iy) := u(x, y) + iv(x, y)$ be holomorphic on $\mathbb{C}$.

ii) For values of $a, b$ found at i), express $f$ as a function of the complex variable $z$.

Exercise 35. Consider a Newton equation of type

$$m y'' = F(y).$$

Suppose that force $F$ admits a potential, that is $F(y) = f'(y)$. Define the potential energy

$$E(y, v) := \frac{1}{2}mv^2 - f(y).$$

i) Prove that $E(y, y') = E(y(t), y'(t))$ is a constant function of $t$. Deduce that $y$ solves a first order separable variables equation.

ii) Assume $m = 1$ and let $F(y) = -2y - 3y^2$ (elastic force plus viscosity). Determine the motion of the mass with $y(0) = -2, y'(0) = \sqrt{8}$. 

Exercise 36. Consider the equation
\[ y'' = -9y + 6\sin(3t). \]
This equation represents the motion of a unitary mass particle subject to an elastic force (constant of elasticity \( k = -9 \)) and to an external force \( F(t) = 6\sin(3t) \).

i) Determine the general solution of the equation.
ii) Solve the Cauchy problem \( y(0) = y'(0) = 0 \).
iii) Describe the long time (that is \( t \to +\infty \)) of the general solution. In particular: are there solutions for which \( \exists \lim_{t \to +\infty} y(t) \)? are there solutions which are bounded, that is \( |y(t)| \leq M \) for all \( t \geq 0 \) for some constant \( M \)? Justify carefully.

Exercise 37. Let
\[ f(x, y) := 3xy + x^2y + xy^2, \quad (x, y) \in D := \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ 0 \leq y \leq 1 - x\}. \]

ii) Discuss the problem of determining min/max (if any) of \( f \) on \( D \).

Exercise 38. Let \( a, b, c, d \in \mathbb{R} \) and
\[ \vec{F}(x, y) := \begin{cases} \frac{ax + by}{(x^2 + y^2)^2}, & (x, y) \in D := \mathbb{R}^2 \setminus \{(0, 0)\}. \end{cases} \]

i) Determine \( a, b, c, d \in \mathbb{R} \) in such a way that \( \vec{F} \) be irrotational on \( D \).
ii) Determine \( a, b, c, d \) such that \( \vec{D} \) be conservative on \( D \). For these values (if any), determine all possible potentials of \( \vec{F} \) on \( D \).
iii) Let \( y = \gamma(t) \subset D \) be the segment joining \((1, 0)\) to \((0, 2)\). For \((a, b, c, d) = (2, 0, 0, 2)\) compute
\[ \int_{\gamma} \vec{F}. \]

Exercise 39. Let \( D := \{(x, y, z) \in \mathbb{R}^3 : 1 - (x^2 + z^2) \leq y \leq \sqrt{1 - (x^2 + z^2)}\} \).

i) Draw \( D \). Is \( D \) a rotation solid?
ii) Compute the volume of \( D \).

Exercise 40. Let \( f = u + iv : \mathbb{C} \to \mathbb{C} \) be a \( \mathbb{C} \)–differentiable function. What are the Cauchy-Riemann equations? How are these equations related to \( \mathbb{C} \)–differentiability of \( f \)? Write a precise statement.

Discuss the following questions:

i) Assume that \( \text{Re} \ f \) or \( \text{Im} \ f \) is constant. What can be drawn on \( f \)?
ii) Assume that \( |f| \) is constant. What can be drawn on \( f \) (hint: \( |f|^2 = u^2 + v^2 \equiv k \ldots \))
Exercise 41. Consider the equation

\[ y' = y(y^2 + 1). \]

i) Determine the general integral of the equation.
ii) Determine the solution of the Cauchy problem \( y(0) = 1 \).

Exercise 42. Let \( D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, x + y + z = 1\} \).

i) Show that \( D \) is the zero set of a submersion.
ii) Is \( D \) compact?
iii) Determine, if any, min/max points for \( f(x, y, z) = x^2 - x + y^2 + xy + yz - y \) on \( D \).

Exercise 43. Let \( D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2x - \sqrt{x^2 + y^2}\} \).

ii) Compute the area of \( D \).

Exercise 44. Let \( u(x, y) := x^5 - 10x^3y^2 + 5xy^4 \).

i) Determine all possible \( v = v(x, y) \) in such a way that \( f(x+iy) := u(x, y) + iv(x, y) \) be holomorphic on \( \mathbb{C} \).
ii) For the \( f \) found at i), determine the analytical expression of \( f(z) \) as function of \( z \in \mathbb{C} \).

Exercise 45. What does it mean that a set \( S \subset \mathbb{R}^d \) is open? Let \( \tilde{f} : \mathbb{R}^d \rightarrow \mathbb{R}^m \) be a continuous function on \( \mathbb{R}^d \). Prove that the following property holds:

\[ \tilde{f}^{-1}(S) \text{ is open, } \forall S \subset \mathbb{R}^m \text{ open.} \]

(recall that \( \tilde{f}^{-1}(S) = \{\tilde{x} \in \mathbb{R}^d : \tilde{f}(\tilde{x}) \in S\} \)). Hint: suppose that for some \( S \) open, \( \tilde{f}^{-1}(S) \) is not open...
Solutions

Exercise 1. i) We have a separable vars eqn, \( y' = a(t)f(y) \) where \( f(y) = y^2 - 4 \) and \( a(t) = \frac{1}{t} \). Since \( a \in \mathbb{C} \) and \( f \in \mathbb{C}^1 \). According to a general result, solutions of the differential equation are either constant or not, in this last case can be determined by separation of variables. Constant solutions are \( y \equiv C \) iff \( y' \equiv 0 = \frac{C^2 - 4}{t} \) iff \( C^2 = 4 \), iff \( C = \pm 2 \). Since the solution of CP is \( y(1) = 0 \), certainly \( y \) is not constant (otherwise \( y \equiv \pm 2 \)). Thus, the solution of proposed CP can be determined by separation of vars:

\[
y' = \frac{y^2 - 4}{t}, \quad \iff \quad \frac{y'}{y^2 - 4} = \frac{1}{t}, \quad \iff \quad \int \frac{y'}{y^2 - 4} \, dt = \int \frac{1}{t} \, dt + C = \log |t| + C.
\]

Now,

\[
\int \frac{y'}{y^2 - 4} \, dt = \int \frac{1}{u^2 - 4} \, du = \int \frac{1}{4} \left( \frac{1}{u - 2} - \frac{1}{u + 2} \right) \, du = \frac{1}{4} \log \left| \frac{u - 2}{u + 2} \right| = \frac{1}{4} \log \left| \frac{y(t) - 2}{y(t) + 2} \right|.
\]

In this way, we have the implicit form for the solution

\[
\frac{1}{4} \log \left| \frac{y(t) - 2}{y(t) + 2} \right| = \log |t| + C.
\]

Imposing the initial/ passage condition we have

\[
\frac{1}{4} \log 1 = \log |1| + C, \quad \iff \quad C = 0.
\]

Thus, for the solution of the CP we have

\[
\frac{1}{4} \log \left| \frac{y(t) - 2}{y(t) + 2} \right| = \log |t|, \quad \iff \quad \frac{y(t) - 2}{y(t) + 2} = t^4, \quad \iff \quad \frac{y(t) - 2}{y(t) + 2} = \pm t^4.
\]

Since \( y(1) = 0 \) we have \(-1 = \pm 1^4 = \pm 1\), thus the appropriate sign is \(-\), and

\[
\frac{y(t) - 2}{y(t) + 2} = -t^4, \quad \iff \quad y(t) - 2 = -t^4(y(t)+2), \quad \iff \quad y(t)(1+t^4) = 2(1-t^4), \quad \iff \quad y(t) = \frac{2 - t^4}{1 + t^4}.
\]

ii) The formula found at i) for \( y \) is defined for every \( t \in \mathbb{R} \). However, since the equation does not make any sense at \( t = 0 \), the solution must be defined either on \((0, \infty) \) or \([0, +\infty) \). Since \( y \) is defined at \( t = 1 \) we conclude that the domain of the solution is \([0, +\infty) \). About limits,

\[
\lim_{t \to 0} y(t) = 2, \quad \lim_{t \to +\infty} y(t) = -2. \quad \Box
\]

Exercise 2. i) For instance \((0, 0, 0) \in D\) iff \( z^2 = 1 \), thus \((0, 0, \pm 1) \in D\) and \( D = \emptyset \). \( D \) is also the zero set of \( g(x, y, z) := x^2 + y^2 + z^2 - xy - 1 \). This is a submersion on \( D \) iff

\[ \nabla g \neq 0, \text{ on } D. \]

We have

\[
\nabla g = \bar{0}, \quad \iff \quad \begin{cases} 2x - y = 0, \\ 2y - x = 0, \\ 2z = 0, \end{cases} \quad \iff \quad (x, y, z) = (0, 0, 0) \not\in D,
\]

from which it follows that \( g \) is a submersion on \( D \).
ii) Certainly, \(D = \{ g = 0 \}\) is closed \((g \in \mathcal{C})\). Is it also bounded? We may see this by using spherical coordinates:
\[
\begin{align*}
x &= \rho \cos \theta \sin \varphi, \\
y &= \rho \sin \theta \sin \varphi, \\
z &= \rho \cos \varphi.
\end{align*}
\]
Then, if \((x, y, z) \in D\) we have
\[
\rho^2 = 1 + \rho^2 \cos \theta \sin (\sin \varphi)^2 = 1 + \frac{1}{2} \rho^2 \sin (2\theta) (\sin \varphi)^2 \leq 1 + \rho^2 \frac{2}{2},
\]
from which
\[
\frac{\rho^2}{2} \leq 1, \quad \iff \quad \rho^2 = \| (x, y, z) \|^2 \leq 2.
\]
Thus, \(D\) is bounded, hence compact.

iii) We have to minimize/maximize \(f(x, y, z) = \sqrt{x^2 + y^2 + z^2}\) or, which is equivalent (same min/max points), \(f(x, y, z) = x^2 + y^2 + z^2\). According to i), we are in condition to apply Lagrange multipliers theorem. According to this result, at min/max points \((x, y, z) \in D\) we have
\[
\nabla f = \lambda \nabla g, \quad \iff \quad \text{rk} \begin{bmatrix} \nabla f(x, y, z) \\ \nabla g(x, y, z) \end{bmatrix} = \text{rk} \begin{bmatrix} 2x & 2y & 2z \\ 2x - y & 2y - x & 2z \end{bmatrix} < 2.
\]
This happens iff all \(2 \times 2\) subdeterminants equal 0:
\[
\begin{align*}
2x(2y - x) - 2y(2x - y) &= 0, \\
2x2z - 2z(2x - y) &= 0, \\
2y2z - 2z(2y - x) &= 0,
\end{align*}
\]
The first leads to \(y = \pm x\), the second \(y = 0\) (then \(x = 0\)) or \(z = 0\). That is we have points \((0, 0, z)\) and \((x, \pm x, 0)\). Now
\begin{itemize}
\item \((0, 0, z) \in D\) iff \(z^2 = 1\), that is \((0, 0, \pm 1)\).
\item \((x, \pm x, 0) \in D\) iff \(2x^2 = 1 \pm x^2\). If \(+\), \(2x^2 = 1 + x^2\), we get \(x = \pm 1\), that is points \((1, 1, 0)\) and \((-1, -1, 0)\). It \(-\), \(x^2 = \frac{1}{3}\), thus points \(\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)\) and \(\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)\).
\end{itemize}
From these we see that \((1, 1, 0)\) and \((-1, -1, 0)\) are points at max distance to \(\vec{0}\) while \(\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)\) and \(\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)\) are points of \(D\) at min distance to \(\vec{0}\). 

\textbf{Exercise 3.} i) \(D \cap \{x = 0\} = \{(0, y, z) : \sqrt{|y|} \leq z \leq 2 - y^2\}\). Thus, in the plane \(yz, D \cap \{x = 0\}\) is the plane region between \(z = \sqrt{|y|}\) and the parabola \(z = 2 - y^2\) (see figure). Since \((x, y, z) \in D\) depends on \((x, y)\) through \(x^2 + y^2\), \(D\) is invariant by rotations around the \(z\)-axis.
ii) We have

\[ \lambda_3(D) = \int_D 1 \, dx \, dy \, dz = \int_{\sqrt{x^2+y^2} \leq 2-(x^2+y^2)} 1 \, dx \, dy \, dz = \int_{\sqrt{x^2+y^2} \leq 2-(x^2+y^2)} 2-(x^2+y^2) \, dx \, dy \]

\[ = \int_{\sqrt{x^2+y^2} \leq 2-(x^2+y^2)} (2 - (x^2 + y^2) - \sqrt{x^2 + y^2}) \, dx \, dy \]

\[ = CV \int_{\sqrt{\rho^2 - \rho^2}, \theta \in [0,2\pi]} \left( \sqrt{\rho} - (2 - \rho^2) \right) \rho \, d\rho \, d\theta. \]

Now, \( \sqrt{\rho} \leq 2 - \rho^2 \) might be hard to solve. However, here \( \rho \geq 0 \); \( \sqrt{\rho} \) is increasing while \( 2 - \rho^2 \) decreases. Since \( \rho = 1 \) they are equal, we conclude that \( \sqrt{\rho} \leq 2 - \rho^2 \) iff \( 0 < \rho < 1 \). We can continue previous chain by the RF:

\[ = 2\pi \left( 1 - \frac{1}{4} - \frac{3}{10} \right) = \frac{7\pi}{10}. \quad \Box \]

**Exercise 4.** i) \( f = u + iv \) is \( \mathbb{C} \)–differentiable on \( \mathbb{C} \) iff \( u, v \) are \( \mathbb{R} \)–differentiable on \( \mathbb{R}^2 \) and \( u, v \) fulfill the CR conditions. Clearly \( v \) is differentiable. Thus we have to look at \( u = u(x, y) \) \( \mathbb{R} \)–differentiable such that

\[ \begin{align*}
\partial_x u &= \partial_y v = -e^{-y}(y \cos x + x \sin x) + e^{-y} \cos x, \\
\partial_y u &= -\partial_x v = -e^{-y}(-y \sin x + \sin x + x \cos x). 
\end{align*} \]

From the first equation,

\[ u(x, y) = \int \partial_x u(x, y) \, dx + c(y) = -e^{-y}(y \sin x - x \cos x) + c(y). \]

We have

\[ \partial_y u = e^{-y}(y \sin x - x \cos x) - e^{-y} \sin x + c'(y) = e^{-y}(y \sin x - x \cos x + \sin x) + c'(y) \]

thus \( \partial_y u = -\partial_x v \) iff \( c'(y) = 0 \), that is \( c(y) \) is constant. We conclude that

\[ u(x, y) = -e^{-y}(y \sin x - x \cos x) + c + e^{-y}(y \cos x + x \sin x). \]

ii) We have

\[ f = u + iv = -e^{-y}(y \sin x - x \cos x) + i e^{-y}(y \cos x + x \sin x) \]

\[ = e^{-y}(y(-\sin x + i \cos x) + x(\cos x + i \sin x)) \]

\[ = e^{-y}(iye^{ix} + xe^{ix}) \]

\[ = e^{ix-y}(iy + x) = e^{i(x+iy)}(x + iy) = e^{iz}. \quad \Box \]
**Exercise 5.** Let $\vec{F} := f \nabla f = (f \partial_x f, f \partial_y f) =: (F_1, F_2)$. According to Green formula,

$$
\int_{\partial D} f \nabla f = \int_D \vec{F} = \int_D (\partial_y F_1 - \partial_x F_2) 
\text{d}x \text{d}y.
$$

Now, since

$$
\partial_x F_1 = \partial_y (f \partial_x f) = \partial_y f \partial_x f + f \partial_y x f, \quad \partial_x F_2 = \partial_x (f \partial_y f) = \partial_x f \partial_y f + f \partial_{xy} f
$$

we easily deduce that $\partial_y F_1 - \partial_x F_2 \equiv 0$ being $f \in C^2(\mathbb{R}^2)$. 

**Exercise 6.** i) We have a separable variables equation $y' = a(t) f(y)$ where $a(t) = \frac{1}{t}$ and $f(y) = e^y - 1$. $y \equiv C$ is a solution iff $0 = \frac{1}{t}(e^C - 1)$, iff $e^C = 1$ that is, $C = 0$. There is a unique constant solution, $y \equiv 0$.

ii) Since $y(1) = -1$, $y$ is not constant. Furthermore, since $a \in C$ and $f \in C^1$, the solution can be found by separating vars:

$$
y' = \frac{e^y - 1}{t}, \quad \iff \quad \frac{y'}{e^y - 1} = \frac{1}{t}, \quad \iff \quad \int \frac{y'(t)}{e^y(t) - 1} 
\text{d}t = \int \frac{1}{t} 
\text{d}t + c = \log |t| + c.
$$

On the lhs

$$
\int \frac{y'(t)}{e^y(t) - 1} 
\text{d}t = \int \frac{du}{e^u - 1} \quad \text{with } u=e^t, \quad u=\log v, \quad du=dv/v
$$

$$
= \log |v - 1| - \log |v| = \log |\frac{e^t - 1}{e^t}|
$$

Thus,

$$
\log |\frac{e^{y(t)} - 1}{e^{y(t)}}| = \log |1 - \frac{1}{e^{y(t)}}| = \log |t| + c.
$$

By imposing the initial condition, we find

$$
c = \log (e - 1),
$$

and

$$
\left| 1 - \frac{1}{e^{y(t)}} \right| = (e - 1)|t|, \quad \iff \quad 1 - \frac{1}{e^{y(t)}} = \pm (e - 1)t.
$$

A check with the initial condition shows that the sign is $-$, thus

$$
1 - \frac{1}{e^{y(t)}} = -(e - 1)t, \quad \iff \quad 1 + (e - 1)t = \frac{1}{e^{y(t)}} = e^{-y(t)}, \quad \iff \quad y(t) = -\log (1 + (e - 1)t).
$$

iii) The domain of definition for the solution is

$$
1 + (e - 1)t > 0, \quad \iff \quad t > \frac{-1}{e - 1}.
$$

However, since at $t = 0$ the solution cannot be defined (because the equation does not make sense at $t = 0$), and the solution is defined on an interval, we conclude that the domain is $]0, +\infty[$. We have

$$
\lim_{t \to 0^+} y(t) = \log 1 = 0, \quad \lim_{t \to +\infty} y(t) = -\infty. \quad \square
$$
Exercise 7. i) Point \((0, y, 0) \in D\) iff \(y^2 = 1\) and \(y^2 = 1\), that is \(y = \pm 1\), so \((0, \pm 1, 0) \in D\). \(D\) is the zero set of \((g_1, g_2) = (x^2 + y^2 - z^2 - 1, y^2 + z - 1)\). According to the Definition, 
\[
(g_1, g_2) \text{ is a submersion on } D \iff \mathrm{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} = 2 \text{ on } D.
\]
Since this is a \(2 \times 3\) matrix, its rank is \(< 2\) iff all \(2 \times 2\) sub determinant equal 0, or
\[
\begin{align*}
4xy &= 0, \\
2x &= 0, \\
2y(-1 + 2z) &= 0,
\end{align*}
\begin{align*}
\iff & \begin{cases} x = 0, \\
y(1 + 2z) = 0.
\end{cases}
\end{align*}
\]
Now,
\begin{itemize}
  \item (0, 0, z) \in D iff \(-z^2 = 1\) and \(z = 1\), impossible;
  \item (0, y, \(-\frac{1}{2}) \in D iff \(y^2 = \frac{5}{3}\) and \(y^2 = \frac{3}{2}\), impossible.
\end{itemize}
Conclusion: at no point of \(D\) the rank of the matrix \(\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix}\) is less than 2, thus \((g_1, g_2)\) is a submersion on \(D\).

ii) \(D\) is certainly closed being defined by equations involving continuous functions. Is it also bounded? From the second equation \(y^2 = 1 - z\), thus \(y = \pm \sqrt{1-z}\) for \(z \leq 1\). Plugging this into the first equation
\[
x^2 = z^2 - (1 - z) + 1 = z^2 + 1 = z^2 + z = z(\gamma + 1), \quad \implies \ x = \pm \sqrt{z^2 + z} \text{ for } z \leq 0 \lor \ z \geq 1.
\]
In particular, for \(z \leq 0\) points
\[
(\pm \sqrt{z^2 + z}, \pm \sqrt{1-z}, z) \in D, \ \forall z \leq 0.
\]
These points are unbounded because
\[
\| (\pm \sqrt{z^2 + z}, \pm \sqrt{1-z}, z) \|^2 = z^2 + z + (1 - z) + z^2 = 2z^2 + 1 \longrightarrow +\infty, \ z \longrightarrow -\infty.
\]
We conclude that \(D\) is unbounded.

iii) By ii) \(D\) is closed and unbounded. We have to min/max \(\sqrt{x^2 + y^2 + z^2}\) or, equivalently, \(f := x^2 + y^2 + z^2\), which is continuous on \(D\) and such that \(\lim_{z \to 0} f = +\infty\). We conclude \(f\) has no max point on \(D\) while it has min points. By i) and according to the Lagrange multipliers theorem, at min point we must have
\[
\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \implies \ \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} < 3.
\]
This happens iff the determinant of the previous jacobian matrix equals 0, that is
\[
8xy(x + z) = 0, \quad \iff \ x = 0, \lor \ y = 0, \lor \ z = -x.
\]
This leads to points \((0, y, z), (x, 0, z)\) and \((x, y, -x)\). Now,
\begin{itemize}
  \item (0, y, z) \in D iff \(y^2 - z^2 = 1\) and \(y^2 + z = 1\). From these, \(z^2 + z = 0\) that is, \(z = 0\) or \(z = -1\), thus we have points \((0, \pm 1, 0)\) and \((0, \pm \sqrt{2}, -1)\);  
\end{itemize}
• \((x, 0, z) \in D\) iff \(x^2 - z^2 = 1\) and \(z = 1\), that is \((\pm \sqrt{2}, 0, 1)\).
• \((x, y, -x) \in D\) iff \(x^2 + y^2 - x = 1\) and \(y^2 - x = 1\), that is \(y^2 = 1\) and \(x = 0\), from which we have points \((0, \pm 1, 0)\).

**Conclusion:** min points are among \((0, \pm 1, 0)\), \((0, \pm \sqrt{2}, -1)\), \((\pm \sqrt{2}, 0, 1)\), and clearly those at min distance to \(\tilde{0}\) are \((0, \pm 1, 0)\). \(\square\)

**Exercise 8.** i) Figures are straightforward. \(D\) is not invariant by any rotation because one part of the inequality \((z \geq x^2 + y^2)\) is invariant by rotations around \(z\)-axis while the second part \((z \leq 1 - y^2)\) is not.

ii) We have

\[
\begin{align*}
\lambda(2) & = \frac{\pi}{2} - (2) \cdot \pi + \frac{\pi}{2} = \frac{\pi}{2} - \frac{\pi}{2} = 0,
\end{align*}
\]

Exercise 9. i) \(\vec{F}\) is irrotational on \(D\) iff

\[
\frac{\partial y}{(x^2 + y^2)^2} ax^2 + by^2 = \frac{\partial x}{(x^2 + y^2)^2} \quad \text{on } D.
\]

By computing derivatives, the previous is equivalent to

\[
\frac{2by(x^2 + y^2) - (ax^2 + by^2)4y}{(x^2 + y^2)^3} = \frac{y(x^2 + y^2) - 4x^2y}{(x^2 + y^2)^3}
\]

that is, iff

\[
(2b - 4a)xy^2 - 2by^3 = -3x^2y + y^3, \quad \iff \quad 2b = -1, -1 - 4a = -3, \quad \iff \quad b = -\frac{1}{2}, a = \frac{1}{2}.
\]

ii) To be conservative, \(\vec{F}\) must be irrotational, hence, necessarily, \(a = \frac{1}{2} = -b\). Thus,

\[
\vec{F} = \left(\frac{1}{2} \frac{x^2 - y^2}{(x^2 + y^2)^2}, \frac{xy}{(x^2 + y^2)^2}\right) = \nabla f,
\]

Looking at the second equation,

\[
f(x, y) = \int \frac{xy}{(x^2 + y^2)^2} \ dy + c(x) = \frac{x}{2} \int 2y(x^2 + y^2)^{-2} \ dy + c(x) = \frac{x (x^2 + y^2)^{-1}}{2} + c(x) = -\frac{1}{2(x^2 + y^2)} + c(x).
\]
Now, by imposing also the first equation we get
\[ c'(x) = 0, \quad \iff \quad c(x) \equiv \text{constant}. \]

Thus, all the potentials of \( \widetilde{F} \) are
\[ f(x, y) = -\frac{1}{2(x^2 + y^2)} + c. \]

**Exercise 10.** About the CR equations see the course notes. Assume that \( f = u + iv \) is \( \mathbb{C} \) differentiable on \( \mathbb{C} \). Then, \( u, v \) are \( \mathbb{R} \) differentiable and the CR eqns hold,
\[
\begin{cases}
\partial_x u = \partial_y v, \\
\partial_y u = -\partial_x v.
\end{cases}
\]
If also \( \overline{f} = u - iv = u + i(-v) \) is \( \mathbb{C} \) differentiable, \( u, -v \) fulfill the CR eqns,
\[
\begin{cases}
\partial_x u = \partial_y (-v) = -\partial_y v, \\
\partial_y u = -\partial_x (-v) = +\partial_x v.
\end{cases}
\]
But then, combining the two CR eqns, we get
\[ \partial_x u = -\partial_y v = -\partial_x u, \quad \iff \quad 2\partial_x u \equiv 0, \]
and, similarly, \( \partial_y u \equiv 0 \). From this \( \nabla u \equiv 0 \) hence \( u \) is constant. Similar conclusion holds for \( v \). We conclude that both \( u \) and \( v \) must be constant, hence also \( f \) must be constant.

Alternative solution: you may remind that we have seen that if a \( \mathbb{C} \) differentiable function is real (or imaginary) valued, then, necessarily, the function must be constant (this is again a consequence of the CR eqns). Now, if both \( f \) and \( \overline{f} \) are \( \mathbb{C} \) differentiable, also \( f + \overline{f} = 2u \) is \( \mathbb{C} \) differentiable. But since \( 2u \) is real valued, \( f + \overline{f} \) (hence \( u \)) must be constant. Same conclusion for \( f - \overline{f} = i2v \), hence \( v \) is constant. \( \square \)

**Exercise 11.** i) The general integral is
\[ y(t) = c_1 w_1(t) + c_2 w_2(t) + u(t), \]
where \((w_1, w_2)\) is a fundamental system of solutions for the homogeneous equation \( y'' - 2y' + y = 0 \) and \( u \) is a particular solution of the equation. The characteristic equation is
\[ \lambda^2 - 2\lambda + 1 = 0, \quad \iff \quad (\lambda - 1)^2 = 0, \quad \iff \quad \lambda_{1,2} = 1. \]
Therefore, the fundamental system of solutions is \( w_1 = e^t, w_2 = te^t \). To compute the particular solution \( u \) we apply the Lagrange formula
\[ u(t) = \left(-\int \frac{w_2}{W} f \, dt\right) w_1 + \left(\int \frac{w_1}{W} f \, dt\right) w_2, \]
where \( W \) is the wronskian
\[ W = \det \begin{bmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{bmatrix} = \det \begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix} = (t+1)e^{2t} - te^{2t} = e^{2t}, \]
and \( f = f(t) = e^{2t} \). Thus
\[
    u(t) = \left( -\int \frac{te'^t}{e^{2t}} dt \right) e^t + \left( \int \frac{e'^t e^{2t}}{e^{2t}} dt \right) (te'^t) = -\left( te'^t - \int e'^t dt \right) e^t + e'te'^t = e^{2t}.
\]

Conclusion: the general integral is
\[
y(t) = c_1 e^t + c_2 te^t + e^{2t}, \quad c_1, c_2 \in \mathbb{R}.
\]

ii) To solve the Cauchy problem we impose the initial conditions \( y(0) = 1 \) and \( y'(0) = 0 \) to the general integral. First notice that
\[
y' = c_1 e^t + c_2 (t + 1) e^t + 2e^{2t},
\]
thus
\[
\begin{align*}
    \begin{cases}
        y(0) = 1, \\
        y'(0) = 0,
    \end{cases} \quad \iff \quad \begin{cases}
        c_1 + 1 = 1, \\
        c_1 + c_2 + 2 = 0,
    \end{cases} \quad \iff \quad \begin{cases}
        c_1 = 0, \\
        c_2 = -2,
    \end{cases}
\end{align*}
\]
and the solution is \( y(t) = -2te^t + e^{2t} \).

iii) Again, we impose the passage conditions
\[
\begin{align*}
    \begin{cases}
        c_1 + 1 = 0, \\
        c_1 e + c_2 e + e^2 = a,
    \end{cases} \quad \iff \quad \begin{cases}
        c_1 = -1, \\
        c_2 = \frac{a - e^2 e}{e},
    \end{cases}
\end{align*}
\]
We conclude that: for every \( a \in \mathbb{R} \) there exists a unique solution to the proposed problem. \( \square \)

**Exercise 12.** i) Clearly \( f(x, 0) = x^6 - x^4 \to +\infty \) for \( |x| \to +\infty \). So, if a limit exists it must be \( +\infty \).
We check this changing coordinates and using polar coords:
\[
f(x, y) = \rho^6 - (\rho \cos \theta)^4 + (\rho \sin \theta)^4 \geq \rho^6 - 2\rho^4 \to +\infty, \quad \text{if} \quad \rho = \|(x, y)\| \to +\infty.
\]

ii) By i) and a consequence of Weierstrass theorem, \( f \) has global minimum on \( \mathbb{R}^2 \) but not any global maximum. Since every point of \( \mathbb{R}^2 \) lies in its interior, according to Fermat theorem (clearly \( \partial_x f = 6x(x^2 + y^2)^2 - 4x^3 \) and \( \partial_y f = 6y(x^2 + y^2)^2 + 4y^3 \) are both continuous on \( \mathbb{R}^2 \), hence \( f \) is differentiable on \( \mathbb{R}^2 \) according to the differentiability test), at min we have \( \nabla f = \mathbf{0} \). Now,
\[
\nabla f = \mathbf{0}, \quad \iff \quad \begin{cases}
    6x(x^2 + y^2)^2 - 4x^3 = 0, \\
    6y(x^2 + y^2)^2 + 4y^3 = 0
\end{cases} \quad \iff \quad \begin{cases}
    x (6(x^2 + y^2)^2 - 4x^2) = 0, \\
    y (6(x^2 + y^2)^2 + 4y^2) = 0
\end{cases}
\]
Now, looking at second equation, we see that either \( y = 0 \) or \( 6(x^2 + y^2)^2 + 4y^2 = 0 \). In the second case we obtain trivially \( x = 0 \) and \( y = 0 \), thus the point \((0, 0)\). Plugging \( y = 0 \) into the first equation we get
\[
x (6x^4 - 4x^2) = 0, \quad \iff \quad x^3 (3x^2 - 2) = 0, \quad \iff \quad x = 0, \; \forall \; x = \pm \sqrt{\frac{2}{3}}.
\]
Thus we have again \((0, 0)\) and two more points \( \left( \pm \sqrt{\frac{2}{3}}, 0 \right) \). Since \( f(0, 0) = 0 \) while
\[
f(\pm \sqrt{\frac{2}{3}}, 0) = \frac{8}{27} - \frac{4}{9} = -\frac{28}{27} < f(0, 0) = 0,
\]
we conclude that \( \left( \pm \sqrt{\frac{2}{3}}, 0 \right) \) are global minimums. Finally, since \( \mathbb{R}^2 \) is connected,
\[
 f(\mathbb{R}^2) = \left[ -\frac{28}{27}, +\infty \right]. \quad \square
\]

**Exercise 13.** ii)
\[
\lambda_3(D) = \int_{x^2 + 2y^2 \leq 4 - 3(x^2 + 2y^2)} 1 \, dxdydz
\]
\[
= R F \int_{x^2 + 2y^2 \leq 4 - 3(x^2 + 2y^2)} \int_{x^2 + 2y^2}^{4 - 3(x^2 + 2y^2)} 1 \, dz \, dxdy
\]
\[
= \int_{x^2 + 2y^2 \leq 4 - 3(x^2 + 2y^2)} 4 \left( 1 - (x^2 + 2y^2) \right) \, dxdy.
\]
Noticed that \( x^2 + 2y^2 \leq 4 - 3(x^2 + 2y^2) \) iff \( x^2 + 2y^2 \leq 1 \), we have
\[
\lambda_3(D) = \int_{x^2 + 2y^2 \leq 1} 4 \left( 1 - (x^2 + 2y^2) \right) \, dxdy.
\]
Changing variables to adapted polar coordinates
\[
x = \rho \cos \theta, \quad \sqrt{2}y = \rho \sin \theta,
\]
we have
\[
\lambda_3(D) = \int_{0 \leq \rho \leq 1, \ 0 \leq \theta \leq 2\pi} 4 \left( 1 - \rho^2 \right) \frac{\rho}{\sqrt{2}} \, d\rho d\theta
\]
\[
= R F \int_{0}^{1} \left( \rho - \rho^3 \right) \, d\rho
\]
\[
= 8\pi \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{4\pi}{\sqrt{2}}. \quad \square
\]

**Exercise 14.** i) Let \( u = x^2 + y^2 \). From CR equations, \( v = v(x, y) \) is such that \( f = u + iv \) is \( \mathbb{C} \)-differentiable iff \( u, v \) are \( \mathbb{R} \)-differentiable and CR equations hold,
\[
\begin{cases}
\partial_x u = \partial_y v, \\
\partial_y u = -\partial_x v.
\end{cases}
\]
Clearly \( u \) is \( \mathbb{R} \)-differentiable. Thus we seek for \( v \) \( \mathbb{R} \)-differentiable such that
\[
\begin{cases}
\partial_x v = -\partial_y u = -2y, \\
\partial_y v = \partial_x u = 2x.
\end{cases}
\]
From the first equation \( v(x, y) = -\int 2y \, dx + c(y) = -2xy + c(y) \). Plugging this into the second equation we have \( \partial_y v = -2x + c'(y) = 2x \), that is \( c'(y) = 4x \), which is impossible since \( c \) does not depend on \( y \). We conclude that such \( v \) does not exist.

ii) Since there is no \( v \) such that \( f = u + iv \) is \( \mathbb{C} \)-differentiable, there is no \( f \) to be found. \quad \square

**Exercise 15.** See notes for the statement. We may formally set the optimization problem in the following way. The set \( y = f(x) \) is also \( f(x) - y = 0 \). Setting \( g(x, y) := f(x) - y \) we see that \( g \) is a submersion on \( \{g = 0\} \). Indeed \( \nabla g = (\partial_x g, \partial_y g) = (f'(x), -1) \neq 0 \), whatever is \( x \). Let now
\[
d(x, y) := (x - a)^2 + (y - b)^2,
\]
the square of distance from \((a, b)\) to \((x, y)\). At minimum \((x, y)\) on the curve, that is \(y = f(x)\), according to Lagrange theorem we have
\[
\nabla d = \lambda \nabla g = \lambda (f'(x), -1).
\]
Since
\[
\nabla d = (2(x - a), 2(y - b)) = 2(x - a, y - b) = 2Q - P,
\]
we have
\[
Q - P = \frac{\lambda}{2} (f'(x), -1).
\]
Now, since the tangent direction to \(y = f(x)\) at point \((x, f(x))\) is \((1, f'(x))\), and clearly \((f'(x), -1) \perp (1, f'(x))\), we have that
\[
Q - P \parallel (f'(x), -1) \perp (1, f'(x)) \parallel \text{tangent to } f,
\]
we obtain the conclusion.

\[\square\]

**Exercise 16.**

i) The equation can be written as
\[
y' = \frac{t}{1 + t^2} \frac{1 - y^2}{y} =: a(t)f(y),
\]
with obvious definition of \(a\) and \(f\). \(y \equiv C\) is a solution iff
\[
0 = y' = \frac{t}{1 + t^2} \frac{1 - C^2}{C}, \quad \iff \quad 1 - C^2 = 0, \quad \iff \quad C = \pm 1.
\]

ii) Since \(y(0) = 2\), \(y\) cannot be constant (otherwise: \(y \equiv \pm 1\) thus, in particular, \(y(0) = \pm 1\) but \(y(0) = 2\)). Therefore, \(y\) can be determined by separation of variables:
\[
\int \frac{y}{1 - y^2} y' \, dt = \int \frac{t}{1 + t^2} \, dt + c = \frac{1}{2} \log(1 + t^2) + c.
\]
Now,
\[
\int \frac{y}{1 - y^2} y' \, dt = \int \frac{u}{1 - u^2} \, du = -\frac{1}{2} \log |1 - u^2| = -\frac{1}{2} \log |1 - y(t)^2|,
\]
hence
\[
-\frac{1}{2} \log |1 - y(t)^2| = \frac{1}{2} \log(1 + t^2) + c, \quad \iff \quad \log |1 - y(t)^2| = -\log(1 + t^2) + c.
\]
(we relabeled \(2c\) by \(c\)). Imposing \(y(0) = 2\),
\[
\log 3 = -\log 1 + c, \quad \iff \quad c = \log 3.
\]
Therefore
\[
|1 - y(t)^2| = \frac{3}{1 + t^2},
\]
that is
\[
1 - y(t)^2 = \pm \frac{3}{1 + t^2}.
\]
When \(t = 0\) lhs is \(-3\), thus sign is \(-\) and
\[
y(t)^2 = 1 + \frac{3}{1 + t^2}, \quad \iff \quad y(t) = \pm \sqrt{1 + \frac{3}{1 + t^2}},
\]
and, again by imposing \( y(0) = 2 \), we see that sign is +. \( \square \)

**Exercise 17.** i) We have \((x, y, 0) \in \Gamma\) iff \(x^2 + y^2 = 1 \) and \(x^2 = 1\), thus \(x = \pm 1\) and \(y^2 = 0\), hence \((\pm 1, 0, 0) \in \Gamma\). Now, \(\Gamma = \{g_1 = 0, \ g_2 = 0\}\), where \(g_1 = x^2 + y^2 - 1\), and \(g_2 = x^2 + z^2 - xz - 1\). Clearly \(g_1, g_2 \in \mathcal{C}^1\) and \((g_1, g_2)\) is a submersion on \(\Gamma\) iff

\[
\text{rank } \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \text{rank } \begin{bmatrix} 2x & 2y & 0 \\ 2x - z & 0 & 2z - x \end{bmatrix} = 2, \ \forall (x, y, z) \in \Gamma.
\]

This is false iff all \(2 \times 2\) submatrices have determinant \(= 0\), that is

\[
\begin{cases}
2y(2x - z) = 0, \\
2x(2z - x) = 0, \\
2y(2z - x) = 0.
\end{cases}
\]

Working on the first equation, we have the alternatives \(y = 0\) or \(2x - z = 0\). In the first case, the system reduces to \(x(2z - x) = 0\) that is \(x = 0\) (points \((0, 0, z)\)) or \(x = 2z\) (points \((2z, 0, z)\)). In the second case, the system reduces to

\[
\begin{cases}
z = 2x, \\
3x^2 = 0, \\
3yz = 0,
\end{cases} \implies (0, y, 0).
\]

Thus, rank is less than 2 at points \((0, 0, z)\), \((2z, 0, z)\) and \((0, y, 0)\). Now:

- \((0, 0, z) \in \Gamma\) iff \(0 = 1\) (first condition), impossible;
- \((2z, 0, z) \in \Gamma\) iff \(4z^2 = 1\) and \(5z^2 = 2z^2 + 1\), that is \(z^2 = \frac{1}{4}\) and \(z^2 = \frac{1}{3}\) which are impossible together.
- \((0, y, 0) \in \Gamma\) iff \(y^2 = 1\) and \(0 = 1\), which is, again, impossible.

Conclusion: none of points where rank is \(\geq 2\) belong to \(\Gamma\), this meaning that rank \(\leq 2\) on \(\Gamma\), hence \((g_1, g_2)\) is a submersion on \(\Gamma\).

ii) Clearly \(\Gamma\) is closed because defined by equations involving continuous functions. Boundedness: from first equation we deduce \(x^2, y^2 \leq 1\). From second equation, recalling that \(ab \leq \frac{a^2 + b^2}{2}\) we have

\[
x^2 + z^2 = xz + 1 \leq \frac{x^2 + z^2}{2} + 1, \quad \iff \quad \frac{x^2 + z^2}{2} \leq 1,
\]

from which, in particular, \(z^2 \leq 2\). Therefore \(\| (x, y, z) \| = \sqrt{x^2 + y^2 + z^2} \leq \sqrt{1 + 1 + 2} = \sqrt{4} = 2\), for every \((x, y, z) \in \Gamma\). Conclusion: \(\Gamma\) is bounded, hence compact.

iii) We have to minimize/maximize \(f(x, y, z) = \sqrt{x^2 + y^2 + z^2}\) or, equivalently, \(f(x, y, z) = x^2 + y^2 + z^2\). By ii), \(\Gamma\) is compact and obviously \(f \in \mathcal{C}\), thus existence of min and max for \(f\) is ensured by Weierstrass’ theorem. To determine min/max points we apply Lagrange’s thm. According to i), this thm can be applied on \(\Gamma\). We deduce that, at min/max points \((x, y, z) \in \Gamma\),

\[
\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \iff \quad \text{rank } \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \\ 2x - z & 0 & 2z - x \end{bmatrix} = 2,
\]

\[
\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \iff \quad \text{rank } \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \\ 2x - z & 0 & 2z - x \end{bmatrix} = 2,
\]
or, equivalently, the determinant of this last matrix equals 0. We obtain

\[ 2z \cdot (-2y(2x - z)) = 0, \quad \iff \quad yz(2x - z) = 0, \quad \iff \quad y = 0, \quad \forall z = 0, \quad \forall z = 2x. \]

Thus possible min/max points are among points \((x, 0, z), (x, y, 0)\) and \((x, y, 2x)\). Now,

- \((x, 0, z) \in \Gamma\) iff \(x^2 = 1\) and \(x^2 + z^2 = xz + 1\), or, equivalently, \(x^2 = 1\) and \(z^2 = xz + 1\). For \(x = 1\) we get \(z^2 = z + 1\), that is \(z = \frac{1+\sqrt{5}}{2}\), namely points \((1, 0, \frac{1+\sqrt{5}}{2})\). For \(x = -1\) we get \(z^2 = -z + 1\), that is \(z = \frac{-1+\sqrt{5}}{2}\), namely points \((-1, 0, \frac{-1+\sqrt{5}}{2})\).
- \((x, y, 0) \in \Gamma\) iff \(x^2 + y^2 = 1\) and \(x^2 = 1\), that is \(x = \pm 1\) and \(y^2 = 0\), namely points \((\pm 1, 0, 0)\).
- \((x, y, 2x) \in \Gamma\) iff \(x^2 + y^2 = 1\) and \(x^2 + 4x^2 = 2x^2 + 1\), from which \(x^2 = \frac{1}{3}\), \(x = \pm \frac{1}{\sqrt{3}}\) and \(y^2 = \frac{2}{3}\), \(y = \pm \sqrt{\frac{2}{3}}\), thus we get points \(\left(\frac{1}{\sqrt{3}}, \pm \sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right)\) and \(\left(-\frac{1}{\sqrt{3}}, \pm \sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right)\) (4 points).

We have

- \(f(1, 0, \frac{1+\sqrt{5}}{2}) = 1 + \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{10 \pm 2\sqrt{5}}{4}, \quad f(-1, 0, \frac{-1+\sqrt{5}}{2}) = 1 + \left(\frac{-1+\sqrt{5}}{2}\right)^2 = \frac{10 \pm 2\sqrt{5}}{4} \approx f(1, 0, 0) = 1;\)
- \(f\left(\frac{1}{\sqrt{3}}, \pm \sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = \frac{1}{3} + \frac{2}{\sqrt{3}} + \frac{4}{3} = \frac{7}{3}\) and \(f\left(-\frac{1}{\sqrt{3}}, \pm \sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{1}{3} + \frac{2}{\sqrt{3}} + \frac{4}{3} = \frac{7}{3}.\)

From this we see that \((1, 0, \frac{1+\sqrt{5}}{2})\) and \((-1, 0, \frac{-1+\sqrt{5}}{2})\) are maximum points while \((\pm 1, 0, 0)\) are min points.

**Exercise 18.** ii) \(D\) is closed (because defined by large inequalities involving continuous functions) and bounded (the root imposes \(x^2 + y^2 \leq 1\) and, consequently, \(0 \leq 1 - (x^2 + y^2) \leq z \leq \sqrt{1 - (x^2 + y^2)} \leq \sqrt{1}\), that is \(0 \leq z \leq 1\)). Thus \(D\) is compact, hence \(1_D\) is integrable on \(D\). Furthermore, noticed that, calling \(\rho^2 = x^2 + y^2\),

\[ 1 - \rho^2 \leq \sqrt{1 - \rho^2}, \quad \iff \quad \sqrt{1 - \rho^2} \leq 1, \]

which is always true, thus \(1 - (x^2 + y^2) \leq \sqrt{1 - (x^2 + y^2)}\) always when defined. Then

\[
\text{Vol } D = \int_D 1 \, dxdydz = \underbrace{RF}_{pol. \, coords} \int_{x^2+y^2 \leq 1} \int_{1-(x^2+y^2)}^{\sqrt{1-(x^2+y^2)}} 1 \, dz \, dxdy \\
= \int_{x^2+y^2 \leq 1} \left(\sqrt{1-(x^2+y^2)} - (1-(x^2+y^2))\right) \, dxdy \\
= 2\pi \int_{0<\rho<1} \rho (1 - \rho^2)^{1/2} - \rho + \rho^3 \, d\rho = 2\pi \left[ -\frac{1}{3}(1 - \rho^2)^{3/2} \right]_{\rho=0}^{\rho=1} - \left[ \frac{\rho^2}{2} \right]_{\rho=0}^{\rho=1} + \left[ \frac{\rho^4}{4} \right]_{\rho=0}^{\rho=1} \\
= 2\pi \left[ \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \right] = \frac{\pi}{6}. \quad \Box
\]
Exercise 19. i) In order $f = u + iv$ is holomorphic on $\mathbb{C}$ we need that $u, v \in \mathbb{C}^1$ (true, $u$ and $v$ are polynomials) and they fulfill the CR equations:

$$
\begin{align*}
\partial_x u &= \partial_y v, \\
\partial_y u &= -\partial_x v,
\end{align*}
$$

$$
\begin{align*}
2ax + by &= x, \\
2cx + dy &= y,
\end{align*}
$$

Thus,

$$
u = \frac{1}{2}x^2 - \frac{1}{2}y^2, \quad v = xy,
$$

and $f = u + iv$ is holomorphic on $\mathbb{C}$.

ii) Notice that

$$f = u + iv = \frac{1}{2}x^2 - \frac{1}{2}y^2 + ixy = \frac{1}{2} (x^2 - y^2 + i2xy) = \frac{1}{2} (x + iy)^2 \equiv \frac{z^2}{2}, \quad z \in \mathbb{C}.
$$

Exercise 20. Clearly $f \in \mathcal{C}(\mathbb{R}^d)$ and moreover $f \geq 0$ (trivial) and

$$
\lim_{\tilde{x} \to \infty_d} f(\tilde{x}) = +\infty.
$$

Just notice that $f(\tilde{x}) \geq ||\tilde{x} - \tilde{a}_1||^2 \longrightarrow +\infty$ when $\tilde{x} \longrightarrow \infty_d$. Thus $f$ cannot have a maximum but it has a minimum according to Weierstrass’ thm. Now, $f$ is differentiable on $\mathbb{R}^d$,

$$
\nabla f = \sum_{j=1}^{N} \nabla \|\tilde{x} - \tilde{a}_j\| \quad^2
$$

and

$$
\nabla \|\tilde{x} - \tilde{a}_j\|^2 = \left( \partial_1 \|\tilde{x} - \tilde{a}_j\|^2, \ldots, \partial_d \|\tilde{x} - \tilde{a}_j\|^2 \right),
$$

so, writing

$$
\|\tilde{x} - \tilde{a}_j\|^2 = \sum_{k=1}^{d} (x_k - a_{j,k})^2, \quad \implies \partial_i \|\tilde{x} - \tilde{a}_j\|^2 = \partial_i \sum_{k=1}^{d} (x_k - a_{j,k})^2 = 2(x_i - a_{j,i}),
$$

we deduce

$$
\nabla \|\tilde{x} - \tilde{a}_j\|^2 = \left( 2(x_1 - a_{j,1}), 2(x_2 - a_{j,2}), \ldots, 2(x_d - a_{j,d}) \right) = 2(\tilde{x} - \tilde{a}_j).
$$

Therefore, $\nabla f \in \mathcal{C}$ and $f$ is differentiable. According to Fermat thm, at min point we must have

$$
\nabla f = \tilde{0}, \quad \iff \sum_{j=1}^{N} 2(\tilde{x} - \tilde{a}_j) = 0, \quad \iff N\tilde{x} - \sum_{j=1}^{N} \tilde{a}_j = \tilde{0}, \quad \iff \tilde{x} = \frac{1}{N} \sum_{j=1}^{N} \tilde{a}_j.
$$

Exercise 21. i) $y \equiv C$ is a solution iff $0 = C \log C$, from which $C > 0$ (to be $\log C$ well defined), thus $\log C = 0$, that is $C = 1$.

ii) If $y(0) = 1$, then $y(t) \equiv 1$ (constant solution. For $a \neq 1$ (but $a > 0$ because of the equation), solution is non constant and it can be determined by separation of variables:

$$
y = y \log y, \quad \iff \frac{y'}{y \log y} = 1, \quad \iff \int \frac{y'}{y \log y} \ dt = t + c.
$$
Since
\[
\int \frac{y'}{y \log y} \, dt \quad \text{and} \quad \int \frac{1}{u \log u} \, du = \int \frac{(\log u)'}{\log u} \, du = \log |\log u| = \log |\log y(t)|.
\]
Therefore,
\[
\log |\log y(t)| = t + c.
\]
By imposing \(y(0) = a\) we have \(c = \log |\log a|\), hence
\[
|\log y(t)| = |\log a|e^t, \quad \iff \quad \log y(t) = \pm (\log a)e^t.
\]
Because of the initial condition we have \(\log y(t) = (\log a)e^t\), hence
\[
y(t) = e^{(\log a)e^t}.
\]
iii) We have \(\lim_{t \to +\infty} y(t) = 0\) iff \(\log a < 0\), that is \(a < 1\). \(\square\)

**Exercise 22.** i) Let \(g_1 := x^2 - y^2 - z^2\) and \(g_2 := x^2 + y^2 - xy - 1\). Then, \(\tilde{g} = (g_1, g_2)\) is a submersion on \(D\) iff \(\text{rk} \tilde{g}'(x, y, z) = 2\) for all \((x, y, z) \in D\). Now,
\[
\text{rk} \tilde{g}'(x, y, z) = \text{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \text{rk} \begin{bmatrix} 2x & -2y & -2z \\ 2x - y & 2y - x & 0 \end{bmatrix} < 2, \quad \iff \quad \begin{cases} 2x(2y - x) + 2y(2x - y) = 0, \\ 2z(2x - y) = 0, \\ 2z(2y - x) = 0. \end{cases}
\]
Simplifying, we get the system
\[
\begin{align*}
&x^2 + y^2 - 4xy = 0, \\
z(2x - y) = 0, \\
z(2y - x) = 0.
\end{align*}
\]
Choosing the second equation, we have the alternative \(z = 0\) or \(2x - y = 0\). In the first case the system reduces to
\[
\begin{align*}
z &= 0, \\
x^2 + y^2 - 4xy &= 0.
\end{align*}
\]
These points belong to \(D\) iff
\[
\begin{cases} 
x^2 = y^2, \\
4xy = xy + 1 \quad \iff \quad y = \pm x, \\
3xy = 1.
\end{cases}
\]
However, since \(x^2 + y^2 = 4xy\) implies that, for \(y = \pm x\), that \(x = 0 = y\), it is impossible that \(3xy = 1\), thus no solutions are in \(D\).

In the second case, namely, \(z \neq 0\) and \(2x - y = 0\) or \(y = 2x\), condition \(\text{rk} \tilde{g}'(x, y, z) < 2\) reduces to
\[
\begin{cases} 
y - 2x, \\
x(2y - x) = 0, \\
2y - x = 0,
\end{cases}
\]
we easily get \( x = y = 0 \), that is a point of type \((0, 0, z)\). Now,

\[
(0, 0, z) \in D, \iff \begin{cases} z = 0, \\ 0 = 1, \end{cases}
\]

clearly impossible. Conclusion: rank of \( \mathcal{g}'(x, y, z) \) is never less than 2 on \( D \), that is \( \mathcal{g} \) is a submersion on \( D \).

ii) \( D \) is clearly closed being defined by equalities involving continuous functions. To determine whether
\( D \) is bounded or less, we look first at constraint \( x^2 + y^2 = xy + 1 \). Writing \( x = \rho \cos \theta \) and \( y = \rho \sin \theta \), this reads as

\[
\rho^2 = \rho^2 \cos \theta \sin \theta + 1 = \frac{\rho^2}{2} \sin(2\theta) + 1, \leq \frac{\rho^2}{2} + 1, \iff \frac{\rho^2}{2} \leq 1, \implies x^2 + y^2 \leq 2, \forall (x, y, z) \in D.
\]

But then, by the first equation,

\[
z^2 = x^2 - y^2 \leq x^2 + y^2 \leq 2, \implies x^2 + y^2 + z^2 \leq 4, \implies \| (x, y, z) \| \leq 2, \forall (x, y, z) \in D.
\]

This means that \( D \) is bounded, hence compact.

iii) We have to minimize/maximize \( f(x, y, z) = \|(x, y, z)\| \) or, which is the same, \( f(x, y, z) = \|(x, y, z)\|^2 = x^2 + y^2 + z^2 \). The existence of min and max is ensured by the Weierstrass theorem being \( D \) compact by ii).

To determine min/max points, we apply Lagrange multipliers theorem. By i), assumptions of this theorem are verified. Thus, at min/max point \((x, y, z) \in D \) we must have

\[
\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \text{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \iff \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 0.
\]

Now,

\[
0 = \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & -2y & -2z \\ 2x - y & 2y - x & 0 \end{bmatrix} = -(2y - z)(-8xz) = 8xz(2y - z),
\]

iff \( x = 0 \), or \( z = 0 \) or \( 2y - z = 0 \). Thus, we have points \((0, y, z)\), \((x, y, 0)\) and \((x, y, 2y)\). Now:

- \((0, y, z) \in D \) iff \( 0 = y^2 + z^2 \) and \( y^2 = 1 \), and of course this is impossible.
- \((x, y, 0) \in D \) iff \( x^2 = y^2 \) and \( x^2 + y^2 = xy + 1 \). From the first we have \( y = \pm x \). For \( y = x \), second condition becomes \( 2x^2 = x^2 = 1 \), thus \( x = 1 \), so \( z = \pm 1 \) and we have points \((\pm 1, \pm 1, 0)\) (same sign). For \( y = -x \), second condition becomes \( 2x^2 = -x^2 + 1 \), that is \( x^2 = \frac{1}{2} \), that is \( x = \pm \frac{1}{\sqrt{2}} \), from which we have points \( \left( \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, 0 \right) \) (opposite sign).
- \((x, y, 2y) \in D \) iff \( x^2 = y^2 + 4y^2 = 5y^2 \) and \( x^2 + y^2 = xy + 1 \). From first equation we get \( x = \pm \sqrt{5}y \). In the case \( x = \sqrt{5}y \), from second eqn we have \( 5y^2 + y^2 = \sqrt{5}y^2 + 1 \), that is \( (6 - \sqrt{5})y^2 = 1 \), that is \( y = \pm \frac{1}{\sqrt{6 - \sqrt{5}}} \), this yielding to points \( \left( \pm \frac{\sqrt{5}}{\sqrt{6 - \sqrt{5}}}, \pm \frac{1}{\sqrt{6 - \sqrt{5}}}, 0 \right) \) (same sign). In the case \( x = -\sqrt{5}y \),
second condition yields to \(5y^2 + y^2 = -\sqrt{5}y\), that is \(y^2 = \frac{1}{5+\sqrt{5}}\), or \(y = \pm \frac{1}{\sqrt{5+\sqrt{5}}}\), from which we get points \(\left(\pm \frac{\sqrt{5}}{\sqrt{5+\sqrt{5}}}, \pm \frac{1}{\sqrt{5+\sqrt{5}}}\right)\) (opposite sign).

Previous analysis figured out possible min/max points. To decide which are min and which max it suffices to compute \(f\) at these points. We have:

- \(f(\pm 1, \pm 1, 0) = 2;\)
- \(f\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, 0\right) = \frac{2}{3} = 0, 6;\)
- \(f\left(\pm \frac{\sqrt{5}}{\sqrt{6-\sqrt{5}}}, \pm \frac{1}{\sqrt{6-\sqrt{5}}}, 0\right) = \frac{6}{6-\sqrt{5}} \approx 1, 59 \ldots\)
- \(f\left(\pm \frac{\sqrt{7}}{\sqrt{5+\sqrt{5}}}, \pm \frac{1}{\sqrt{5+\sqrt{5}}}, 0\right) = \frac{6}{5+\sqrt{5}} \approx 0, 83 \ldots\)

From this it is clear that \((\pm 1, \pm 1, 0)\) are points of \(D\) at max distance to \(\mathbf{0}\), while \(\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, 0\right)\) are points of \(D\) at min distance to \(\mathbf{0}\).

\(\square\)

**Exercise 23.** i) To be irrotational, the field must verify

\[
\frac{\partial}{\partial y} \frac{ax + by}{\sqrt{x^2 + y^2}} = \frac{\partial}{\partial x} \frac{cx + dy}{\sqrt{x^2 + y^2}}, \quad \forall (x, y) \in D = \mathbb{R}^2 \setminus \{0\}.
\]

We have

\[
\frac{\partial}{\partial y} \frac{ax + by}{\sqrt{x^2 + y^2}} = \frac{b(x^2 + y^2) - (ax + by)}{2(x^2 + y^2)^{3/2}} = \frac{bx^2 - axy}{(x^2 + y^2)^{3/2}},
\]

and, similarly

\[
\frac{\partial}{\partial x} \frac{cx + dy}{\sqrt{x^2 + y^2}} = \frac{cy^2 - dxy}{(x^2 + y^2)^{3/2}}.
\]

Thus, the field is irrotational iff

\[
\frac{bx^2 - axy}{(x^2 + y^2)^{3/2}} = \frac{cy^2 - dxy}{(x^2 + y^2)^{3/2}}, \quad \iff \quad bx^2 - axy = cy^2 - dxy, \quad \forall (x, y) \in \mathbb{R}^2 \setminus \{0\}.
\]

Since the identity is trivally verified at \((x, y) = \mathbf{0}\), we may say that the field is irrotational iff

\[
bx^2 - axy \equiv cy^2 - dxy, \quad \iff \quad b = c = 0, a = d.
\]

ii) By i), to be conservative \(\tilde{F}\) must have the form

\[
\tilde{F} = \left(\frac{ax}{\sqrt{x^2 + y^2}}, \frac{ay}{\sqrt{x^2 + y^2}}\right).
\]
Now, such a $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is
\[
\begin{aligned}
\partial_x f &= \frac{ax}{\sqrt{x^2 + y^2}}, \\
\partial_y f &= \frac{ay}{\sqrt{x^2 + y^2}}.
\end{aligned}
\]
From first equation,
\[
f(x, y) = \int \frac{ax}{\sqrt{x^2 + y^2}} \, dx + k(y) = \frac{a}{2} \int (x^2 + y^2)^{-1/2} (2x) \, dx + k(y) = a(x^2 + y^2)^{1/2} + k(y).
\]
Plugging this into the second equation we have
\[
\partial_y f = a \frac{1}{2} (x^2 + y^2)^{-1/2} 2y + k'(y) = \frac{ay}{\sqrt{x^2 + y^2}}, \quad \iff k'(y) = 0.
\]
Thus, we deduce that
\[
f(x, y) = a \sqrt{x^2 + y^2} + k, \quad k \in \mathbb{R},
\]
are all the potentials for $\vec{F}$.

**Exercise 24.** For the volume, we may notice that
\[
\lambda_3(D) = \int_D 1 \, dx dy dz = \int_0^1 \left( \int_{x^2 + 4y^2 < 1 + z^2} dx dy \right) dz.
\]
By using adapted polar coordinates, $x = \rho \cos \theta, y = \frac{1}{2} \rho \sin \theta$, in such a way that $x^2 + 4y^2 = \rho^2$, we have
\[
\int_{x^2 + 4y^2 < 1 + z^2} dx dy = \int_{0 \leq \rho \leq \sqrt{1 + z^2}} \int_{0 \leq \theta \leq 2\pi} \frac{1}{2} \rho \, d\rho d\theta = \pi \int_0^{\sqrt{1 + z^2}} \rho \, d\rho = \pi \left[ \frac{\rho^2}{2} \right]_{\rho = 0}^{\rho = \sqrt{1 + z^2}} = \frac{\pi}{2} (1 + z^2).
\]
Therefore
\[
\lambda_3(D) = \int_0^1 \frac{\pi}{2} (1 + z^2) \, dz = \frac{\pi}{2} \left( 1 + \left[ \frac{z^3}{3} \right]_{z=0}^{z=1} \right) = \frac{2}{3} \pi.
\]

**Exercise 25.** i) If $u(x, y) = \text{Re} \, f(x + iy)$ and $v(x, y) = \text{Im} \, f(x + iy)$, then
\[
g(x + iy) = \overline{f(x - iy)} = u(x, -y) + iv(x, -y) = u(x, -y) - iv(x, -y),
\]
from which we see that
\[
U(x, y) = \text{Re} \, g(x + iy) = u(x, -y), \quad V(x, y) = \text{Im} \, g(x + iy) = -v(x, -y).
\]
ii) $g$ is holomorphic iff $U, V$ are $\mathbb{R}$-differentiable and they verify CR equations. Clearly, since $f$ is holomorphic, $u, v$ are $\mathbb{R}$-differentiable, hence also $U, V$ are $\mathbb{R}$-differentiable. Therefore, we have to verify if $U, V$ fulfill also the CR equations, that is
\[
\begin{aligned}
\partial_x U &= \partial_y V, \\
\partial_y U &= -\partial_x V.
\end{aligned}
\]
We have,
\[ \partial_x U = \partial_x (u(x, -y)) = \partial_x u(x, -y), \quad \partial_y V = \partial_y (-v(x, -y)) = -\partial_y v(x, -y)(-1) = \partial_y v(x, -y). \]

And since \( \partial_x u \equiv \partial_y v \) we deduce that also \( \partial_x U = \partial_y V \). Similarly, \( \partial_y U = -\partial_x V \) and the check is completed.

**Exercise 26.** i) We have a second order equation. The homogeneous equation is \( y'' + 2y' + y = 0 \), whose characteristic equation is \( \lambda^2 + 2\lambda + 1 = 0 \), or \( (\lambda + 1)^2 = 0 \). The fundamental system of solutions for the homogeneous equation is \( w_1 = e^{-t}, w_2 = te^{-t} \), whose wronskian is
\[ W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}(1 - t) \end{bmatrix} = e^{-2t}(1 - t) + te^{-2t} = e^{-2t}. \]
The general solution of the original equation is then
\[ y(t) = \left( c_1 - \int \frac{w_2}{W} (t + 1) \, dt \right) w_1 + \left( c_2 + \int \frac{w_1}{W} (t + 1) \, dt \right) w_2 \]
We have
\[ \int \frac{w_2}{W} (t + 1) \, dt = \int \frac{te^{-t}}{e^{-2t}} (t + 1) \, dt = \int e^t (t^2 + t) \, dt = e^t (t^2 + t) - \int e^t (2t + 1) \, dt = e^t (t^2 + t - 1) + \int 2e^t \, dt = e^t (t^2 - t + 1), \]
and
\[ \int \frac{w_1}{W} (t + 1) \, dt = \int \frac{-e^{-t}}{e^{-2t}} (t + 1) \, dt = \int e^t (t + 1) \, dt = e^t (t + 1) - \int e^t \, dt = te^t. \]
Therefore, the general integral is
\[ y(t) = \left( c_1 - e^t (t^2 - t + 1) \right) e^{-t} + \left( c_2 + te^t \right) te^{-t} = c_1 e^{-t} + c_2 te^{-t} + t - 1, \quad c_1, c_2 \in \mathbb{R}. \]

ii) Imposing \( y(0) = 0 \) we get \( c_1 - 1 = 0 \), that is \( c_1 = 1 \), so
\[ y(t) = e^{-t} + c_2 t e^{-t} + t - 1. \]
To determine also \( c_2 \), we impose \( y'(0) = 1 \), that is, since
\[ y'(t) = -e^{-t} + c_2 e^{-t} (1 - t) + 1, \quad \Rightarrow \quad -1 + c_2 + 1 = 1, \quad \iff \quad c_2 = 1. \]
The solution of the Cauchy problem is then,
\[ y(t) = e^{-t} + te^{-t} + t - 1, \quad c_1, c_2 \in \mathbb{R}. \]

iii) From \( y(0) = 0 \) we get
\[ y(t) = e^{-t} + c_2 t e^{-t} + t - 1, \]
and imposing also \( y(1) = 0 \) we get
\[ 0 = e^{-1} + c_2 e^{-1}, \quad \iff \quad c_2 = 1. \]
The solution is the same of that one found at ii). \( \square \)
Exercise 27. i) For \( D \neq \emptyset \) we consider a point of type \((x, y, 2)\). Then \((x, y, 2) \in D\) iff \(x^2 + y^2 = 4\) and \(y^2 = 1\), thus \(y = \pm 1\) and \(x^2 = 3\), that is \(x = \pm \sqrt{3}\). We conclude that points \((\pm \sqrt{3}, \pm 1, 2)\) (four points, all possible combinations of sign) belong to \(D\).

We have that \(D = \{g_1 = 0, g_2 = 0\}\) where \(g_1 = x^2 + y^2 - z^2\), and \(g_2 = y^2 + (z - 2)^2 - 1\). Clearly, both \(g_1\) and \(g_2\) are differentiable functions (they are polynomials). In order \(\tilde{g} = (g_1, g_2)\) be a submersion on \(D\) we need to verify that

\[
\text{rk } \tilde{g}' = \text{rk } \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \text{rk } \begin{bmatrix} 2x & 2y & -2z \\ 0 & 2y & 2(z - 2) \end{bmatrix} = 2, \forall (x, y, z) \in D.
\]

Now, this is false iff all \(2 \times 2\) sub-determinants of the Jacobian matrix \(\tilde{g}'\) vanish, that is iff

\[
\begin{cases}
4xy = 0, \\
4x(z - 2) = 0, \\
8y(z - 1) = 0.
\end{cases}
\]

The first subsystem has solutions \((0, 0, z)\) and \((0, y, 1)\) \((x, y \in \mathbb{R})\); the second, \((0, 0, z)\) and \((x, 0, 2)\), \((x, z \in \mathbb{R})\). Now:

- \((0, 0, z) \in D\) iff \(z^2 = 0\) and \((z - 2)^2 = 1\), impossible;
- \((0, y, 1) \in D\) iff \(y^2 = 1\) and \(y^2 + 1 = 1\), again impossible;
- \((x, 0, 2) \in D\) iff \(x^2 = 4\) and \(0 = 1\), impossible.

Conclusion: there is no point on \(D\) at which rank of \(\tilde{g}'\) is less than 2, therefore rank of \(\tilde{g}'(x, y, z)\) is 2 for every \((x, y, z) \in D\), that is \(\tilde{g}'\) is a submersion on \(D\).

ii) \(D\) is defined by equalities involving continuous functions, it is therefore closed. From the second equation

\[
y^2 + (z - 2)^2 = 1, \quad \implies \quad y^2 \leq 1, \quad (z - 2)^2 \leq 1.
\]

In particular, \(-1 \leq z - 2 \leq 1\), that is \(1 \leq z \leq 3\), thus \(z^2 \leq 9\). Plugging this into the first equation,

\[
x^2 + y^2 = z^2, \quad x^2 + y^2 + 9 = 19, \quad \text{for every } (x, y, z) \in D, \text{ from which we see that } D \text{ is bounded.}
\]

We conclude that \(D\) is compact.

iii) Points at min/max distance to \(\tilde{0}\) minimize/maximize the function \(f = x^2 + y^2 + z^2\). Since \(f\) is continuous and \(D\) is compact, according to the Weierstrass theorem, \(f\) has both min and max on \(D\).

To determine these points, we apply the Lagrange multipliers’ theorem. By i), hypotheses of the theorem are fulfilled. Thus, at every \((x, y, z) \in D\) min/max point for \(f\) in \(D\) we must have

\[
\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \iff \quad \text{rk } \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \quad \iff \quad \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 2(z - 2) \end{bmatrix} = 0.
\]

By computing the determinant we get

\[
0 = 2x \cdot 4y(z - 2 + z) - 2x \cdot 4y(z - 2 - z) = 16xyz,
\]

whose solutions are points \((0, y, z), (x, 0, z)\) and \((x, y, 0)\). Now,
• $(0, y, z) \in D$ iff $y^2 = z^2$ and $y^2 + (z - 2)^2 = 1$, from which $z^2 + (z - 2)^2 = 1$, or $2z^2 - 2z + 3 = 0$, and since $\Delta < 0$ there are no solutions to this equation;

• $(x, 0, z) \in D$ iff $x^2 = z^2$ and $(z - 2)^2 = 1$, from which $z = 1, 3$ and $x^2 = 1$ (that is $x = \pm 1$), or $x^2 = 9$ (that is $x = \pm 3$). We obtain points $(\pm 1, 0, 1)$ and $(\pm 3, 0, 3)$;

• $(x, y, 0) \in D$ iff $x^2 + y^2 = 0, y^2 + 4 = 1$ which is impossible.

Since $f(\pm 1, 0, 1) = 2$ and $f(\pm 3, 0, 3) = 18$ we deduce that $(\pm 1, 0, 1)$ are points of $D$ at min distance to 0, $(\pm 3, 0, 3)$ are points of $D$ at max distance to 0. □

**Exercise 28 ii)** The change of variable is given in the form $(u, v) = \Phi(x, y) = (y - x^3, y + x^3)$. According to the change of variable formula,

$$\int_D f(x, y) \, dx \, dy = \int_{\Phi(D)} f(\Phi^{-1}(u, v)) \left| \det(\Phi^{-1}) \right| \, du \, dv.$$

We need to determine $\Phi^{-1}$. Notice that

\[
\begin{align*}
\left\{ \begin{array}{c}
u = y - x^3, \\
v = y + x^3,
\end{array} \right. & \iff \left\{ \begin{array}{c}u + v = 2y, \\
v - u = 2x^3,
\end{array} \right. \iff \left\{ \begin{array}{c}y = u + v, \\
x^3 = \frac{v - u}{2},
\end{array} \right. \iff \left\{ \begin{array}{c}y = \frac{u + v}{2}, \\
x = \left(\frac{v - u}{2}\right)^{1/3},
\end{array} \right.
\end{align*}
\]

Therefore

$$\Phi^{-1}(u, v) = \left(\left(\frac{v - u}{2}\right)^{1/3}, \frac{u + v}{2}\right).$$

Moreover,

$$(x, y) \in D \iff \left\{ \begin{array}{c}x \geq 1, \\
x^3 \leq y \leq 3,
\end{array} \right. \iff \left\{ \begin{array}{c}(\frac{v - u}{2})^{1/3} \geq 1, \\
\frac{v - u}{2} \leq \frac{u + v}{2} \leq 3
\end{array} \right. \iff \left\{ \begin{array}{c}v - u \geq 2, \\
v - u \leq v + u \leq 6
\end{array} \right.$$ that is

$$\Phi(D) = \{(u, v) : 2 \leq v - u \leq v + u \leq 6\}.$$ 

Now, to be $v - u \leq v + u$ it must be $u \geq 0$, and from $2 \leq v - u \leq v + u \leq 6$ we get $2 + u \leq v \leq 6 - u$ provided $2 + u \leq 6 - u$, that is $u \leq 2$. In conclusion

$$\Phi(D) = \{(u, v) : 0 \leq u \leq 2, 2 + u \leq v \leq 6 - u\}.$$

About $f$, in coordinates $(u, v)$ we have

$$f(\Phi^{-1}(u, v)) = \left(\frac{v - u}{2}\right)^{2/3} u e^u,$$

while

$$\det(\Phi^{-1})' = \det \left[ \begin{array}{ccc}
\frac{1}{3} (\frac{v - u}{2})^{-2/3} & \frac{1}{3} (\frac{v - u}{2})^{-2/3} \\
\frac{1}{3} (\frac{v - u}{2})^{2/3} & \frac{1}{3} (\frac{v - u}{2})^{2/3}
\end{array} \right] = -\frac{1}{6} (\frac{v - u}{2})^{-2/3}.$$
In conclusion

\[
\int_D f \, dx \, dy = \int_{0 \leq u \leq 2, 2 + u \leq v \leq 6-u} \frac{(v-u)^{2/3}}{2} \, du \, dv = \int_{0 \leq u \leq 2, 2 + u \leq v \leq 6-u} u \, du \, dv
\]

\[RF = \frac{1}{6} \int_0^2 \int_{2+u}^{6-u} u \, dv \, du = \frac{1}{6} \int_0^2 \int_{2+u}^{6-u} v \, dv \, du = \frac{1}{6} \int_0^2 u \, [e^v]_{v=2+u}^v \, du\]

\[= \frac{1}{6} \int_0^2 u (e^{6-u} - e^{2+u}) \, du = \frac{1}{6} \left( e^{2} \int_0^2 u e^{-u} \, du - e^2 \int_0^2 e^u \, du \right)\]

\[= \frac{1}{6} \left( e^{2} \left( [ue^{-u}]_{u=0}^2 + \int_0^2 e^{-u} \, du \right) - e^2 \left( [ue^u]_{u=0}^2 - \int_0^2 e^u \, du \right) \right)\]

\[= \frac{1}{6} \left( e^{2} (2e^2 - e^2 - 1) - e^2 (2e^2 - (e^2 - 1)) \right)\]

\[= \frac{\epsilon^2}{6} (2e^2 - e^4 - 1). \quad \square \]

**Exercise 29.** In order \( f = u + iv \) be holomorphic, we need that \( u, v \) are both \( \mathbb{R} \)--differentiable (and certainly \( v \) it is), and they verify the CR equations,

\[
\begin{align*}
\partial_x u &= \partial_y v, \\
\partial_y u &= -\partial_x v.
\end{align*}
\]

Thus we have to look for an \( \mathbb{R} \)--differentiable \( u \) such that

\[
\begin{align*}
\partial_x u &= 3y^2 - 3x^2 + 4x, \\
\partial_y u &= -(6xy + 4y - 1).
\end{align*}
\]

From the first equation we get,

\[
u(x, y) = \int (3y^2 - 3x^2 + 4x) \, dx + k(y) = 3y^2x - x^3 + 2x^2 + k(y).
\]

Plugging this into the second equation we have

\[6xy + k'(y) = 6xy - 4y + 1, \quad \iff \quad k'(y) = -4y + 1, \quad \iff \quad k(y) = -2y^2 + y + k, \quad k \in \mathbb{R}.
\]

Thus, all the possible \( u \) that verify the CR eqns together with \( v \) are

\[u(x, y) = 3y^2x - x^3 + 2x^2 - 2y^2 + y + k.
\]

Since such \( u \) are clearly \( \mathbb{R} \)--differentiable, \( f = u + iv \) is \( \mathbb{C} \)--differentiable (holomorphic) on \( \mathbb{R}^2 \).
To determine the analytical expression for \( f \) as a function of complex variable \( z = x + iy \), we may notice that

\[
f = u + iv = 3y^2x - x^3 + 2x^2 - 2y^2 + y + k + i(y^3 - 3x^2y + 4xy - x)
\]

\[
= -i(x + iy) + 2(x^2 - y^2 + i2xy) - \left(x^3 - 3y^2x + i3x^2y\right) + k
\]

\[
= -z^3 + 2z^2 - iz + k. \quad \square
\]

**Exercise 30.** See notes for definitions and characterizations.

Let’s focus on the resuire property. We first notice that \( \partial S = \emptyset, \partial S \) is closed. We assume then that \( \partial S \neq \emptyset \). To verify that \( \partial S \) is closed, we use the Cantor characterization. Let \( (x_n) \subset \partial S \) be such that \( x_n \to \bar{x} \in \mathbb{R}^d \). We prove that \( \bar{x} \in \partial S \). Fix \( r > 0 \). Since \( x_n \to \bar{x} \), we have that for \( n \geq N \), \( \|x_n - \bar{x}\| \leq \frac{r}{2} \).

Now, since \( \bar{x} \in \partial S \),

\[
B(\bar{x}, r/2] \cap S \neq \emptyset, \land B(\bar{x}, r/2] \cap S^c \neq \emptyset.
\]

Since \( \|\bar{x} - x\| \leq \frac{r}{2} \), we have that

\[
B(\bar{x}, r/2] \subset B(\bar{x}, r],
\]

therefore

\[
B(\bar{x}, r] \cap S \supset B(\bar{x}, r/2] \cap S \neq \emptyset,
\]

and, similarly, \( B(\bar{x}, r] \cap S^c \neq \emptyset \). We conclude that \( \bar{x} \in \partial S \), thus \( \partial S \) is closed. \( \square \)

**Exercise 31.** First of all let \( z \neq 0 \). Setting \( w = \frac{1}{z} \), we have to solve

\[
\sinh w = 0, \iff \frac{e^w - e^{-w}}{2} = 0, \iff e^{2w} = 1, \iff 2w = \log |1 + i(0 + k2\pi)| = ik2\pi, k \in \mathbb{Z}.
\]

Thus

\[
\frac{1}{z} = w = ik\pi, \iff z = \frac{1}{ik\pi} = \frac{-i}{k\pi} = \frac{i}{k\pi}, k \in \mathbb{Z} \setminus \{0\}. \quad \square
\]

**Exercise 32.** The problem asks to determine

\[
\min_{(x,y,z) \in D} \max_{(x,y,z) \in D} \sqrt{(x - 1)^2 + (y - 2)^2 + (z + 3)^2}.
\]

Previous problem has the same min/max points (if any) of

\[
\min_{(x,y,z) \in D} \max_{(x,y,z) \in D} \left( (x - 1)^2 + (y - 2)^2 + (z + 3)^2 \right),
\]

which is the problem we solve here.

We start discussing existence. \( D \) is certainly a closed set (defined by an equality of a continuous function). Let’s see if \( D \) is also bounded. Since no condition on \( z \) is given, it means that if \( (x, y, z_0) \in D \) then \( (x, y, z) \in D \) for every \( z \in \mathbb{R} \). In particular \( (x, x, z) \in D \) for every \( x, z \in \mathbb{R} \). We deduce that \( D \) is unbounded. Thus, \( D \) is not compact. The function \( f(x, y, z) = \|x - 1, y - 2, z + 3\|^2 \) is clearly continuous, and since

\[
\lim_{(x,y,z) \to \infty} f = +\infty,
\]
we conclude that \( f \) has no maximum on \( D \) but it has global minimum on \( D \).

To determine the minimum, we wish to apply the Lagrange multipliers’ theorem. To this aim, we need first to check if \( D = \{ g = 0 \} \) where \( g = (x - y)^2 + (x - y) \), and \( g \) is a submersion on \( D \) if \( \nabla g \neq 0 \) on \( D \). We have

\[
\nabla g = (2(x - y) - 1, -2(x - y + 1, 0) = \tilde{0}, \iff 2(x - y) - 1 = 0, \iff x - y = \frac{1}{2}.
\]

However, if \( x - y = \frac{1}{2} \) we easily see that the condition characterizing \( D \) is not fulfilled. Thus, \( \nabla g \neq 0 \) always. Thus, in particular, \( g \) is a submersion on \( D \). Therefore, according to Lagrange multipliers’ theorem, at \( (x, y, z) \in D \) min point for \( f \),

\[
\nabla f = \lambda \nabla g, \iff \text{rk} \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \text{rk} \begin{bmatrix} 2(x - 1) & 2(y - 2) & 2(z + 3) \\ 2(x - y) - 1 & -2(x - y) + 1 & 0 \end{bmatrix} < 2.
\]

This happens iff all \( 2 \times 2 \) sub-determinants vanish, that is

\[
\begin{cases} (1 - 2(x - y)) (x + y - 3) = 0, \\ 2(z + 3)(2(x - y) - 1) = 0, \\ 2(z + 3)(1 - 2(x - y)) = 0. \end{cases}
\]

The first equation yields to the alternative \( x - y = \frac{1}{2} \), and plugging this into the other two equations we get identities \( 0 = 0 \). Thus, we get points \( (x, x, \frac{1}{2}, z) \). Now these points belong to \( D \) iff \( \frac{1}{4} - \frac{1}{2} = 0 \), which is false.

In the second case, \( x + y = 3 \), and plugging this into the other two equations we get \( z = -3 \), thus points \( (x, 3 - x, -3) \). Now,

\( (x, 3 - x, -3) \in D, \iff (2x - 3)^2 - (2x - 3) = 0, \iff (2x - 3)(2x - 4) = 0, \iff x = \frac{3}{2}, \forall x = 2. \)

We get points \( (\frac{3}{2}, \frac{3}{2}, -3) \) and \( (2, 1, -3) \). Since \( f(\frac{3}{2}, \frac{3}{2}, -3) = \frac{1}{3} + \frac{1}{3} = \frac{1}{2} \) and \( f(2, 1, -3) = 1 + 1 = 2 \), we see that the points of \( D \) at minimum distance to \( (1, 2, -3) \) is \( (\frac{3}{2}, \frac{3}{2}, -3) \).

**Exercise 33.** i) \( D \) is closed because it is defined by large inequalities. It is not open because \( D \neq \emptyset, \mathbb{R}^3 \). It is unbounded since \( (n, n, \frac{1}{\cosh(2n\pi)}) \in D \) for every \( n \in \mathbb{N} \), therefore it is not compact.

ii) We have

\[
\lambda_3 (D) = \int_D 1 \, dx \, dy \, dz = \int_{\mathbb{R}^2} \left( \int_0^{1/\cosh(x^2 + y^2)} \frac{1}{\cosh(x^2 + y^2)} \, dz \right) \, dx \, dy = \int_{\mathbb{R}^2} \frac{1}{\cosh(x^2 + y^2)} \, dx \, dy.
\]

By introducing polar coordinates,

\[
\lambda_3 (D) = \int_{\rho > 0, \ 0 < \theta < 2\pi} \frac{1}{\cosh \rho^2} \rho \, d\rho \, d\theta = 2\pi \int_0^{+\infty} \frac{\rho}{\cosh \rho^2} \, d\rho.
\]

Notice that

\[
\frac{\rho}{\cosh \rho^2} = \frac{2\rho}{e^{\rho^2} + e^{-\rho^2}} = \frac{2\rho e^{\rho^2}}{1 + e^{2\rho^2}} = \partial_{\rho} \arctan(e^{\rho^2}),
\]
thus
\[
\lambda_3(D) = 2\pi \left[ \arctan(e^{\rho^2}) \right]_{\rho=0}^{\rho=\infty} = 2\pi \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = \frac{\pi^2}{2}.
\]

iii) Proceeding as in ii), we have
\[
I_{\alpha} := \int_D e^{\alpha(x^2+y^2)} \, dxdydz = \int_{\mathbb{R}^2} \left( \int_0^R e^{\alpha(x^2+y^2)\rho^2} \, d\rho \right) dx dy = \int_{\mathbb{R}^2} e^{\alpha(x^2+y^2)} \, dxdy.
\]
Changing vars to polar coords,
\[
I_{\alpha} = \int_{\rho \geq 0, \ 0 \leq \theta \leq 2\pi} \frac{e^{\alpha\rho^2}}{\cosh \rho^2} \, d\rho d\theta = 2\pi \int_0^{+\infty} 2\rho e^{\alpha \rho^2} \, d\rho.
\]
Notice that
\[
\frac{2\rho e^{\alpha \rho^2}}{1 + e^{2\rho^2}} \sim_{+\infty} 2\rho e^{(\alpha-1)\rho^2}
\]
and
\[
\exists \int_0^{+\infty} \rho e^{(\alpha-1)\rho^2} \, d\rho \iff \alpha - 1 < 0, \quad \implies \alpha < 1. \quad \Box
\]

**Exercise 34.** i) In order \( f = u + iv \) be \( \mathbb{C} \)-differentiable on \( \mathbb{C} \) we need 1. that \( u, v \) are \( \mathbb{R} \) differentiable on \( \mathbb{R}^2 \) (which is true, being \( u, v \) polynomials) and 2. \( u, v \) fulfil the CR equations, namely
\[
\begin{cases}
\partial_x u \equiv \partial_y v, \\
\partial_x u \equiv -\partial_y v,
\end{cases}
\]
\[
\begin{cases}
3x^2 + ay^2 \equiv bx^2 - 3y^2, \\
2axy \equiv -2bxy,
\end{cases}
\]
\[
\iff b = 3, \quad a = -3.
\]

ii) We have
\[
f = (x^3 - 3xy^2) + i(3x^2y - y^3) = (x + iy)^3 = z^3. \quad \Box
\]

**Exercise 35.** i) To prove that \( \phi(t) := E(y(t), y'(t)) \) is constant we show that the derivative of \( \phi \) w.r.t. \( t \) vanishes. According to the total derivative formula, we have
\[
\phi'(t) = \frac{d}{dt} E(y, y') = \partial_y E(y, y')y' + \partial_{y'} E(y, y')y''.
\]
Now,
\[
E(y, v) = \frac{1}{2} mv^2 - f(y), \quad \implies \partial_y E = -f'(y) = -F(y), \quad \partial_v E = mv,
\]
thus
\[
\phi'(t) = -F(y)y' + my'y'' = y' (my'' - F(y)) \equiv 0.
\]
Therefore
\[
E(y, y') \equiv k, \iff \frac{1}{2} m(y')^2 - f(y) \equiv k, \iff (y')^2 = \frac{2}{m} (f(y) + k), \iff y' = \pm \sqrt{\frac{2}{m} (f(y) + k)}.
\]
The last one is a separable variables equation.
ii) If \( m = 1 \) and \( F(y) = -2y - 3y^2 \), then \( f(y) = \int F(y)\,dy = \int (-2y - 3y^2) = -y^2 - y^3 \). Therefore

\[
y' = ±\sqrt{2(k - y^2 - y^3)},
\]

where \( E(y, y') \equiv k \). In particular, \( E(y(0), y'(0)) = k \), and since \( y(0) = -2, \ y'(0) = \sqrt{8} \) we have

\[
E(-2, \sqrt{8}) = \frac{1}{2}(\sqrt{8})^2 - ((-2)^2 - (-2)^3) = 4 - (-4 + 8) = 0.
\]

Thus \( k = 0 \) and \( y \) solves the equation

\[
y' = ±\sqrt{-2(y^3 + y^2)} = ±\sqrt{-2y^2(y + 1)} = ±\sqrt{2}y\sqrt{-y - 1}.
\]

Since at \( t = 0 \) we have \( y'(0) = \sqrt{8} > 0, \ y(0) = -2 < 0 \) the previous equation is

\[
y' = \sqrt{2}y\sqrt{-y - 1}.
\]

We can now solve this by separation of variables once we notice that \( y \) is not a constant solution. We have

\[
\int \frac{y'}{y\sqrt{-y - 1}} \,dt = -\int \sqrt{2} \,dt = -\sqrt{2}t + c.
\]

We have

\[
\int \frac{y'}{y\sqrt{-y - 1}} \,dt \quad \text{substitute} \quad u = y(t), \ du = y'(t) \,dt \quad \Rightarrow \quad \int \frac{1}{u\sqrt{u-1}} \,du = \int \frac{1}{\sqrt{v}} \,(2v) \,dv
\]

\[
= 2 \int \frac{1}{1+u^2} \,dv = 2 \arctan v = 2 \arctan \sqrt{-y - 1}.
\]

Therefore

\[
2 \arctan \sqrt{-y - 1} = -\sqrt{2}t + c.
\]

For \( t = 0 \) we have

\[
2 \arctan \sqrt{1} = c, \quad \iff \quad c = \frac{\pi}{2}.
\]

We conclude that

\[
2 \arctan \sqrt{-y - 1} = -\sqrt{2}t + \frac{\pi}{2}, \iff \sqrt{-y - 1} = \tan\left(-\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right), \iff \quad y(t) = -1 - \tan^2\left(-\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right). \quad \Box
\]

**Exercise 36.** i) We have a second order linear equation

\[
y'' + 9y = 6 \sin(3t).
\]

The homogeneous equation associated to this is \( y'' + 9y = 0 \), whose characteristic equation is \( \lambda^2 + 9 = 0 \), that is \( \lambda = ±3i \). The fundamental system of solutions for the homogeneous equation is then \( w_1(t) = \sin(3t), \ w_2(t) = \cos(3t) \), whose wronskian is

\[
W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w'_1 & w'_2 \end{bmatrix} = \det \begin{bmatrix} \sin(3t) & \cos(3t) \\ 3\cos(3t) & -3\sin(3t) \end{bmatrix} = -3(\sin^2(3t) + \cos^2(3t)) = -3.
\]
Therefore, the general solution for the original equation is

\[ y(t) = \left( c_1 - \int \frac{w_2}{W} 6 \sin(3t) \, dt \right) w_1 + \left( c_2 + \int \frac{w_1}{W} 6 \sin(3t) \, dt \right) w_2. \]

We have

\[ 6 \int \frac{w_2}{W} \sin(3t) \, dt = 6 \int \frac{\cos(3t)}{\sin(3t)} \sin(3t) \, dt = -\int \sin(6t) \, dt = \frac{1}{6} \cos(6t), \]

\[ 6 \int \frac{w_1}{W} \sin(3t) \, dt = 6 \int \frac{\sin(3t)}{\sin(3t)} \sin(3t) \, dt = -2 \int \sin^2(3t) \, dt. \]

Now

\[ \int \sin^2(3t) \, dt = \int (\sin(3t)) \left( -\frac{\cos(3t)}{3} \right)' \, dt = -\frac{1}{3} \sin(3t) \cos(3t) + \int \cos^2(3t) \, dt \]

thus

\[ \int \sin^2(3t) \, dt = \frac{1}{2} \left( t - \frac{\sin(6t)}{6} \right). \]

In conclusion,

\[ y(t) = \left( c_1 - \frac{\cos(6t)}{6} \right) \sin(3t) + \left( c_2 - t + \frac{\sin(6t)}{6} \right) \cos(3t), \quad c_1, c_2 \in \mathbb{R}. \]

ii) Imposing \( y(0) = 0 \) we get

\[ c_2 = 0, \]

thus

\[ y(t) = \left( c_1 - \frac{\cos(6t)}{6} \right) \sin(3t) - \left( t - \frac{\sin(6t)}{6} \right) \cos(3t). \]

Computing \( y'(t) \) we have

\[ y'(t) = \sin(6t) \sin(3t) + \left( c_1 - \frac{\cos(6t)}{6} \right) 3 \cos(3t) - (1 - \cos(6t)) \cos(3t) + \left( t - \frac{\sin(6t)}{6} \right) 3 \sin(3t), \]

and, by imposing \( y'(0) = 0 \) we get

\[ 3 \left( c_1 - \frac{1}{6} \right) = 0, \quad \iff \quad c_1 = \frac{1}{6}. \]

The solution of the CP is then

\[ y(t) = \frac{1}{6} \left( 1 - \cos(6t) \right) \sin(3t) - \left( t - \frac{\sin(6t)}{6} \right) \cos(3t). \]

iii) We may write the general solution in the form

\[ y(t) = \begin{cases} \text{bounded} & 0 \leq c_1, c_2, 0 < t < \infty, \\ \text{unbounded} & \end{cases} \]

and since the unbounded component is independent of \( c_1, c_2 \) we deduce that all the solutions are unbounded for \( t \rightarrow \pm \infty \).

\[ \square \]
Exercise 37. i) $D$ is closed being defined by large inequalities involving continuous functions of $(x, y)$. It is not open since $D \neq \emptyset, \mathbb{R}^2$. It is bounded because $x \geq 0$ and from $0 \leq y \leq 1 - x$, in particular $1 - x \geq 0$, that is $x \leq 1$, so $0 \leq x \leq 1$ and, at same time, $0 \leq y \leq 1 - x \leq 1$. Thus $0 \leq x, y \leq 1$ and this implies that $D$ is bounded. Since $D$ is closed and bounded it is also compact.

ii) Since $f$ is clearly continuous on $D$ and $D$ is compact, $f$ admits both global min/max on $D$. To determine min/max points, we may argue in the following way. If $(x, y) \in D$ is a min/max point for $f$ then

- either $(x, y) \in \text{Int} D$
- or $(x, y) \in D \setminus \text{Int} D = \partial D$.

In the first case, since

$$
\begin{align*}
\partial_x f &= 3y + 2xy + y^2, \\
\partial_y f &= 3x + x^2 + 2xy
\end{align*}
$$

so $\partial_x f, \partial_y f \in \mathcal{C}(D)$, $f$ is then differentiable on $D$, according to Fermat theorem, at min/max points

$$
\nabla f(x, y) = 0, \quad \iff \quad \begin{cases} 3y + 2xy + y^2 = 0, \\
3x + x^2 + 2xy = 0. \end{cases}
$$

The first equation leads to the alternative $y = 0$ or $3 + 2x + y = 0$. In the first case, the second equation becomes $x(3 + x) = 0$. whose solutions are $x = 0$ and $x = -3$. This produces points $(0, 0)$ and $(-3, 0)$.

In any case these do not belong to $\text{Int} D$. In the second case, $y = -2x - 3$, from the second equation we obtain $x(-3 - 3x) = 0$, that is $x = 0$ or $x = -1$. This yields points $(0, -3), (-1, -1) \notin D$. In conclusion, no stationary point for $f$ is in the interior of $D$.

Thus, min/max points for $f$ are on $\partial D = A \cup B \cup C$ where $A = \{(0, y) : 0 \leq y \leq 1\}$, $B = \{(x, 0) : 0 \leq x \leq 1\}$ and, finally, $C = \{(x, 1 - x) : 0 \leq x \leq 1\}$. On $A$ we have

$$
f(0, y) = 0,
$$

thus every point is min/max point for $f$ on $A$. On $B$, similarly, we have $f(x, 0) = 0$, thus every point of $B$ is at same time min/max for $f$ on $B$. Finally, on $C$

$$
f(x, 1 - x) = 3x(1 - x) + x^2(1 - x) + x(1 - x)^2 = 3x - 3x^2 + x^2 - x^3 + x - 2x^2 + x^3 = -4x^2 + 4x =: g(x).
$$

Let’s determine min/max points for $g$ with $x \in [0, 1]$. We have $g'(x) = -8x + 4 \geq 0$ iff $x \leq \frac{1}{2}$. Thus $x = \frac{1}{2}$ is max point for $g$ and $x = 0, 1$ are min points for $g$. This means that

- $\left(\frac{1}{2}, \frac{1}{2}\right)$ is max point for $f$ on $C$
- $(0, 1), (1, 0)$ are min points for $f$ on $C$.

We can now draw the conclusion:

- for minimum, candidates are points $(x, 0), (0, y)$ with $0 \leq x, y \leq 1$ where $f = 0$. All these are min points for $f$ on $D$;
- for maximum, candidates are points $\left(\frac{1}{2}, \frac{1}{2}\right)$ (where $f = 1$) and $(x, 0)$ and $(0, y)$ with $0 \leq x, y \leq 1$ (where $f = 0$). Thus, the max point is $\left(\frac{1}{2}, \frac{1}{2}\right)$. 

Exercise 38. i) Let $\vec{F} = (\phi, \psi)$. In order $\vec{F}$ be irrotational on $D$ we need
\[ \partial_y \phi \equiv \partial_x \psi, \text{ on } D. \]

We have
\[ \partial_y \phi = \frac{b(x^2+y^2)^2-(ax+by)2(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{b(x^2+y^2)^2-4y(ax+by)}{(x^2+y^2)^2} = \frac{bx^2-4axy-3by^2}{(x^2+y^2)^2}, \]
\[ \partial_x \psi = \frac{c(x^2+y^2)^2-(cx+dy)2(x^2+y^2)2x}{(x^2+y^2)^4} = \frac{c(x^2+y^2)^2-4x(cx+dy)}{(x^2+y^2)^2} = \frac{-3cx^2-4dxy+cy^2}{(x^2+y^2)^2}. \]

Hence,
\[ \partial_y \phi \equiv \partial_x \psi, \iff bx^2-4axy-3by^2 \equiv -3cx^2-4dxy+cy^2, \iff \begin{cases} b = -3c, \\ a = d, \\ -3b = c \end{cases} \]
from which $b = c = 0$ and $a = d \in \mathbb{R}$. Thus
\[ \vec{F} = \left( \frac{ax}{(x^2+y^2)^2}, \frac{ay}{(x^2+y^2)^2} \right), \forall (x, y) \in D. \]

ii) Necessary condition to be conservative is that $\vec{F}$ be irrotational, thus $\vec{F}$ is given as at the end of i). Now, such $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is
\[ \begin{cases} \partial_x f = \frac{ax}{(x^2+y^2)^2}, \\ \partial_y f = \frac{ay}{(x^2+y^2)^2}. \end{cases} \]

From the first equation
\[ f(x, y) = \int \frac{ax}{(x^2+y^2)^2} \, dx + k(y) = \frac{a}{2} \int \partial_x - (x^2+y^2)^{-1} \, dx + k(y) = -\frac{a}{2} (x^2+y^2)^{-1} + k(y). \]

Plugging this into the second equation we have
\[ \partial_y f = \frac{ay}{(x^2+y^2)^2}, \iff \frac{ay}{(x^2+y^2)^2} + k'(y) = \frac{ay}{(x^2+y^2)^2}, \iff k'(y) = 0, \iff .k(y) = k \in \mathbb{R}. \]

Thus, $\vec{F}$ is conservative with potentials
\[ f(x, y) = -\frac{a}{2} (x^2+y^2)^{-1} + k, \ k \in \mathbb{R}. \]

iii) By previous discussion, when $(a, b, c, d) = (2, 0, 0, 2)$, field $\vec{F}$ is conservative. Thus
\[ \int_y \vec{F} = f(0, 2) - f(1, 0) = -\frac{1}{4} - (-1) = \frac{3}{4}. \]

Exercise 39. i) Since $x^2 + z^2$ is invariant by rotations around the $y$–axis, $D$ is invariant by rotations around such axis. We can draw any section containing the $y$ axis, for instance $D \cap \{x = 0\}$ (section of $D$ in plane $yz$). We have
\[ D \cap \{x = 0\} = \{(0, y, z) : 1 - z^2 \geq y \leq \sqrt{1-z^2}\}. \]

Figure:
Multiplying the first relation by $u$ and because of CR equations

Now, $1 - \rho^2 \leq 1$ always true, the condition on $\rho$ is $\rho^2 \leq 1$, that is $0 \leq \rho < 1$. In conclusion,

$$
\lambda_3(D) = 2\pi \int_0^1 \rho \left( \sqrt{1 - \rho^2} - (1 - \rho^2) \right) d\rho = 2\pi \int_0^1 \rho (1 - \rho^2)^{1/2} - \rho + \rho^3 \, d\rho
$$

$$
= 2\pi \left( \left[ -\frac{1}{3} (1 - \rho^2)^{3/2} \right]_{\rho=0}^{\rho=1} - \left[ \frac{\rho^2}{2} \right]_{\rho=0}^{\rho=1} + \left[ \frac{\rho^4}{4} \right]_{\rho=0}^{\rho=1} \right) = 2\pi \left( \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right) = \frac{\pi}{6}. \quad \square
$$

**Exercise 40.** See notes for CR equations and connection with $\mathbb{C}$–differendiability.

i) If $f = u + iv$ with, for example, $u$ constant, then, by the CR eqns,

$$
\begin{cases}
0 \equiv \partial_x u \equiv \partial_y v, \\
0 = \partial_y u \equiv -\partial_x v,
\end{cases} \implies \begin{cases}
\partial_x v \equiv 0, \\
\partial_y v \equiv 0.
\end{cases}
$$

From this it follows that $v$ is constant.

iii) Suppose now that $f = u + iv$ be $\mathbb{C}$–differendiable and such that $|f| = \sqrt{u^2 + v^2} = k$ or, equivalently, $u^2 + v^2 \equiv k^2$. If $k = 0$ the conclusion is trivial. Assume $k \neq 0$. By computing $\partial_x$ we have

$$
2u \partial_x u + 2v \partial_x v \equiv 0,
$$

and because of CR equations

$$
u \partial_x u - v \partial_y u = 0. 
$$

Similarly, computing $\partial_y$

$$
2u \partial_y u + 2v \partial_y v = 0, \iff u \partial_y u + v \partial_x u = 0.
$$

Multiplying the first relation by $\partial_x u$ and the second by $\partial_y u$ we obtain

$$
u (\partial_x u)^2 \equiv v \partial_x u \partial_x u = -u (\partial_y u)^2, \iff u \left( (\partial_x u)^2 + (\partial_y u)^2 \right) \equiv 0. \iff u^2 \left( (\partial_x u)^2 + (\partial_y u)^2 \right) \equiv 0.
$$

Similarly,

$$
v^2 \left( (\partial_x v)^2 + (\partial_y v)^2 \right) \equiv 0.$$
By CR eqns, \((\partial_x u)^2 + (\partial_y u)^2 \equiv (\partial_x v)^2 + (\partial_y v)^2\), thus summing up the two previous relations we get
\[
(u^2 + v^2) \left((\partial_x u)^2 + (\partial_y u)^2\right) \equiv 0, \quad \iff \quad k^2 \left((\partial_x u)^2 + (\partial_y u)^2\right) \equiv 0, \quad \iff \quad (\partial_x u)^2 + (\partial_y u)^2 \equiv 0,
\]
which means \(\partial_x u \equiv \partial_y u \equiv 0\). Thus \(u\) is constant and we can now conclude by ii).

**Exercise 41.** i) We have a separable variables equation. Solutions are either constant or obtained by separation of variables. In the first case, \(y \equiv C\) is a solution iff \(C(C^2 + 1) = 0\), that is \(C = 0\). Other solution are obtained by separation of variables:
\[
y' = y(y^2 + 1), \quad \iff \quad \frac{y'}{y(y^2 + 1)} = 1, \quad \iff \quad \int \frac{y'}{y(y^2 + 1)} \, dt = t + k.
\]
Now,
\[
\int \frac{y'}{y(y^2 + 1)} \, dt = \frac{u(y(t))}{u(y^2 + 1)} \, du = \int \frac{1}{u(u^2 + 1)} \, du.
\]
According to Hermite decomposition,
\[
\frac{1}{u(u^2 + 1)} = \frac{A}{u} + \frac{Bu + C}{u^2 + 1}
\]
from which \(A = 1, B = -1\) and \(C = 0\). Therefore
\[
\int \frac{1}{u(u^2 + 1)} \, du = \log |u| - \frac{1}{2} \log(u^2 + 1) = \log \left|\frac{u}{\sqrt{u^2 + 1}}\right|.
\]
Thus we have
\[
\log \left|\frac{y}{\sqrt{y^2 + 1}}\right| = t + k,
\]
that is
\[
\frac{|y|}{\sqrt{y^2 + 1}} = ke^t, \quad \iff \quad \frac{y^2}{y^2 + 1} = ke^{2t}, \quad (k > 0) \quad \iff \quad y^2 = \frac{ke^{2t}}{1 - ke^{2t}}, \quad \iff \quad y = \pm \sqrt{\frac{ke^{2t}}{1 - ke^{2t}}}.
\]
i) The solution for which \(y(0) = 1\) cannot be a constant solution. Since \(y(0) = 1\), we have
\[
y(t) = \sqrt{\frac{ke^{2t}}{1 - ke^{2t}}},
\]
and \(y(0) = 1\) means \(\sqrt{\frac{k}{1-k}} = 1\), that is \(k = \frac{1}{2}\).

**Exercise 42.** i) Let \((g_1, g_2) := (x^2 + y^2 - 1, x + y + z - 1)\) in such a way \(D = \{g_1 = 0, \ g_2 = 0\}\). To check that \((g_1, g_2)\) is a submersion on \(D\) we have to verify that
\[
\text{rk} \begin{pmatrix} g_1 \\ \nabla g_2 \end{pmatrix} = \text{rk} \begin{pmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{pmatrix} = 2, \ \forall (x, y, z) \in D.
\]
Now, rank is \(< 2\) iff the two gradients are linearly dependent. This is manifestly impossible because of their third component.
ii) $D$ is closed being defined by equalities involving continuous functions. $D$ is also bounded: indeed, by first equation we have $x^2, y^2 < 1$, thus $-1 \leq x, y \leq 1$, and by the second

$$-1 \geq z = 1 - (x + y) \leq 3,$$

thus $z^2 \leq 9$ and $x^2 + y^2 + z^2 \leq 11$.

iii) Function $f$ is continuous on $D$ compact: existence of min/max is ensured by Weierstrass thm. To determine these points, we apply Lagrange multipliers thm. By i), $D$ fulfills the assumption of the thm. Thus, at $(x, y, z)$ min/max point for $f$ on $D$ we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \iff \quad \text{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \text{rk} \begin{bmatrix} 2x + y - 1 & 2y + x + z - 1 & y \\ 2x & 2y & 0 \\ 1 & 1 & 1 \end{bmatrix} < 3,$$

that is iff the determinant of previous matrix vanishes. We get the condition

$$2y(x - y) + 2(y(2x + y - 1) - x(2y + x + z - 1)) = 0,$$

from which, simplifying,

$$y(y - x) + (y^2 - y - x^2 + x - xz) = 0.$$

Since we are looking for solutions $(x, y, z) \in D$, we must have $z = 1 - x - y$, and plugging this into previous equation yields,

$$y(2y - 1) = 0, \quad \iff \quad y = 0, \forall y = \frac{1}{2}.$$

Thus we get points $(x, 0, 1 - x)$ and $(x, \frac{1}{2}, \frac{1}{2} - x)$, to which we have still to impose the condition $x^2 + y^2 = 1$. In the first case $x^2 + 0^2 = 1$, thus $x = \pm 1$, that is points $(\pm 1, 0, \mp 1)$ (two points). In the second case, $x^2 + \frac{1}{4} = 1$, thus $x = \frac{3}{4}$ and $x = \pm \frac{\sqrt{3}}{2}$, that is points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1 - \sqrt{3}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1 + \sqrt{3}}{2}\right)$. We have

- $f(1, 0, -1) = 0$
- $f(-1, 0, 1) = 2$
- $f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1 - \sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} - 2)$
- $f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1 + \sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} + 2)$

From this we see that $(-1, 0, 1)$ is max point, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1 - \sqrt{3}}{2}\right)$ is min point. □

**Exercise 43.** i) $D$ is closed, being defined by large inequalities involving continuous functions. Let’s check that $D$ is bounded (hence compact). Denoting by $\rho = \sqrt{x^2 + y^2} = \|(x, y)\|$ we have

$$(x, y) \in D, \quad \implies \rho^2 \leq 2\rho \cos \theta - \rho = \rho(2\cos \theta - 1), \quad \iff \quad \rho \leq 2\cos \theta - 1 \leq 1.$$

Therefore, $D$ is bounded. In particular, $D$ cannot be be open: only $\emptyset, \mathbb{R}^2$ are both open and closed, and $(0, 0) \in D$ (thus $D \neq \emptyset$), and $D$ is bounded, thus $D \subseteq \mathbb{R}^2$.

ii) The area of $D$ is

$$\lambda_2(D) = \int_D 1 \, dxdy = \int_{x^2+y^2 \leq 2x - \sqrt{x^2+y^2}} 1 \, dxdy = \int_{\rho \leq 2\cos \theta - 1} \rho \, d\rho d\theta.$$
Now, notice that since \( \rho \geq 0 \), this imposes \( 2 \cos \theta - 1 \geq 0 \), that is \( \cos \theta \geq \frac{1}{2} \). In one period this means \(-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}\). Thus

\[
\lambda_2(D) = \int_{\rho \leq 2 \cos \theta - 1, -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}} \rho \ d\rho d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{0}^{2 \cos \theta - 1} \rho \ d\rho \ d\theta = \frac{1}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2 \cos \theta - 1)^2 \ d\theta
\]

\[
= \frac{1}{2} \left( \frac{2\pi}{3} - 4 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos \theta \ d\theta + 4 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos \theta)^2 \ d\theta \right)
\]

\[
= \frac{4}{3} - 2\sqrt{3} + 2 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos \theta)^2 \ d\theta.
\]

About this last integral we have

\[
\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos \theta)^2 \ d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos \theta)(\sin \theta)' \ d\theta = [\sin \theta \cos \theta]_{\theta=-\pi/3}^{\theta=\pi/3} + \int_{0}^{2\pi} (\cos \theta)^2 \ d\theta = \frac{\sqrt{3}}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos \theta)^2 \ d\theta,
\]

from which \( \frac{\pi}{3} (\cos \theta)^2 \ d\theta = \frac{\sqrt{3}}{4} \). We conclude that \( \lambda_2(D) = \frac{4}{3} - \frac{3\sqrt{3}}{2} \).

**Exercise 44.**

i) In order \( f = u + iv \) be \( \mathbb{C} \)-differentiable on \( \mathbb{C} \), we need \( u, v \) \( \mathbb{R} \)-differentiable on \( \mathbb{R}^2 \) and fulfilling the CR equations. About \( u \) it is clear that, being \( \partial_x u, \partial_y u \in \mathcal{C}(\mathbb{R}^2) \), \( u \) is \( \mathbb{R} \)-differentiable on \( \mathbb{R}^2 \) by the differentiability test. Thus, we look for a \( v \) differentiable such that

\[
\begin{align*}
\partial_x u &= \partial_y v, \\
\partial_y u &= -\partial_x v,
\end{align*}
\]

\[
\Leftrightarrow \quad \begin{align*}
\partial_x v &= -\partial_y u = -(20x^3y + 20xy^3), \\
\partial_y v &= \partial_x u = 5x^4 - 30x^2y^2 + 5y^4.
\end{align*}
\]

From first equation,

\[
v(x, y) = \int 20x^3y - 20xy^3 \ dx + k(y) = 5x^4y - 10x^2y^3 + k(y),
\]

and plugging this into the second equation we have

\[
5x^4 - 30x^2y^2 + k'(y) = 5x^4 - 30x^2y^2 + 5y^4, \quad \Leftrightarrow \quad k'(y) = 5y^4, \quad \Leftrightarrow \quad k(y) = y^5 + k,
\]

where \( k \) is now a constant. Thus, the \( v \) that fulfils CR eqns together with \( u \) is

\[
v(x, y) = 5x^4y - 10x^2y^3 + 5y^4 + k,
\]

and since this is also differentiable (being \( \partial_x v, \partial_y v \in \mathcal{C}(\mathbb{R}^2) \)), we conclude that \( f = u + iv \) is \( \mathbb{C} \)-differentiable.

ii) We have

\[
f = \left(x^5 - 10x^3y^2 + 5xy^4\right) + i \left(5x^4y - 10x^2y^3 + 5y^4 + k\right)
\]

Noticed that, for \( z = x + iy \),

\[
z^5 = (x + iy)^5 = x^5 + 5ix^4y - 10x^3y^2 - i10x^2y^3 + 5xy^4 + iy^5
\]

thus \( f = z^5 + ik, k \in \mathbb{R} \).

**Exercise 45.** See notes for definitions. We aim to prove that \( f^{-1}(S) \) is open if \( S \) it is. Suppose this is false. There exists then a point \( x \in f^{-1}(S) \) for which

\[
\not\exists B(x, r) \subset f^{-1}(S).
\]
This means that:

$$\forall r > 0, \ B(x, r) \cap f^{-1}(S)^c \neq \emptyset.$$  

Taking \( r = \frac{1}{n} \)

$$\forall n \in \mathbb{N}, \ n \geq 1, \ \exists x_n \in B(x, 1/n] \cap f^{-1}(S)^c.$$ 

This means that \( \|x_n - x\| \leq \frac{1}{n} \), thus \( x_n \rightarrow x \). By continuity, \( f(x_n) \rightarrow f(x) \). Furthermore, by construction of \( (x_n) \), we have that \( x_n \in f^{-1}(S)^c \), that is \( f(x_n) \notin S \) for every \( n \). However, since \( f(x) \in S \) (recall that \( x \in f^{-1}(S) \)), and \( S \) is supposed to be open,

$$\exists B(f(x), \rho) \subset S.$$ 

And since \( f(x_n) \rightarrow f(x) \), we have that

$$\exists N : f(x_n) \in B(f(x), \rho) \subset S, \ \forall n \geq N,$$

which is in contradiction with the construction of \( (x_n) \). We deduce that the initial assumption must be false, that is \( f^{-1}(S) \) is open.  \( \Box \)