Calculus 2 — Final Exam

Exercise 1. Consider the Cauchy problem

$$\begin{cases} y' = \dfrac{y^2 - 4}{t}, \\ y(1) = 0. \end{cases}$$

i) Determine the solution.
ii) Determine the domain of definition $]a, b[$ of the solution and the limits of $y(t)$ when $t \longrightarrow a$ and $t \longrightarrow b$.

Exercise 2. Let

$$D := \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1 + xy \right\}.$$

i) Show that $D \neq \emptyset$ is the zero set of a submersion.
ii) Is D compact?
iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 3. Let

$$D := \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ (x^2 + y^2)^{1/4} \leqslant z \leqslant 2 - x^2 - y^2 \right\}.$$

i) Draw $D \cap \{x = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 4. Let

$$v(x, y) := e^{-y} \left(y \cos x + x \sin x \right), \quad (x, y) \in \mathbb{R}^2.$$

i) Determine all possible $u = u(x, y)$ in such a way that $f(x + iy) := u(x, y) + iv(x, y)$ be $\mathbb{C}-$differentiable on $\mathbb{R}^2$.
ii) Express the f found at i) as function of complex number z, that is $f = f(z)$.

Exercise 5. State the Green formula. Let $f \in \mathscr{C}(\mathbb{R}^2)$ with $\partial_i f, \partial_j(\partial_i f) \in \mathscr{C}(\mathbb{R}^2)$, for all $i, j = 1, 2$. Prove that

$$\oint_{\partial D} f \nabla f = 0.$$

Exercise 6. Consider the equation
$$y' = \frac{e^y - 1}{t},\ t \neq 0.$$

i) Determine the constant solutions.
ii) Determine the solution of the Cauchy problem $y(1) = -1$.
iii) Determine in particular the domain of definition $]a, b[$ of the solution and its limits when $t \longrightarrow a+$ and $t \longrightarrow b-$.

Exercise 7. Let
$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1,\ y^2 + z = 1\}.$$

i) Show that $D \neq \emptyset$ is the zero set of a submersion (g_1, g_2).
ii) Is D compact?
iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 8. Let
$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 \leqslant z \leqslant 1 - y^2\}.$$

i) Draw $D \cap \{x = 0\}$ and $D \cap \{y = 0\}$. Is D invariant by some rotation? Justify your answer. Draw D as best as you can.
ii) Compute the volume of D.

Exercise 9. Let
$$\vec{F} := \left(\frac{ax^2 + by^2}{(x^2 + y^2)^2}, \frac{xy}{(x^2 + y^2)^2}\right)$$
on $D = \mathbb{R}^2 \backslash \{(0, 0)\}$. Here $a, b \in \mathbb{R}$ are constants.

i) Determine all possible values for a, b in such a way $\vec{F}$ be irrotational on D.
ii) Determine values of a, b, c in such a way $\vec{F}$ be conservative on D, in this case determining also all the possible potentials.

Exercise 10. What are the Cauchy–Riemann equations (or conditions)? State precisely. Then, let $f = u + iv$ ($u = \operatorname{Re} f$ and $v = \operatorname{Im} f$) be a $\mathbb{C}$ differentiable function on the entire plane $\mathbb{C}$. Assume that also $\overline{f} = u - iv = u + i(-v)$ is $\mathbb{C}$ differentiable on $\mathbb{C}$. What conclusion can you draw on f?

February 2023

Exercise 11. Consider the second order equation

$$y'' - 2y' + y = e^{2t}.$$

i) Determine the general integral.
ii) Solve the Cauchy problem $y(0) = 1$, $y'(0) = 0$.
iii) For which $a \in \mathbb{R}$ there exists a solution such that $y(0) = 0$ and $y(1) = a$?

Exercise 12. Let

$$f(x,y) := (x^2 + y^2)^3 - x^4 + y^4, \ (x,y) \in \mathbb{R}^2.$$

i) Compute, if it exists, $\lim_{(x,y)\to\infty_2} f(x,y)$.
ii) Discuss existence of min/max of f on $\mathbb{R}^2$ and find the eventual min/max points of f. What about $f(\mathbb{R}^2)$?

Exercise 13. Let $D := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + 2y^2 \leqslant z \leqslant 4 - 3(x^2 + 2y^2)\}$.

i) Draw the set D. Someone says: "D is a rotation volume with respect to the z−axis". Is it true or false?
ii) Compute the volume of D.

Exercise 14. Let

$$u(x,y) := x^2 + y^2.$$

i) Determine, if any, $v = v(x,y)$ in such a way that $f(x+iy) := u(x,y) + iv(x,y)$ be $\mathbb{C}$−differentiable on $\mathbb{C}$.
ii) For the f you found at i), write $f = f(z)$ as function of $z \in \mathbb{C}$.

Exercise 15. State the Lagrange multipliers theorem. Then, consider a curve $y = f(x)$ defined by a function $f = f(x) : \mathbb{R} \longrightarrow \mathbb{R}$, $f \in \mathscr{C}^1(\mathbb{R})$. Let $P = (a,b)$ a point in the cartesian plane not belonging to the curve $y = f(x)$. Prove that if Q is a point of the curve $y = f(x)$ where the distance to P is minimum, then the segment $P - Q$ is perpendicular to the tangent to f.

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Exercise 16. Consider the differential equation

$$y' = \frac{t - ty^2}{y + t^2 y}.$$

i) Show that it is a separable variables equation and determine all possible constant solutions.
ii) Determine the solution of the Cauchy Problem with passage condition $y(0) = 2$.

Exercise 17. Let $\Gamma \subset \mathbb{R}^3$ the set described by equations

$$\Gamma \ : \ \begin{cases} x^2 + y^2 = 1, \\ \\ x^2 + z^2 = xz + 1. \end{cases}$$

i) Show that $\Gamma \neq \emptyset$ is the zero set of a submersion on Γ.
ii) Is Γ compact? Justify your answer.
iii) Determine points of Γ at minimum/maximum distance to $(0, 0, 0)$ (if any).

Exercise 18. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ 1 - (x^2 + y^2) \leqslant z \leqslant \sqrt{1 - (x^2 + y^2)}\}$.
i) Draw $D \cap \{y = 0\}$ and deduce a figure for D.
ii) Compute the volume of D.

Exercise 19. Let $f = u + iv$ where

$$u(x, y) := ax^2 + bxy + cy^2, \quad v(x, y) := xy, \quad x + iy \in \mathbb{C}.$$

(a, b, c are real constant)
i) Determine all possible a, b, c such that f be holomorphic on $\mathbb{C}$.
ii) For values found at i), determine the analytical expression for $f = f(z)$ in terms of variable $z \in \mathbb{C}$.

Exercise 20. Let $\vec{a}_1, \ldots, \vec{a}_N \in \mathbb{R}^d$ be N fixed vectors, $\vec{a}_i \neq \vec{a}_j$ for $i \neq j$. Define

$$f(\vec{x}) := \sum_{j=1}^{N} \|\vec{x} - a_j\|^2.$$

Discuss the problem of determining, if any, points of min/max for f on $\mathbb{R}^d$. Justify carefully, state all general facts you use.

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Exercise 21. Consider the equation
$$y' = y \log y.$$

i) Determine, if any, all constant solutions.
ii) Solve the Cauchy problem with $y(0) = a$.
iii) Determine, if any, values of a such that $\lim_{t\to+\infty} y(t) = 0$.

Exercise 22. Let $D := \{(x, y, z) \in \mathbb{R}^3 : \quad x^2 = y^2 + z^2, \quad x^2 + y^2 = xy + 1\}$.

i) Show that D is the zero set of a submersion on D itself.
ii) Is D compact? Justify your answer.
iii) Determine, if any, the points of D at the min / max distance to the origin.

Exercise 23. Consider the vector field
$$\vec{F}(x, y) := \left(\frac{ax + by}{\sqrt{x^2 + y^2}}, \frac{cx + dy}{\sqrt{x^2 + y^2}}\right),\ (x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}.$$

i) Find all possible values of $a, b, c, d \in \mathbb{R}$ such that $\vec{F}$ is irrotational.
ii) Find all possible values for a, b, c, d such that $\vec{F}$ is conservative. For such values, determine the potentials of $\vec{F}$.

Exercise 24. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + 4y^2 - z^2 \leqslant 1,\ 0 \leqslant z \leqslant 1\}$. Draw D and calculate its volume.

Exercise 25. Let $f = u + iv$ be holomorphic on $D \subset \mathbb{C}$. Define
$$g(z) := \overline{f(\overline{z})},\ z \in \overline{D} := \{w \in \mathbb{C} \ : \ \overline{w} \in D\}$$

i) Express real and imaginary part of g in terms of real and imaginary parts u and v of f.
ii) Use i) to discuss whether g is holomorphic on $\overline{D}$ or not.

Exercise 26. Consider the differential equation

$$y" + 2y' + y = t + 1.$$

i) Determine the general integral of the equation.
ii) Solve the Cauchy problem $y(0) = 0$, $y'(0) = 1$.
iii) Discuss the boundary value problem $y(0) = 0$, $y(1) = 0$.

Exercise 27. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = z^2,\ y^2 + (z-2)^2 = 1\right\}.$$

i) Show that $D \neq \emptyset$ and it is the zero set of a submersion.
ii) Is D compact? Prove or disprove.
iii) Find points of D at min/max distance to $\vec{0}$.

Exercise 28. Let $D := \{(x, y) \in \mathbb{R}^2 \ : \ x \geqslant 1,\ x^3 \leqslant y \leqslant 3\}$.

i) Draw D.
ii) By using the change of variables $u = y - x^3$, $v = y + x^3$, compute the integral

$$\int_D x^2(y - x^3)e^{y+x^3}\, dxdy.$$

Exercise 29. Let $v(x, y) := y^3 - 3x^2y + 4xy - x$, $(x, y) \in \mathbb{R}^2$. Determine all possible $u = u(x, y)$ such that

$$f(x + iy) := u(x, y) + iv(x, y),$$

be holomorphic on $\mathbb{C}$. What is $f(z)$ as a function of z?

Exercise 30. What does it mean that a set $C \subset \mathbb{R}^d$ is closed? What is the Cantor characterization of closed sets?

Given a generic set $S \subset \mathbb{R}^d$, we define the frontier of S as the set

$$\partial S := \left\{\vec{x} \in \mathbb{R}^d \ : \ \forall r > 0,\ B(\vec{x}, r] \cap S \neq \varnothing,\ B(\vec{x}, r] \cap S^c \neq \varnothing\right\}.$$

Is ∂S always closed? Justify your answer providing a proof if yes, a counterexample if no.

Exam Simulation

Exercise 31. Solve the following equation in the unknown $z \in \mathbb{C}$:

$$\sinh \frac{1}{z} = 0.$$

Exercise 32. Consider the set (surface)

$$D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - 2xy + y^2 - x + y = 0\}.$$

Determine, if any, points of D at min/max distance to the point $(1, 2, -3)$. Justify carefully the method you use.

Exercise 33. Let

$$D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ 0 \leqslant z \leqslant \frac{1}{\cosh(x^2 + y^2)}\right\}.$$

i) Draw $D \cap \{x = 0\}$ and deduce the figure of D. Is D closed? Open? Bounded? Compact? Justify your answer.
ii) Determine the volume of D.
iii) Determine for which values of α the following integral has a finite value:

$$\int_D e^{\alpha(x^2+y^2)}\, dxdydz.$$

Exercise 34. Let

$$u(x, y) := x^3 + axy^2, \quad v(x, y) := bx^2y - y^3, \quad (x, y) \in \mathbb{R}^2.$$

i) Determine $a, b \in \mathbb{R}$ in such a way that $f(x + iy) := u(x, y) + iv(x, y)$ be holomorphic on $\mathbb{C}$.
ii) For values of a, b found at i), express f as a function of the complex variable z.

Exercise 35. Consider a Newton equation of type

$$my'' = F(y).$$

Suppose that force F admits a potential, that is $F(y) = f'(y)$. Define the potential energy

$$E(y, v) := \frac{1}{2}mv^2 - f(y).$$

i) Prove that $E(y, y') = E(y(t), y'(t))$ is a constant function of t. Deduce that y solves a first order separable variables equation.
ii) Assume $m = 1$ and let $F(y) = -2y - 3y^2$ (elastic force plus viscosity). Determine the motion of the mass with $y(0) = -2$, $y'(0) = \sqrt{8}$.

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Exercise 36. Consider the equation

$$y'' = -9y + 6\sin(3t).$$

This equation represents the motion of a unitary mass particle subject to an elastic force (constant of elasticity $k = -9$) and to an external force $F(t) = 6\sin(3t)$.

i) Determine the general solution of the equation.
ii) Solve the Cauchy problem $y(0) = y'(0) = 0$.
iii) Describe the long time (that is $t \longrightarrow +\infty$) of the general solution. In particular: are there solutions for which $\exists \lim_{t\to+\infty} y(t)$? are there solutions which are bounded, that is $|y(t)| \leqslant M$ for all $t \geqslant 0$ for some constant M? Justify carefully.

Exercise 37. Let

$$f(x, y) := 3xy + x^2y + xy^2,\ \ (x, y) \in D := \{(x, y) \in \mathbb{R}^2 \ :\ x \geqslant 0,\ 0 \leqslant y \leqslant 1 - x\}.$$

i) Draw D. Is D closed? open? bounded? compact? Justify carefully.
ii) Discuss the problem of determining min/max (if any) of f on D.

Exercise 38. Let $a, b, c, d \in \mathbb{R}$ and

$$\vec{F}(x, y) := \left(\frac{ax + by}{(x^2 + y^2)^2}, \frac{cx + dy}{(x^2 + y^2)^2} \right),\ \ (x, y) \in D := \mathbb{R}^2 \setminus \{(0, 0)\}.$$

i) Determine $a, b, c, d \in \mathbb{R}$ in such a way that $\vec{F}$ be irrotational on D.
ii) Determine a, b, c, d such that $\vec{D}$ be conservative on D. For these values (if any), determine all possible potentials of $\vec{F}$ on D.
iii) Let $\gamma = \gamma(t) \subset D$ be the segment joining $(1, 0)$ to $(0, 2)$. For $(a, b, c, d) = (2, 0, 0, 2)$ compute

$$\int_\gamma \vec{F}.$$

Exercise 39. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ :\ 1 - (x^2 + z^2) \leqslant y \leqslant \sqrt{1 - (x^2 + z^2)}\}$.

i) Draw D. Is D a rotation solid?
ii) Compute the volume of D.

Exercise 40. Let $f = u + iv : \mathbb{C} \longrightarrow \mathbb{C}$ be a $\mathbb{C}-$differentiable function. What are the Cauchy-Riemann equations? How are these equations relatived to $\mathbb{C}-$differentiability of f? Write a precise statement.

Discuss the following questions:

i) Assume that Re f or Im f is constant. What can be drawn on f?
ii) Assume that $|f|$ is constant. What can be drawn on f? (hint: $|f|^2 = u^2 + v^2 \equiv k \dots$)

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Exercise 41. Consider the equation

$$y' = y(y^2 + 1).$$

i) Determine the general integral of the equation.
ii) Determine the solution of the Cauchy problem $y(0) = 1$.

Exercise 42. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = 1,\ x + y + z = 1\}$.

i) Show that D is the zero set of a submersion.
ii) Is D compact?
iii) Determine, if any, min/max points for $f(x, y, z) = x^2 - x + y^2 + yx + yz - y$ on D.

Exercise 43. Let

$$D := \left\{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant 2x - \sqrt{x^2 + y^2}\right\}.$$

i) Is D closed? open? bounded? compact? Justify carefully.
ii) Compute the area of D.

Exercise 44. Let

$$u(x, y) := x^5 - 10x^3y^2 + 5xy^4.$$

i) Determine all possible $v = v(x, y)$ in such a way that $f(x+iy) := u(x, y)+iv(x, y)$ be holomorphic on $\mathbb{C}$.
ii) For the f found at i), determine the analytical expression of $f(z)$ as function of $z \in \mathbb{C}$.

Exercise 45. What does it mean that a set $S \subset \mathbb{R}^d$ is open? Let $\vec{f} : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a continuous function on $\mathbb{R}^d$. Prove that the following property holds:

$$\vec{f}^{-1}(S) \text{ is open, } \forall S \subset \mathbb{R}^m \text{ open.}$$

(recall that $\vec{f}^{-1}(S) = \{\vec{x} \in \mathbb{R}^d \ : \ \vec{f}(\vec{x}) \in S\}$). Hint: suppose that for some S open, $\vec{f}^{-1}(S)$ is not open...

Solutions

Exercise 1. i) We have a separable vars eqn, $y' = a(t)f(y)$ where $f(y) = y^2 - 4$ and $a(t) = \frac{1}{t}$. Since $a \in \mathscr{C}$ and $f \in \mathscr{C}^1$. According to a general result, solutions of the differential equation are either constant or not, in this last case can be determined by separation of variables. Constant solutions are $y \equiv C$ iff $y' \equiv 0 = \frac{C^2-4}{t}$ iff $C^2 = 4$, iff $C = \pm 2$. Since the solution of CP is $y(1) = 0$, certainly y is not constant (otherwise $y \equiv \pm 2$). Thus, the solution of proposed CP can be determined by separation of vars:

$$y' = \frac{y^2-4}{t}, \iff \frac{y'}{y^2-4} = \frac{1}{t}, \iff \int \frac{y'}{y^2-4}\,dt = \int \frac{1}{t}\,dt + C = \log|t| + C.$$

Now,

$$\int \frac{y'}{y^2-4}\,dt \overset{u=y'(t)}{=} \int \frac{1}{u^2-4}\,du = \int \frac{1}{4}\left(\frac{1}{u-2} - \frac{1}{u+2}\right)\,du = \frac{1}{4}\log\left|\frac{u-2}{u+2}\right| = \frac{1}{4}\log\left|\frac{y(t)-2}{y(t)+2}\right|.$$

In this way, we have the implicit form for the solution

$$\frac{1}{4}\log\left|\frac{y(t)-2}{y(t)+2}\right| = \log|t| + C.$$

Imposing the initial/passage condition we have

$$\frac{1}{4}\log 1 = \log|1| + C, \iff C = 0.$$

Thus, for the solution of the CP we have

$$\frac{1}{4}\log\left|\frac{y(t)-2}{y(t)+2}\right| = \log|t|, \iff \left|\frac{y(t)-2}{y(t)+2}\right| = t^4, \iff \frac{y(t)-2}{y(t)+2} = \pm t^4.$$

Since $y(1) = 0$ we have $-1 = \pm 1^4 = \pm 1$, thus the appropriate sign is $-$, and

$$\frac{y(t)-2}{y(t)+2} = -t^4, \iff y(t)-2 = -t^4(y(t)+2), \iff y(t)(1+t^4) = 2(1-t^4), \iff y(t) = 2\frac{1-t^4}{1+t^4}.$$

ii) The formula found at i) for y is defined for every $t \in \mathbb{R}$. However, since the equation does not make any sense at $t = 0$, the solution must be defined either on $]-\infty, 0[$ or $]0, +\infty[$. Since y is defined at $t = 1$ we conclude that the domain of the solution is $]0, +\infty[$. About limits,

$$\lim_{t\to 0} y(t) = 2, \quad \lim_{t\to+\infty} y(t) = -2. \quad \square$$

Exercise 2. i) For instance $(0,0,z) \in D$ iff $z^2 = 1$, thus $(0,0,\pm 1) \in D$ and $D \neq \emptyset$. D is also the zero set of $g(x,y,z) := x^2 + y^2 + z^2 - xy - 1$. This is a submersion on D iff

$$\nabla g \neq \vec{0}, \text{ on } D.$$

We have

$$\nabla g = \vec{0}, \iff \begin{cases} 2x - y = 0, \\ 2y - x = 0, \\ 2z = 0, \end{cases} \iff (x,y,z) = (0,0,0) \notin D,$$

from which it follows that g is a submersion on D.

ii) Certainly, $D = \{g = 0\}$ is closed ($g \in \mathscr{C}$). Is it also bounded? We may see this by using spherical coordinates:

$$\begin{cases} x = \rho \cos\theta \sin\varphi, \\ y = \rho \sin\theta \sin\varphi, \\ z = \rho \cos\varphi. \end{cases} \quad \rho^2 = x^2 + y^2 + z^2 = \|(x, y, z)\|^2.$$

Then, if $(x, y, z) \in D$ we have

$$\rho^2 = 1 + \rho^2 \cos\theta \sin\theta (\sin\varphi)^2 = 1 + \frac{1}{2}\rho^2 \sin(2\theta)(\sin\varphi)^2 \leqslant 1 + \frac{\rho^2}{2},$$

from which

$$\frac{\rho^2}{2} \leqslant 1, \quad \Longleftrightarrow \quad \rho^2 = \|(x, y, z)\|^2 \leqslant 2.$$

Thus, D is bounded, hence compact.

iii) We have to minimize/maximize $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ or, which is equivalent (same min/max points), $f(x, y, z) = x^2 + y^2 + z^2$. According to i), we are in condition to apply Lagrange multipliers theorem. According to this result, at min/max points $(x, y, z) \in D$ we have

$$\nabla f = \lambda \nabla g, \quad \Longleftrightarrow \quad \text{rk} \begin{bmatrix} \nabla f(x, y, z) \\ \\ \nabla g(x, y, z) \end{bmatrix} = \text{rk} \begin{bmatrix} 2x & 2y & 2z \\ \\ 2x - y & 2y - x & 2z \end{bmatrix} < 2.$$

This happens iff all 2×2 subdeterminats equal 0:

$$\begin{cases} 2x(2y - x) - 2y(2x - y) = 0, \\ \\ 2x2z - 2z(2x - y) = 0, \\ \\ 2y2z - 2z(2y - x) = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y^2 - x^2 = 0, \\ yz = 0, \\ xz = 0. \end{cases}$$

The first leads to $y = \pm x$, the second $y = 0$ (then $x = 0$) or $z = 0$. That is we have points $(0, 0, z)$ and $(x, \pm x, 0)$. Now

- $(0, 0, z) \in D$ iff $z^2 = 1$, that is $(0, 0, \pm 1)$.
- $(x, \pm x, 0) \in D$ iff $2x^2 = 1 \pm x^2$. If $+$, $2x^2 = 1 + x^2$, we get $x = \pm 1$, that is points $(1, 1, 0)$ and $(-1, -1, 0)$. It $-$, $x^2 = \frac{1}{3}$, thus points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$.

Prom these we see that $(1, 1, 0)$ and $(-1, -1, 0)$ are points at max distance to $\vec{0}$ while $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$ are points of D at min distance to $\vec{0}$. □

Exercise 3. i) $D \cap \{x = 0\} = \{(0, y, z) \ : \ \sqrt{|y|} \leqslant z \leqslant 2 - y^2\}$. Thus, in the plane yz, $D \cap \{x = 0\}$ is the plane region between $z = \sqrt{|y|}$ and the parabola $z = 2 - y^2$ (see figure). Since $(x, y, z) \in D$ depends on (x, y) through $x^2 + y^2$, D is invariant by rotations around the $z-$axis.

ii) We have

$$\begin{aligned}\lambda_3(D) &= \int_D 1\,dxdydz = \int_{\sqrt[4]{x^2+y^2}\leqslant z\leqslant 2-(x^2+y^2)} 1\,dxdydz \overset{RF}{=} \int_{\sqrt[4]{x^2+y^2}\leqslant 2-(x^2+y^2)} \int_{\sqrt[4]{x^2+y^2}}^{2-(x^2+y^2)} 1\,dz\,dxdy \\ &= \int_{\sqrt[4]{x^2+y^2}\leqslant 2-(x^2+y^2)} \left(2-(x^2+y^2)-\sqrt[4]{x^2+y^2}\right)\,dxdy \\ &\overset{CV}{=} \int_{\sqrt{\rho}\leqslant 2-\rho^2,\ \theta\in[0,2\pi]} \left(\sqrt{\rho}-\left(2-\rho^2\right)\right)\rho\,d\rho d\theta.\end{aligned}$$

Now, $\sqrt{\rho}\leqslant 2-\rho^2$ might be hard to solve. However, here $\rho\geqslant 0$; $\sqrt{\rho}$ is increasing while $2-\rho^2$ decreases. Since at $\rho=1$ they are equal, we conclude that $\sqrt{\rho}\leqslant 2-\rho^2$ iff $0\leqslant\rho\leqslant 1$. We can continue previous chain by the RF:

$$\begin{aligned}&\overset{RF}{=} \int_0^1\int_0^{2\pi}\left(2\rho-\rho^3-\rho^{3/2}\right)\,d\theta\,d\rho = 2\pi\left(-\left[\rho^2\right]_{\rho=0}^{\rho=1}-\left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}-\left[\frac{\rho^{5/2}}{5/2}\right]_{\rho=0}^{\rho=1}\right) \\ &= 2\pi\left(1-\frac{1}{4}-\frac{2}{5}\right)=\frac{7\pi}{10}. \quad \square\end{aligned}$$

Exercise 4. i) $f=u+iv$ is $\mathbb{C}-$differentiable on $\mathbb{C}$ iff u,v are $\mathbb{R}-$differentiable on $\mathbb{R}^2$ and u,v fulfill the CR conditions. Clearly v is differentiable. Thus we have to look at $u=u(x,y)$ $\mathbb{R}-$differentiable such that

$$\begin{cases}\partial_x u &= \partial_y v = -e^{-y}(y\cos x + x\sin x) + e^{-y}\cos x, \\ \partial_y u &= -\partial_x v = -e^{-y}\left(-y\sin x+\sin x + x\cos x\right).\end{cases}$$

From the first equation,

$$u(x,y)=\int \partial_x u(x,y)\,dx + c(y) = -e^{-y}\left(y\sin x - x\cos x\right)+c(y).$$

We have

$$\partial_y u = e^{-y}\left(y\sin x - x\cos x\right) - e^{-y}\sin x + c'(y) = e^{-y}\left(y\sin x - x\cos x+\sin x\right)+c'(y)$$

thus $\partial_y u = -\partial_x v$ iff $c'(y)=0$, that is $c(y)$ is constant. We conclude that

$$u(x,y) = -e^{-y}\left(y\sin x - x\cos x\right)+c+e^{-y}\left(y\cos x + x\sin x\right).$$

ii) We have

$$\begin{aligned}f &= u+iv = -e^{-y}\left(y\sin x - x\cos x\right)+ie^{-y}\left(y\cos x + x\sin x\right) \\ &= e^{-y}\left(y\left(-\sin x + i\cos x\right)+x\left(\cos x + i\sin x\right)\right) \\ &= e^{-y}\left(iye^{ix}+xe^{ix}\right) \\ &= e^{ix-y}(iy+x) = e^{i(x+iy)}(x+iy) = e^{iz}z. \quad \square\end{aligned}$$

Exercise 5. Let $\vec{F} := f\nabla f = (f\partial_x f, f\partial_y f) =: (F_1, F_2)$. According to Green formula,

$$\oint_{\partial D} f\nabla f = \oint_{\partial D} \vec{F} = \int_D \left(\partial_y F_1 - \partial_x F_2\right)\, dxdy.$$

Now, since

$$\partial_y F_1 = \partial_y(f\partial_x f) = \partial_y f \partial_x f + f\partial_{yx} f, \quad \partial_x F_2 = \partial_x(f\partial_y f) = \partial_x f\partial_y f + f\partial_{xy} f$$

we easily deduce that $\partial_y F_1 - \partial_x F_2 \equiv 0$ being $f \in \mathscr{C}^2(\mathbb{R}^2)$. □

Exercise 6. i) We have a separable variables equation $y' = a(t)f(y)$ where $a(t) = \frac{1}{t}$ and $f(y) = e^y - 1$. $y \equiv C$ is a solution iff $0 = \frac{1}{t}(e^C - 1)$, iff $e^C = 1$ that is, $C = 0$. There is a unique constant solution, $y \equiv 0$.

ii) Since $y(1) = -1$, y is not constant. Furthermore, since $a \in \mathscr{C}$ and $f \in \mathscr{C}^1$, the solution can be found by separating vars:

$$y' = \frac{e^y - 1}{t}, \iff \frac{y'}{e^y - 1} = \frac{1}{t}, \iff \int \frac{y'(t)}{e^{y(t)} - 1}\, dt = \int \frac{1}{t}\, dt + c = \log|t| + c.$$

On the lhs

$$\int \frac{y'(t)}{e^{y(t)}-1}\, dt \overset{u=y(t)}{=} \int \frac{du}{e^u - 1} \overset{v=e^u,\ u=\log v,\ du=dv/v}{=} \int \frac{1}{v(v-1)}\, dv = \int -\frac{1}{v} + \frac{1}{v-1}\, dv$$

$$= \log|v-1| - \log|v| = \log\left|\frac{e^u - 1}{e^u}\right|$$

$$= \log\left|\frac{e^{y(t)} - 1}{e^{y(t)}}\right|.$$

Thus,

$$\log\left|\frac{e^{y(t)} - 1}{e^{y(t)}}\right| = \log\left|1 - \frac{1}{e^{y(t)}}\right| = \log|t| + c.$$

By imposing the initial condition, we find

$$c = \log(e-1),$$

and

$$\left|1 - \frac{1}{e^{y(t)}}\right| = (e-1)|t|, \iff 1 - \frac{1}{e^{y(t)}} = \pm(e-1)t.$$

A check with the initial condition shows that the sign is $-$, thus

$$1 - \frac{1}{e^{y(t)}} = -(e-1)t, \iff 1 + (e-1)t = \frac{1}{e^{y(t)}} = e^{-y(t)}, \iff y(t) = -\log\left(1 + (e-1)t\right).$$

iii) The domain of definition for the solution is

$$1 + (e-1)t > 0, \iff t > -\frac{1}{e-1}.$$

However, since at $t = 0$ the solution cannot be defined (because the equation does not make sense at $t = 0$), and the solution is defined on an interval, we conclude that the domain is $]0, +\infty[$. We have

$$\lim_{t\to 0+} y(t) = \log 1 = 0, \quad \lim_{t\to+\infty} y(t) = -\infty. \quad \square$$

Exercise 7. i) Point $(0, y, 0) \in D$ iff $y^2 = 1$ and $y^2 = 1$, that is $y = \pm 1$, so $(0, \pm 1, 0) \in D$. D is the zero set of $(g_1, g_2) = (x^2 + y^2 - z^2 - 1, y^2 + z - 1)$. According to the Definition,

$$(g_1, g_2) \text{ is a submersion on } D \iff \operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} = 2 \text{ on } D.$$

Since this is a 2×3 matrix, its rank is < 2 iff all 2×2 sub determinant equal 0, or

$$\begin{cases} 4xy = 0, \\ 2x = 0, \\ 2y(-1 + 2z) = 0, \end{cases} \iff \begin{cases} x = 0, \\ \\ y(1 + 2z) = 0. \end{cases} \iff \begin{array}{l} \begin{cases} x = 0, \\ y = 0, \end{cases} \iff (0, 0, z), \\ \\ \begin{cases} x = 0, \\ \\ z = -\frac{1}{2}, \end{cases} \iff (0, y, -\frac{1}{2}). \end{array}$$

Now,

- $(0, 0, z) \in D$ iff $-z^2 = 1$ and $z = 1$, impossible;
- $(0, y, -\frac{1}{2}) \in D$ iff $y^2 = \frac{5}{4}$ and $y^2 = \frac{3}{2}$, impossible.

Conclusion: at no point of D the rank of the matrix $\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix}$ is less than 2, thus (g_1, g_2) is a submersion on D.

ii) D is certainly closed being defined by equations involving continuous functions. Is it also bounded? From the second equation $y^2 = 1 - z$, thus $y = \pm\sqrt{1 - z}$ for $z \leqslant 1$. Plugging this into the first equation

$$x^2 = z^2 - (1 - z) + 1 = z^2 + z = z(z + 1), \implies x = \pm\sqrt{z^2 + z} \text{ for } z \leqslant 0 \ \vee \ z \geqslant 1.$$

In particular, for $z \leqslant 0$ points

$$(\pm\sqrt{z^2 + z}, \pm\sqrt{1 - z}, z) \in D, \ \forall z \leqslant 0.$$

These points are unbounded because

$$\|(\pm\sqrt{z^2 + z}, \pm\sqrt{1 - z}, z)\|^2 = z^2 + z + (1 - z) + z^2 = 2z^2 + 1 \longrightarrow +\infty, \ z \longrightarrow -\infty.$$

We conclude that D is unbounded.

iii) By ii) D is closed and unbounded. We have to min/max $\sqrt{x^2 + y^2 + z^2}$ or, equivalently, $f := x^2 + y^2 + z^2$, which is continuous on D and such that $\lim_{\infty_3} f = +\infty$. We conclude f has no max point on D while it has min points. By i) and according to the Lagrange multipliers theorem, at min point we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \operatorname{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 1 \end{bmatrix} < 3.$$

This happens iff the determinant of the previous jacobian matrix equals 0, that is

$$8xy(x + z) = 0, \iff x = 0, \ \vee \ y = 0, \ \vee \ z = -x.$$

This leads to points $(0, y, z)$, $(x, 0, z)$ and $(x, y, -x)$. Now,

- $(0, y, z) \in D$ iff $y^2 - z^2 = 1$ and $y^2 + z = 1$. From these, $z^2 + z = 0$ that is, $z = 0$ or $z = -1$, thus we have points $(0, \pm 1, 0)$ and $(0, \pm\sqrt{2}, -1)$;

- $(x, 0, z) \in D$ iff $x^2 - z^2 = 1$ and $z = 1$, that is $(\pm\sqrt{2}, 0, 1)$.
- $(x, y, -x) \in D$ iff $x^2 + y^2 - x^2 = 1$ and $y^2 - x = 1$, that is $y^2 = 1$ and $x = 0$, from which we have points $(0, \pm 1, 0)$.

Conclusion: min points are among $(0, \pm 1, 0)$, $(0, \pm\sqrt{2}, -1)$, $(\pm\sqrt{2}, 0, 1)$, and clearly thos at min distance to $\vec{0}$ are $(0, \pm 1, 0)$. □

Exercise 8. i) Figures are straightforward. D is not invariant by any rotation because one part of the inequality ($z \geqslant x^2 + y^2$) is invariant by rotations around $z-$axis while the second part ($z \leqslant 1 - y^2$) is not.

ii) We have

$$\begin{aligned}
\lambda_3(D) &= \int_D 1\, dxdydz \overset{RF}{=} \int_{x^2+y^2 \leqslant 1-y^2} \int_{x^2+y^2}^{1-y^2} 1\, dz\, dxdy = \int_{x^2+2y^2 \leqslant 1} \left(1 - y^2 - (x^2+y^2)\right)\, dxdy \\
&= \int_{x^2+2y^2 \leqslant 1} \left(1 - (x^2 + 2y^2)\right)\, dxdy \\
&\overset{CV\ x=\rho\cos\theta,\ \sqrt{2}y=\rho\sin\theta}{=} \int_{0\leqslant\rho\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi} \left(1-\rho^2\right) \frac{\rho}{\sqrt{2}}\, d\rho\, d\theta \\
&\overset{RF}{=} \frac{2\pi}{\sqrt{2}} \int_0^1 \rho - \rho^3\, d\rho = \sqrt{2}\pi \left(\left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=1} - \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=1} \right) = \frac{\sqrt{2}\pi}{4}. \quad \square
\end{aligned}$$

Exercise 9. i) $\vec{F}$ is irrotational on D iff

$$\partial_y \frac{ax^2 + by^2}{(x^2+y^2)^2} \equiv \partial_x \frac{xy}{(x^2+y^2)^2} \text{ on } D.$$

By computing derivatives, the previous is equivalent to

$$\frac{2by(x^2+y^2) - (ax^2+by^2)4y}{(x^2+y^2)^3} = \frac{y(x^2+y^2) - 4x^2y}{(x^2+y^2)^3}$$

that is, iff

$$(2b - 4a)yx^2 - 2by^3 = -3x^2y + y^3, \quad \Longleftrightarrow \quad 2b = -1,\ -1 - 4a = -3, \quad \Longleftrightarrow \quad b = -\frac{1}{2},\ a = \frac{1}{2}.$$

ii) To be conservative, $\vec{F}$ must be irrotational, hence, necessarily, $a = \frac{1}{2} = -b$. Thus,

$$\vec{F} = \left(\frac{1}{2}\frac{x^2-y^2}{(x^2+y^2)^2}, \frac{xy}{(x^2+y^2)^2}\right) = \nabla f, \quad \Longleftrightarrow \quad \begin{cases} \partial_x f = \frac{1}{2}\frac{x^2-y^2}{(x^2+y^2)^2}, \\ \\ \partial_y f = \frac{xy}{(x^2+y^2)^2}. \end{cases}$$

Looking at the second equation,

$$f(x,y) = \int \frac{xy}{(x^2+y^2)^2}\, dy + c(x) = \frac{x}{2}\int 2y(x^2+y^2)^{-2}\, dy + c(x) = \frac{x}{2}\frac{(x^2+y^2)^{-1}}{-1} + c(x) = -\frac{1}{2(x^2+y^2)} + c(x).$$

Now, by imposing also the first equation we get

$$c'(x) = 0, \quad \Longleftrightarrow \quad c(x) \equiv \text{constant}.$$

Thus, all the potentials of $\vec{F}$ are

$$f(x,y) = -\frac{1}{2(x^2+y^2)} + c. \qquad \square$$

Exercise 10. About the CR equations see the course notes. Assume that $f = u + iv$ is $\mathbb{C}$ differentiable on $\mathbb{C}$. Then, u, v are $\mathbb{R}$ differentiable and the CR eqns hold,

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases}$$

If also $\overline{f} = u - iv = u + i(-v)$ is $\mathbb{C}$ differentiable, $u, -v$ fulfill the CR eqns,

$$\begin{cases} \partial_x u = \partial_y(-v) = -\partial_y v, \\ \\ \partial_y u = -\partial_x(-v) = +\partial_x v. \end{cases}$$

But then, combining the two CR eqns, we get

$$\partial_x u = -\partial_y v = -\partial_x u, \implies 2\partial_x u \equiv 0,$$

and, similarly, $\partial_y u \equiv 0$. From this $\nabla u \equiv 0$ hence u is constant. Similar conclusion holds for v. We conclude that both u and v must be constant, hence also f must be constant.

Alternative solution: you may remind that we have seen that if a $\mathbb{C}$ differentiable function is real (or imaginary) valued, then, necessarily, the function must be constant (this is again a consequence of the CR eqns). Now, if both f and $\overline{f}$ are $\mathbb{C}$ differentiable, also $f + \overline{f} = 2u$ is $\mathbb{C}$ differentiable. But since $2u$ is real valued, $f + \overline{f}$ (hence u) must be constant. Same conclusion for $f - \overline{f} = i2v$, hence v is constant. $\square$

Exercise 11. i) The general integral is

$$y(t) = c_1 w_1(t) + c_2 w_2(t) + u(t),$$

where (w_1, w_2) is a fundamental system of solutions for the homogeneous equation $y'' - 2y' + y = 0$ and u is a particular solution of the equation. The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0, \quad \Longleftrightarrow \quad (\lambda - 1)^2 = 0, \quad \Longleftrightarrow \quad \lambda_{1,2} = 1.$$

Therefore, the fundamental system of solutions is $w_1 = e^t$, $w_2 = te^t$. To compute the particular solution u we apply the Lagrange formula

$$u(t) = \left(-\int \frac{w_2}{W} f\, dt\right) w_1 + \left(\int \frac{w_1}{W} f\, dt\right) w_2,$$

where W is the wronskian

$$W = \det \begin{bmatrix} w_1 & w_2 \\ \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} e^t & te^t \\ \\ e^t & (t+1)e^t \end{bmatrix} = (t+1)e^{2t} - te^{2t} = e^{2t},$$

and $f = f(t) = e^{2t}$. Thus

$$u(t) = \left(-\int \frac{te^t}{e^{2t}} e^{2t}\, dt\right) e^t + \left(\int \frac{e^t}{e^{2t}} e^{2t}\, dt\right)(te^t) = -\left(te^t - \int e^t\, dt\right) e^t + e^t te^t = e^{2t}.$$

Conclusion: the general integral is

$$y(t) = c_1 e^t + c_2 te^t + e^{2t},\ c_1, c_2 \in \mathbb{R}.$$

ii) To solve the Cauchy problem we impose the initial conditions $y(0) = 1$ and $y'(0) = 0$ to the general integral. First notice that

$$y' = c_1 e^t + c_2(t+1)e^t + 2e^{2t},$$

thus

$$\begin{cases} y(0) = 1, \\ y'(0) = 0, \end{cases} \iff \begin{cases} c_1 + 1 = 1, \\ c_1 + c_2 + 2 = 0, \end{cases} \iff \begin{cases} c_1 = 0, \\ c_2 = -2, \end{cases}$$

and the solution is $y(t) = -2te^t + e^{2t}$.

iii) Again, we impose the passage conditions

$$\begin{cases} c_1 + 1 = 0, \\ c_1 e + c_2 e + e^2 = a, \end{cases} \iff \begin{cases} c_1 = -1, \\ c_2 = \frac{a-e^2+e}{e}. \end{cases}$$

We conclude that: for every $a \in \mathbb{R}$ there exists a unique solution to the proposed problem. □

Exercise 12. i) Clearly $f(x, 0) = x^6 - x^4 \longrightarrow +\infty$ for $|x| \longrightarrow +\infty$. So, if a limit exists it must be $= +\infty$. We check this changing coordinates and using polar coords:

$$f(x, y) = \rho^6 - (\rho\cos\theta)^4 + (\rho\sin\theta)^4 \geqslant \rho^6 - 2\rho^4 \longrightarrow +\infty,\ \text{if } \rho = \|(x, y)\| \longrightarrow +\infty.$$

ii) By i) and a consequence of Weierstrass theorem, f has global minimum on $\mathbb{R}^2$ but not any global maximum. Since every point of $\mathbb{R}^2$ lies in its interior, according to Fermat theorem (clearly $\partial_x f = 6x(x^2 + y^2)^2 - 4x^3$ and $\partial_y f = 6y(x^2 + y^2)2 + 4y^3$ are both continuous on $\mathbb{R}^2$, hence f is differentiable on $\mathbb{R}^2$ according to the differentiability test), at min we have $\nabla f = \vec{0}$. Now,

$$\nabla f = \vec{0}, \iff \begin{cases} 6x(x^2 + y^2)^2 - 4x^3 = 0, \\ 6y(x^2 + y^2)^2 + 4y^3 = 0 \end{cases} \iff \begin{cases} x\left(6(x^2 + y^2)^2 - 4x^2\right) = 0, \\ y\left(6(x^2 + y^2)^2 + 4y^2\right) = 0, \end{cases}$$

Now, looking at second equation, we see that either $y = 0$ or $6(x^2 + y^2)^2 + 4y^2 = 0$. In the second case we obtain trivially $x = 0$ and $y = 0$, thus the point $(0, 0)$. Plugging $y = 0$ into the first equation we get

$$x\left(6x^4 - 4x^2\right) = 0, \iff x^3(3x^2 - 2) = 0, \iff x = 0,\ \vee\ x = \pm\sqrt{\frac{2}{3}}.$$

Thus we have again $(0, 0)$ and two more points $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$. Since $f(0, 0) = 0$ while

$$f\left(\pm\sqrt{\frac{2}{3}}, 0\right) = \frac{8}{27} - \frac{4}{9} = -\frac{28}{27} < f(0, 0) = 0,$$

we conclude that $\left(\pm\sqrt{\frac{2}{3}}, 0\right)$ are global minimums. Finally, since $\mathbb{R}^2$ is connected,

$$f(\mathbb{R}^2) = \left[-\frac{28}{27}, +\infty\right[. \quad \square$$

Exercise 13. ii)

$$\begin{aligned}
\lambda_3(D) &= \int_{x^2+2y^2 \leqslant z \leqslant 4-3(x^2+2y^2)} 1 \, dxdydz \\
&\overset{RF}{=} \int_{x^2+2y^2 \leqslant 4-3(x^2+2y^2)} \int_{x^2+2y^2}^{4-3(x^2+2y^2)} 1 \, dz \, dxdy \\
&= \int_{x^2+2y^2 \leqslant 4-3(x^2+2y^2)} 4\left(1-(x^2+2y^2)\right) \, dxdy.
\end{aligned}$$

Noticed that $x^2+2y^2 \leqslant 4-3(x^2+2y^2)$ iff $x^2+2y^2 \leqslant 1$, we have

$$\lambda_3(D) = \int_{x^2+2y^2 \leqslant 1} 4\left(1-(x^2+2y^2)\right) \, dxdy.$$

Changing variables to adapted polar coordinates

$$x = \rho\cos\theta, \quad \sqrt{2}y = \rho\sin\theta,$$

we have

$$\lambda_3(D) = \int_{0 \leqslant \rho \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi} 4\left(1-\rho^2\right)\frac{\rho}{\sqrt{2}} \, d\rho d\theta \overset{RF}{=} \frac{8\pi}{\sqrt{2}} \int_0^1 (\rho - \rho^3) \, d\rho = \frac{8\pi}{\sqrt{2}}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{4\pi}{\sqrt{2}}. \quad \square$$

Exercise 14. i) Let $u = x^2 + y^2$. From CR equations, $v = v(x, y)$ is such that $f = u + iv$ is $\mathbb{C}-$differentiable iff u, v are $\mathbb{R}-$differentiable and CR equations hold,

$$\begin{cases} \partial_x u = \partial_y v, \\ \partial_y u = -\partial_x v. \end{cases}$$

Clearly u is $\mathbb{R}-$differentiable. Thus we seek for v $\mathbb{R}-$differentiable such that

$$\begin{cases} \partial_x v = -\partial_y u = -2y, \\ \partial_y v = \partial_x u = 2x. \end{cases}$$

From the first equation $v(x, y) = -\int 2y \, dx + c(y) = -2xy + c(y)$. Plugging this into the second equation we have $\partial_y v = -2x + c'(y) = 2x$, that is $c'(y) = 4x$, which is impossible since c does not depend on y. We conclude that such v does not exist.

ii) Since there is no v such that $f = u + iv$ is $\mathbb{C}-$differentiable, there is no f to be found. $\square$

Exercise 15. See notes for the statement. We may formally set the optimization problem in the following way. The set $y = f(x)$ is also $f(x) - y = 0$. Setting $g(x, y) := f(x) - y$ we see that g is a submersion on $\{g = 0\}$. Indeed $\nabla g = (\partial_x g, \partial_y g) = (f'(x), -1) \neq 0$, whatever is x. Let now

$$d(x, y) := (x-a)^2 + (y-b)^2,$$

the square of distance from (a, b) to (x, y). At minimum (x, y) on the curve, that is $y = f(x)$, according to Lagrange theorem we have

$$\nabla d = \lambda \nabla g = \lambda(f'(x), -1).$$

Since

$$\nabla d = (2(x-a), 2(y-b)) = 2(x-a, y-b) = 2Q - P,$$

we have

$$Q - P = \frac{\lambda}{2}(f'(x), -1).$$

Now, since the tangent direction to $y = f(x)$ at point $(x, f(x))$ is $(1, f'(x))$, and clearly $(f'(x), -1) \perp (1, f'(x))$, we have that

$$Q - P \parallel (f'(x), -1) \perp (1, f'(x)) \parallel \text{ tangent to } f,$$

we obtain the conclusion. □

Exercise 16. i) The equation can be written as

$$y' = \frac{t}{1+t^2}\frac{1-y^2}{y} =: a(t)f(y),$$

with obvious definition of a and f. $y \equiv C$ is a solution iff

$$0 = y' = \frac{t}{1+t^2}\frac{1-C^2}{C}, \quad \Longleftrightarrow \quad 1 - C^2 = 0, \quad \Longleftrightarrow \quad C = \pm 1.$$

ii) Since $y(0) = 2$, y cannot be constant (otherwise: $y \equiv \pm 1$ thus, in particular, $y(0) = \pm 1$ but $y(0) = 2$). Therefore, y can be determined by separation of variables:

$$\frac{y}{1-y^2}y' = \frac{t}{1+t^2}, \quad \Longleftrightarrow \quad \int \frac{y}{1-y^2}y'\,dt = \int \frac{t}{1+t^2}\,dt + c = \frac{1}{2}\log(1+t^2) + c.$$

Now,

$$\int \frac{y}{1-y^2}y'\,dt \overset{u=y(t),\ du=y'(t)dt}{=} \int \frac{u}{1-u^2}\,du = -\frac{1}{2}\log|1-u^2| = -\frac{1}{2}\log|1-y(t)^2|,$$

hence

$$-\frac{1}{2}\log|1-y(t)^2| = \frac{1}{2}\log(1+t^2) + c, \quad \Longleftrightarrow \quad \log|1-y(t)^2| = -\log(1+t^2) + c.$$

(we relabeled $2c$ by c). Imposing $y(0) = 2$,

$$\log 3 = -\log 1 + c, \quad \Longleftrightarrow \quad c = \log 3.$$

Therefore

$$|1 - y(t)^2| = \frac{3}{1+t^2},$$

that is

$$1 - y(t)^2 = \pm\frac{3}{1+t^2}.$$

When $t = 0$ lhs is -3, thus sign is $-$ and

$$y(t)^2 = 1 + \frac{3}{1+t^2}, \quad \Longleftrightarrow \quad y(t) = \pm\sqrt{1 + \frac{3}{1+t^2}},$$

and, again by imposing $y(0) = 2$, we see that sign is $+$. □

Exercise 17. i) We have $(x, y, 0) \in \Gamma$ iff $x^2 + y^2 = 1$ and $x^2 = 1$, thus $x = \pm 1$ and $y^2 = 0$, hence $(\pm 1, 0, 0) \in \Gamma$. Now, $\Gamma = \{g_1 = 0,\ g_2 = 0\}$, where $g_1 = x^2 + y^2 - 1$, and $g_2 = x^2 + z^2 - xz - 1$. Clearly $g_1, g_2 \in \mathscr{C}^1$ and (g_1, g_2) is a submersion on Γ iff

$$\operatorname{rank} \begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 2x & 2y & 0 \\ \\ 2x - z & 0 & 2z - x \end{bmatrix} = 2,\ \forall (x, y, z) \in \Gamma.$$

This is false iff all 2×2 submatrices have determinant =0, that is

$$\begin{cases} 2y(2x - z) = 0, \\ 2x(2z - x) = 0, \\ 2y(2z - x) = 0. \end{cases}$$

Working on the first equation, we have the alternatives $y = 0$ or $2x - z = 0$. In the first case, the system reduces to $x(2z - x) = 0$ that is $x = 0$ (points $(0, 0, z)$) or $x = 2z$ (points $(2z, 0, z)$). In the second case, the system reduces to

$$\begin{cases} z = 2x, \\ 3x^2 = 0, \\ 3yx = 0, \end{cases} \iff (0, y, 0).$$

Thus, rank is less than 2 at points $(0, 0, z)$, $(2z, 0, z)$ and $(0, y, 0)$. Now:

- $(0, 0, z) \in \Gamma$ iff $0 = 1$ (first condition), impossible;
- $(2z, 0, z) \in \Gamma$ iff $4z^2 = 1$ and $5z^2 = 2z^2 + 1$, that is $z^2 = \frac{1}{4}$ and $z^2 = \frac{1}{3}$ which are impossible together.
- $(0, y, 0) \in \Gamma$ iff $y^2 = 1$ and $0 = 1$, which is, again, impossible.

Conclusion: none of points where rank is ¡2 belong to Γ, this meaning that rank =2 on Γ, hence (g_1, g_2) is a submersion on Γ.

ii) Clearly Γ is closed because defined by equations involving continuous functions. Boundedness: from first equation we deduce $x^2, y^2 \leqslant 1$. From second equation, recalling that $ab \leqslant \frac{a^2+b^2}{2}$ we have

$$x^2 + z^2 = xz + 1 \leqslant \frac{x^2 + z^2}{2} + 1, \implies \frac{x^2 + z^2}{2} \leqslant 1,$$

from which, in particular, $z^2 \leqslant 2$. Therefore $\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2} \leqslant \sqrt{1 + 1 + 2} = \sqrt{4} = 2$, for every $(x, y, z) \in \Gamma$. Conclusion: Γ is bounded, hence compact.

iii) We have to minimize/maximize $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ or, equivalently, $f(x, y, z) = x^2 + y^2 + z^2$. By ii), Γ is compact and obviously $f \in \mathscr{C}$, thus existence of min and max for f is ensured by Weierstrass' theorem. To determine min/max points we apply Lagrange's thm. According to i), this thm can be applied on Γ. We deduce that, at min/max points $(x, y, z) \in \Gamma$,

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \operatorname{rank} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \\ 2x - z & 0 & 2z - x \end{bmatrix} = 2,$$

or, equivalently, the determinant of this last matrix equals 0. We obtain

$$2z \cdot (-2y(2x - z)) = 0, \quad \Longleftrightarrow \quad yz(2x - z) = 0, \quad \Longleftrightarrow \quad y = 0, \ \vee\ z = 0, \ \vee\ z = 2x.$$

Thus possible min/max points are among points $(x, 0, z)$, $(x, y, 0)$ and $(x, y, 2x)$. Now,

- $(x, 0, z) \in \Gamma$ iff $x^2 = 1$ and $x^2 + z^2 = xz + 1$, or, equivalently, $x^2 = 1$ and $z^2 = xz + 1$. For $x = 1$ we get $z^2 = z + 1$, that is $z = \frac{1\pm\sqrt{5}}{2}$, namely points $(1, 0, \frac{1\pm\sqrt{5}}{2})$. For $x = -1$ we get $z^2 = -z + 1$, that is $z = \frac{-1\pm\sqrt{5}}{2}$, namely points $(-1, 0, \frac{-1\pm\sqrt{5}}{2})$.
- $(x, y, 0) \in \Gamma$ iff $x^2 + y^2 = 1$ and $x^2 = 1$, that is $x = \pm 1$ and $y^2 = 0$, namely points $(\pm 1, 0, 0)$.
- $(x, y, 2x) \in \Gamma$ iff $x^2 + y^2 = 1$ and $x^2 + 4x^2 = 2x^2 + 1$, from which $x^2 = \frac{1}{3}$, $x = \pm\frac{1}{\sqrt{3}}$ and $y^2 = \frac{2}{3}$, $y = \pm\sqrt{\frac{2}{3}}$, thus we get points $\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right)$ (4 points).

We have

- $f(1, 0, \frac{1\pm\sqrt{5}}{2}) = 1 + \left(\frac{1\pm\sqrt{5}}{2}\right)^2 = \frac{10\pm 2\sqrt{5}}{4}$, $f(-1, 0, \frac{-1\pm\sqrt{5}}{2}) = 1 + \left(\frac{-1\pm\sqrt{5}}{2}\right)^2 = \frac{10\pm 2\sqrt{5}}{4}$≈;f(±1, 0, 0) = 1;
- $f\left(\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}\right) = \frac{1}{3} + \frac{2}{3} + \frac{4}{3} = \frac{7}{3}$ and $f\left(-\frac{1}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{1}{3} + \frac{2}{3} + \frac{4}{3} = \frac{7}{3}$.

From this we see that $(1, 0, \frac{1+\sqrt{5}}{2})$ and $(-1, 0, \frac{-1-\sqrt{5}}{2})$ are maximum points while $(\pm 1, 0, 0)$ are min points. □

Exercise 18. ii) D is closed (because defined by large inequalities involving continuous functions) and bounded (the root imposes $x^2 + y^2 \leqslant 1$ and, consequently, $0 \leqslant 1 - (x^2 + y^2) \leqslant z \leqslant \sqrt{1 - (x^2 + y^2)} \leqslant \sqrt{1}$, that is $0 \leqslant z \leqslant 1$). Thus D is compact, hence 1_D is integrable on D. Furthermore, noticed that, calling $\rho^2 = x^2 + y^2$,

$$1 - \rho^2 \leqslant \sqrt{1 - \rho^2}, \quad \Longleftrightarrow \quad \sqrt{1 - \rho^2} \leqslant 1,$$

which is always true, thus $1 - (x^2 + y^2) \leqslant \sqrt{1 - (x^2 + y^2)}$ always when defined. Then

$$\begin{aligned}
\text{Vol } D \quad &= \int_D 1\, dxdydz \overset{RF}{=} \int_{x^2+y^2\leqslant 1} \int_{1-(x^2+y^2)}^{\sqrt{1-(x^2+y^2)}} 1\, dz\, dxdy \\
&= \int_{x^2+y^2\leqslant 1} \left(\sqrt{1 - (x^2 + y^2)} - (1 - (x^2 + y^2))\right)\, dxdy \\
&\overset{pol.\ coords}{=} \int_{0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\rho\leqslant 1} \left(\sqrt{1 - \rho^2} - 1 + \rho^2\right) \rho\, d\rho d\theta \\
&\overset{RF}{=} 2\pi \int_0^1 \rho(1 - \rho^2)^{1/2} - \rho + \rho^3\, d\rho = 2\pi \left[\left[-\tfrac{1}{3}(1 - \rho^2)^{3/2}\right]_{\rho=0}^{\rho=1} - \left[\tfrac{\rho^2}{2}\right]_{\rho=0}^{\rho=1} + \left[\tfrac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}\right] \\
&= 2\pi \left[+\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4}\right] = \tfrac{\pi}{6}. \quad \square
\end{aligned}$$

Exercise 19. i) In order $f = u + iv$ is holomorphic on $\mathbb{C}$ we need that $u, v \in \mathscr{C}^1$ (true, u and v are polynomials) and they fulfill the CR equations:

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v, \end{cases} \iff \begin{cases} 2ax + by = x, \\ \\ bx + 2cy = -y, \end{cases} \forall (x, y) \in \mathbb{R}^2., \iff \begin{cases} 2a = 1,\ b = 0, \\ \\ b = 0,\ 2c = -1. \end{cases}$$

Thus,

$$u = \frac{1}{2}x^2 - \frac{1}{2}y^2,\ \ v = xy,$$

and $f = u + iv$ is holomorphic on $\mathbb{C}$.

ii) Notice that

$$f = u + iv = \frac{1}{2}x^2 - \frac{1}{2}y^2 + ixy = \frac{1}{2}\left(x^2 - y^2 + i2xy\right) = \frac{1}{2}(x + iy)^2 \equiv \frac{z^2}{2},\ z \in \mathbb{C}. \quad \square$$

Exercise 20. Clearly $f \in \mathscr{C}(\mathbb{R}^d)$ and moreover $f \geqslant 0$ (trivial) and

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = +\infty.$$

Just notice that $f(\vec{x}) \geqslant \|\vec{x} - \vec{a}_1\|^2 \longrightarrow +\infty$ when $\vec{x} \longrightarrow \infty_d$. Thus f cannot have a maximum but it has a minimum according to Weierstrass' thm. Now, f is differentiable on $\mathbb{R}^d$,

$$\nabla f = \sum_{j=1}^{N} \nabla \|\vec{x} - \vec{a}_j\|^2$$

and

$$\nabla \|\vec{x} - \vec{a}_j\|^2 = \left(\partial_1 \|\vec{x} - \vec{a}_j\|^2, \ldots, \partial_d \|\vec{x} - \vec{a}_j\|^2\right),$$

so, writing

$$\|\vec{x} - \vec{a}_j\|^2 = \sum_{k=1}^{d} (x_k - a_{j,k})^2,\ \implies\ \partial_i \|\vec{x} - \vec{a}_j\|^2 = \partial_i \sum_{k=1}^{d} (x_k - a_{j,k})^2 = 2(x_i - a_{j,i}),$$

we deduce

$$\nabla \|\vec{x} - \vec{a}_j\|^2 = \big(2(x_1 - a_{j,1}), 2(x_2 - a_{j,2}, \ldots, 2(x_d - a_{j,d})\big) = 2(\vec{x} - \vec{a}_j).$$

Therefore, $\nabla f \in \mathscr{C}$ and f is differentiable. According to Fermat thm, at min point we must have

$$\nabla f = \vec{0},\ \iff\ \sum_{j=1}^{N} 2(\vec{x} - \vec{a}_j) = 0,\ \iff\ N\vec{x} - \sum_{j=1}^{N} \vec{a}_j = \vec{0},\ \iff\ \vec{x} = \frac{1}{N}\sum_{j=1}^{N} \vec{a}_j. \quad \square$$

Exercise 21. i) $y \equiv C$ is a solution iff $0 = C \log C$, from which $C > 0$ (to be $\log C$ well defined), thus $\log C = 0$, that is $C = 1$.

ii) If $y(0) = 1$, then $y(t) \equiv 1$ (constant solution. For $a \neq 1$ (but $a > 0$ because of the equation), solution is non constant and it can be determined by separation of variables:

$$y = y \log y,\ \iff\ \frac{y'}{y \log y} = 1,\ \iff\ \int \frac{y'}{y \log y}\, dt = t + c.$$

Since

$$\int \frac{y'}{y\log y}\,dt \overset{u=y(t),\ du=y'(t)dt}{=} \int \frac{1}{u\log u}\,du = \int \frac{(\log u)'}{\log u}\,du = \log|\log u| = \log|\log y(t)|.$$

Therefore,

$$\log|\log y(t)| = t + c.$$

By imposing $y(0) = a$ we have $c = \log|\log a|$, hence

$$|\log y(t)| = |\log a|e^t, \quad \Longleftrightarrow \quad \log y(t) = \pm(\log a)e^t.$$

Because of the initial condition we have $\log y(t) = (\log a)e^t$, hence

$$y(t) = e^{(\log a)e^t}.$$

iii) We have $\lim_{t\to+\infty} y(t) = 0$ iff $\log a < 0$, that is $a < 1$. □

Exercise 22. i) Let $g_1 := x^2 - y^2 - z^2$ and $g_2 := x^2 + y^2 - xy - 1$. Then, $\vec{g} = (g_1, g_2)$ is a submersion on D iff $\mathrm{rk}\vec{g}'(x, y, z) = 2$ for all $(x, y, z) \in D$. Now,

$$\mathrm{rk}\,\vec{g}'(x,y,z) = \mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & -2y & -2z \\ \\ 2x-y & 2y-x & 0 \end{bmatrix} < 2, \iff \begin{cases} 2x(2y-x) + 2y(2x-y) = 0, \\ 2z(2x-y) = 0, \\ 2z(2y-x) = 0. \end{cases}$$

Simplifying, we get the system

$$\begin{cases} x^2 + y^2 - 4xy = 0, \\ z(2x-y) = 0, \\ z(2y-x) = 0. \end{cases}$$

Choosing the second equation, we have the alternative $z = 0$ or $2x - y = 0$. In the first case the system reduces to

$$\begin{cases} z = 0, \\ \\ x^2 + y^2 - 4xy = 0. \end{cases}$$

These points belong to D iff

$$\begin{cases} x^2 = y^2, \\ \\ 4xy = xy + 1, \end{cases} \iff \begin{cases} y = \pm x, \\ \\ 3xy = 1. \end{cases}$$

However, since $x^2 + y^2 = 4xy$ implies that, for $y = \pm x$, that $x = 0 = y$, it is impossible that $3xy = 1$, thus no solutions are in D.

In the second case, namely, $z \neq 0$ and $2x - y = 0$ or $y = 2x$, condition $\mathrm{rk}\vec{g}'(x, y, z) < 2$ reduces to

$$\begin{cases} y - 2x, \\ x(2y-x) = 0, \\ 2y - x = 0, \end{cases}$$

we easily get $x = y = 0$, that is a point of type $(0, 0, z)$. Now,

$$(0,0,z) \in D, \quad \Longleftrightarrow \quad \begin{cases} z = 0, \\ \\ 0 = 1, \end{cases}$$

clearly impossible. Conclusion: rank of $\vec{g}'(x, y, z)$ is never less than 2 on D, that is $\vec{g}$ is a submersion on D.

ii) D is clearly closed being defined by equalities involving continuous functions. To determine whether D is bounded or less, we look first at constraint $x^2 + y^2 = xy + 1$. Writing $x = \rho \cos\theta$ and $y = \rho \sin\theta$, this reads as

$$\rho^2 = \rho^2 \cos\theta \sin\theta + 1 = \frac{\rho^2}{2}\sin(2\theta) + 1, \leqslant \frac{\rho^2}{2} + 1, \implies \frac{\rho^2}{2} \leqslant 1, \implies x^2 + y^2 \leqslant 2, \ \forall (x, y, z) \in D.$$

But then, by the first equation,

$$z^2 = x^2 - y^2 \leqslant x^2 \leqslant x^2 + y^2 \leqslant 2, \implies x^2 + y^2 + z^2 \leqslant 4, \implies \|(x, y, z)\| \leqslant 2, \ \forall (x, y, z) \in D.$$

This means that D is bounded, hence compact.

iii) We have to miunimize/maximize $f(x, y, z) = \|(x, y, z)\|$ or, which is the same, $f(x, y, z) = \|(x, y, z)\|^2 = x^2 + y^2 + z^2$. The existence of min and max is ensured by the Weierstrass theorem being D compact by ii).

To determine min/max points, we apply Lagrange multipliers theorem. By i), assumptions of this theorem are verified. Thus, at min/max point $(x, y, z) \in D$ we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \quad \Longleftrightarrow \quad \mathrm{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \quad \Longleftrightarrow \quad \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 0.$$

Now,

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & -2y & -2z \\ 2x - y & 2y - x & 0 \end{bmatrix} = -(2y - z)(-8xz) = 8xz(2y - z),$$

iff $x = 0$, or $z = 0$ or $2y - z = 0$. Thus, we have points $(0, y, z)$, $(x, y, 0)$ and $(x, y, 2y)$. Now:

- $(0, y, z) \in D$ iff $0 = y^2 + z^2$ and $y^2 = 1$, and of course this is impossible.
- $(x, y, 0) \in D$ iff $x^2 = y^2$ and $x^2 + y^2 = xy + 1$. From the first we have $y = \pm x$. For $y = x$, second condition becomes $2x^2 = x^2 = 1$, thus $x^2 = 1$, so $x = \pm 1$ and we have points $(\pm 1, \pm 1, 0)$ (same sign). For $y = -x$, second condition becomes $2x^2 = -x^2 + 1$, that is $x^2 = \frac{1}{3}$, that is $x = \pm \frac{1}{\sqrt{3}}$, from which we have points $\left(\pm \frac{1}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}, 0\right)$ (opposite sign).
- $(x, y, 2y) \in D$ iff $x^2 = y^2 + 4y^2 = 5y^2$ and $x^2 + y^2 = xy + 1$. From first equation we get $x = \pm\sqrt{5}y$. In the case $x = \sqrt{5}y$, from second eqn we have $5y^2 + y^2 = \sqrt{5}y^2 + 1$, that is $(6 - \sqrt{5})y^2 = 1$, that is $y = \pm \frac{1}{\sqrt{6-\sqrt{5}}}$, this yielding to points $\left(\pm \frac{\sqrt{5}}{\sqrt{6-\sqrt{5}}}, \pm \frac{1}{\sqrt{6-\sqrt{5}}}, 0\right)$ (same sign). In the case $x = -\sqrt{5}y$,

second condition yields to $5y^2 + y^2 = -\sqrt{5}y^1$, that is $y^2 = \frac{1}{5+\sqrt{5}}$, or $y = \pm\frac{1}{\sqrt{5+\sqrt{5}}}$, from which we get points $\left(\mp\frac{\sqrt{5}}{\sqrt{5+\sqrt{5}}}, \pm\frac{1}{\sqrt{5+\sqrt{5}}}, 0\right)$ (opposite sign).

Previous analysis figured out possible min/max points. To decide which are min and which max it suffices to compute f at these points. We have:

- $f(\pm 1, \pm 1, 0) = 2$;
- $f\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, 0\right) = \frac{2}{3} = 0,\bar{6}$;
- $f\left(\pm\frac{\sqrt{5}}{\sqrt{6-\sqrt{5}}}, \pm\frac{1}{\sqrt{6-\sqrt{5}}}, 0\right) = \frac{6}{6-\sqrt{5}} \approx 1,59\dots$
- $f\left(\mp\frac{\sqrt{5}}{\sqrt{5+\sqrt{5}}}, \pm\frac{1}{\sqrt{5+\sqrt{5}}}, 0\right) = \frac{6}{5+\sqrt{5}} \approx 0,83\dots$

From this it is clear that $(\pm 1, \pm 1, 0)$ are points of D at max distance to $\vec{0}$, while $\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}, 0\right)$ are poitns of D at min distance to $\vec{0}$. □

Exercise 23. i) To be irrotational, the field must verify

$$\partial_y \frac{ax+by}{\sqrt{x^2+y^2}} \equiv \partial_x \frac{cx+dy}{\sqrt{x^2+y^2}}, \quad \forall (x,y) \in D = \mathbb{R}^2 \backslash \{\vec{0}\}.$$

We have

$$\partial_y \frac{ax+by}{\sqrt{x^2+y^2}} = \frac{b\sqrt{x^2+y^2} - (ax+by)\frac{2y}{2\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{b(x^2+y^2) - y(ax+by)}{(x^2+y^2)^{3/2}} = \frac{bx^2 - axy}{(x^2+y^2)^{3/2}},$$

and, similarly

$$\partial_x \frac{cx+dy}{\sqrt{x^2+y^2}} = \frac{cy^2 - dxy}{(x^2+y^2)^{3/2}}.$$

Thus, the field is irrotational iff

$$\frac{bx^2 - axy}{(x^2+y^2)^{3/2}} \equiv \frac{cy^2 - dxy}{(x^2+y^2)^{3/2}}, \quad \Longleftrightarrow \quad bx^2 - axy = cy^2 - dxy, \;\; \forall (x,y) \in \mathbb{R}^2 \backslash \{\vec{0}\}.$$

Since the identity is trivally verified at $(x,y) = \vec{0}$, we may say that the field is irrotational iff

$$bx^2 - axy \equiv cy^2 - dxy, \quad \Longleftrightarrow \quad b = c = 0, \; a = d.$$

ii) By i), to be conservative $\vec{F}$ must have the form

$$\vec{F} = \left(\frac{ax}{\sqrt{x^2+y^2}}, \frac{ay}{\sqrt{x^2+y^2}}\right)$$

Now, such a $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = \frac{ax}{\sqrt{x^2+y^2}}, \\ \partial_y f = \frac{ay}{\sqrt{x^2+y^2}}. \end{cases}$$

From first equation,

$$f(x,y) = \int \frac{ax}{\sqrt{x^2+y^2}}\, dx + k(y) = \frac{a}{2}\int (x^2+y^2)^{-1/2}(2x)\, dx + k(y) = a(x^2+y^2)^{1/2} + k(y).$$

Plugging this into the second equation we have

$$\partial_y f = a\frac{1}{2}(x^2+y^2)^{-1/2}2y + k'(y) = \frac{ay}{\sqrt{x^2+y^2}}, \quad \Longleftrightarrow \quad k'(y) = 0.$$

Thus, we deduce that

$$f(x,y) = a\sqrt{x^2+y^2} + k,\ k \in \mathbb{R},$$

are all the potentials for $\vec{F}$. □

Exercise 24. For the volume, we may notice that

$$\lambda_3(D) = \int_D 1\, dxdydz \overset{RF}{=} \int_0^1 \left(\int_{x^2+4y^2 \leqslant 1+z^2} dxdy \right) dz.$$

By using adapted polar coordinates, $x = \rho\cos\theta$, $y = \frac{1}{2}\rho\sin\theta$, in such a way that $x^2+4y^2 = \rho^2$, we have

$$\int_{x^2+4y^2\leqslant 1+z^2} dxdy = \int_{0\leqslant\rho\leqslant\sqrt{1+z^2},\ 0\leqslant\theta\leqslant 2\pi} \frac{1}{2}\rho\, d\rho d\theta \overset{RF}{=} \pi \int_0^{\sqrt{1+z^2}} \rho\, d\rho = \pi\left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=\sqrt{1+z^2}} = \frac{\pi}{2}(1+z^2).$$

Therefore

$$\lambda_3(D) = \int_0^1 \frac{\pi}{2}(1+z^2)\, dz = \frac{\pi}{2}\left(1 + \left[\frac{z^3}{3}\right]_{z=0}^{z=1}\right) = \frac{2}{3}\pi. \quad \square$$

Exercise 25. i) If $u(x,y) = \text{Re}\, f(x+iy)$ and $v(x,y) = \text{Im}\, f(x+iy)$, then

$$g(x+iy) = \overline{f(x-iy)} = \overline{u(x,-y)+iv(x,-y)} = u(x,-y) - iv(x,-y),$$

from which we see that

$$U(x,y) = \text{Re}\, g(x+iy) = u(x,-y),\quad V(x,y) = \text{Im}\, g(x+iy) = -v(x,-y).$$

ii) g is holomorphic iff U, V are $\mathbb{R}-$differentiable and they verify CR equations. Clearly, sunce f is holomorphic, u, v are $\mathbb{R}-$differentiable, hence also U, V are $\mathbb{R}-$differentiable. Therefore, we have to verify if U, V fulfil also the CR equations, that is

$$\begin{cases} \partial_x U \equiv \partial_y V, \\ \partial_y U \equiv -\partial_x V. \end{cases}$$

We have,

$$\partial_x U = \partial_x (u(x,-y)) = \partial_x u(x,-y), \quad \partial_y V = \partial_y \left(-v(x,-y)\right) = -\partial_y v(x,-y)(-1) = \partial_y v(x,-y).$$

And since $\partial_x u \equiv \partial_y v$ we deduce that also $\partial_x U = \partial_y V$. Similarly, $\partial_y U = -\partial_x V$ and the check is completed. □

Exercise 26. i) We have a second order equation. The homogeneous equation is $y'' + 2y' + y = 0$, whose characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, or $(\lambda + 1)^2 = 0$. The fundamental system of solutions for the homogeneous equation is $w_1 = e^{-t}$, $w_2 = te^{-t}$, whose wronskian is

$$W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}(1-t) \end{bmatrix} = e^{-2t}(1-t) + te^{-2t} = e^{-2t}.$$

The general solution of the original equation is then

$$y(t) = \left(c_1 - \int \frac{w_2}{W}(t+1)\, dt\right) w_1 + \left(c_2 + \int \frac{w_1}{W}(t+1)\, dt\right) w_2$$

We have

$$\begin{aligned} \int \tfrac{w_2}{W}(t+1)\, dt &= \int \tfrac{te^{-t}}{e^{-2t}}(t+1)\, dt = \int e^t (t^2+t)\, dt = e^t(t^2+t) - \int e^t(2t+1)\, dt \\ &= e^t(t^2 + t - 2t - 1) + \int 2e^t\, dt = e^t(t^2 - t + 1), \end{aligned}$$

and

$$\int \frac{w_1}{W}(t+1)\, dt = \int \frac{e^{-t}}{e^{-2t}}(t+1)\, dt = \int e^t(t+1)\, dt = e^t(t+1) - \int e^t\, dt = te^t.$$

Therefore, the general integral is

$$y(t) = \left(c_1 - e^t(t^2 - t + 1)\right) e^{-t} + \left(c_2 + te^t\right) te^{-t} = c_1 e^{-t} + c_2 te^{-t} + t - 1, \quad c_1, c_2 \in \mathbb{R}.$$

ii) Imposing $y(0) = 0$ we get $c_1 - 1 = 0$, that is $c_1 = 1$, so

$$y(t) = e^{-t} + c_2 te^{-t} + t - 1.$$

To determine also c_2, we impose $y'(0) = 1$, that is, since

$$y'(t) = -e^{-t} + c_2 e^{-t}(1-t) + 1, \implies -1 + c_2 + 1 = 1, \iff c_2 = 1.$$

The solution of the Cauchy problem is then,

$$y(t) = e^{-t} + te^{-t} + t - 1, \quad c_1, c_2 \in \mathbb{R}.$$

iii) From $y(0) = 0$ we get

$$y(t) = e^{-t} + c_2 te^{-t} + t - 1,$$

and imposing also $y(1) = 0$ we get

$$0 = e^{-1} + c_2 e^{-1}, \iff c_2 = 1.$$

The solution is the same of that one found at ii). □

Exercise 27. i) For $D \neq \emptyset$ we consider a point of type $(x, y, 2)$. Then $(x, y, 2) \in D$ iff $x^2 + y^2 = 4$ and $y^2 = 1$, thus $y = \pm 1$ and $x^2 = 3$, that is $x = \pm\sqrt{3}$. We conclude that points $(\pm\sqrt{3}, \pm 1, 2)$ (four points, all possible combinations of sign) belong to D.

We have that $D = \{g_1 = 0,\ g_2 = 0\}$ where $g_1 = x^2 + y^2 - z^2$, and $g_2 = y^2 + (z-2)^2 - 1$. Clearly, both g_1 and g_2 are differentiable functions (they are polynomials). In order $\vec{g} = (g_1, g_2)$ be a submersion on D we need to verify that

$$\operatorname{rk} \vec{g}' = \operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 2x & 2y & -2z \\ \\ 0 & 2y & 2(z-2) \end{bmatrix} = 2,\ \forall (x, y, z) \in D.$$

Now, this is false iff all 2×2 sub-determinants of the Jacobian matrix $\vec{g}'$ vanish, that is iff

$$\begin{cases} 4xy = 0, \\ 4x(z-2) = 0, \\ 8y(z-1) = 0. \end{cases} \iff \begin{cases} x = 0, \\ \\ y(z-1) = 0, \end{cases} \vee \begin{cases} y = 0, \\ \\ x(z-2) = 0, \end{cases}$$

The first subsystem has solutions $(0, 0, z)$ and $(0, y, 1)$ $(x, y \in \mathbb{R})$; the second, $(0, 0, z)$ and $(x, 0, 2)$, $(x, z \in \mathbb{R})$. Now:

- $(0, 0, z) \in D$ iff $z^2 = 0$ and $(z-2)^2 = 1$, impossible;
- $(0, y, 1) \in D$ iff $y^2 = 1$ and $y^2 + 1 = 1$, again impossible;
- $(x, 0, 2) \in D$ iff $x^2 = 4$ and $0 = 1$, impossible.

Cocnlusion: there is no point on D at which rank of $\vec{g}'$ is less than 2, therefore rank of $\vec{g}'(x, y, z)$ is 2 for every $(x, y, z) \in D$, that is $\vec{g}$ is a submersion on D.

ii) D is defined by equalities involving continuous functions, it is therefore closed. From the second equation

$$y^2 + (z-2)^2 = 1, \implies y^2 \leqslant 1,\ (z-2)^2 \leqslant 1.$$

In particular, $-1 \leqslant z - 2 \leqslant 1$, that is $1 \leqslant z \leqslant 3$, thus $z^2 \leqslant 9$. Plugging this into the first equation,

$$x^2 + y^2 = z^2, \quad x^2 + y^2 9, \implies x^2 9.$$

In conclusion $x^2 + y^2 + z^2 9 + 1 + 9 = 19$, for every $(x, y, z) \in D$, from which we see that D is bounded. We conclude that D is compact.

iii) Points at min/max distance to $\vec{0}$ minimize/maximize the function $f = x^2 + y^2 + z^2$. Since f is continuous and D is compact, according to the Weierstrass theorem, f has both min and max on D.

To determine these points, we apply the Lagrange multipliers' theorem. By i), hypotheses of the theorem are fulfilled. Thus, at every $(x, y, z) \in D$ min/max point for f in D we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \operatorname{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \iff \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \det \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 0 & 2y & 2(z-2) \end{bmatrix} = 0.$$

By computing the determinant we get

$$0 = 2x \cdot 4y(z - 2 + z) - 2x \cdot 4y(z - 2 - z) = 16xyz,$$

whose solutions are points $(0, y, z)$, $(x, 0, z)$ and $(x, y, 0)$. Now,

- $(0, y, z) \in D$ iff $y^2 = z^2$ and $y^2 + (z-2)^2 = 1$, from which $z^2 + (z-2)^2 = 1$, or $2z^2 - 2z + 3 = 0$, and since $\Delta < 0$ there are no solutions to this equation;
- $(x, 0, z) \in D$ iff $x^2 = z^2$ and $(z-2)^2 = 1$, from which $z = 1, 3$ and $x^2 = 1$ (that is $x = \pm 1$), or $x^2 = 9$ (that is $x = \pm 3$). We obtain points $(\pm 1, 0, 1)$ and $(\pm 3, 0, 3)$;
- $(x, y, 0) \in D$ iff $x^2 + y^2 = 0$, $y^2 + 4 = 1$ which is impossible.

Since $f(\pm 1, 0, 1) = 2$ and $f(\pm 3, 0, 3) = 18$ we deduce that $(\pm 1, 0, 1)$ are points of D at min distance to $\vec{0}$, $(\pm 3, 0, 3)$ are points of D at max distance to $\vec{0}$. □

Exercise 28 ii) The change or variable is given in the form $(u, v) = \Phi(x, y) = (y - x^3, y + x^3)$. According to the change of variable formula,

$$\int_D f(x, y)\, dxdy = \int_{\Phi(D)} f(\Phi^{-1}(u, v)) |\det(\Phi^{-1})'(u, v)|\, dudv.$$

We need to determine Φ^{-1}. Notice that

$$\begin{cases} u = y - x^3, \\ v = y + x^3, \end{cases} \iff \begin{cases} u + v = 2y, \\ v - u = 2x^3, \end{cases} \iff \begin{cases} y = \frac{u+v}{2}, \\ x^3 = \frac{v-u}{2}, \end{cases} \iff \begin{cases} y = \frac{u+v}{2}, \\ x = \left(\frac{v-u}{2}\right)^{1/3}, \end{cases}$$

Therefore

$$\Phi^{-1}(u, v) = \left(\left(\frac{v-u}{2} \right)^{1/3}, \frac{u+v}{2} \right).$$

Moreover,

$$(x, y) \in D, \iff \begin{cases} x \geqslant 1, \\ x^3 \leqslant y \leqslant 3, \end{cases} \iff \begin{cases} \left(\frac{v-u}{2}\right)^{1/3} \geqslant 1, \\ \frac{v-u}{2} \leqslant \frac{u+v}{2} \leqslant 3 \end{cases} \iff \begin{cases} v - u \geqslant 2, \\ v - u \leqslant v + u \leqslant 6 \end{cases}$$

that is

$$\Phi(D) = \{(u, v) \ : \ 2 \leqslant v - u \leqslant v + u \leqslant 6\}.$$

Now, to be $v - u \leqslant v + u$ it must be $u \geqslant 0$, and from $2 \leqslant v - u \leqslant v + u \leqslant 6$ we get $2 + u \leqslant v \leqslant 6 - u$ provided $2 + u \leqslant 6 - u$, that is $u \leqslant 2$. In conclusion

$$\Phi(D) = \{(u, v) \ : \ 0 \leqslant u \leqslant 2, \ 2 + u \leqslant v \leqslant 6 - u\}.$$

About f, in coordinates (u, v) we have

$$f(\Phi^{-1}(u, v)) = \left(\frac{v-u}{2} \right)^{2/3} u e^v,$$

while

$$\det(\Phi^{-1})' = \det \begin{bmatrix} \frac{1}{3} \left(\frac{v-u}{2}\right)^{-2/3} \left(-\frac{1}{2}\right) & \frac{1}{3} \left(\frac{v-u}{2}\right)^{-2/3} \left(+\frac{1}{2}\right) \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = -\frac{1}{6} \left(\frac{v-u}{2} \right)^{-2/3}.$$

In conclusion

$$
\begin{aligned}
\int_D f\ dxdy &= \int_{0\leqslant u\leqslant 2,\ 2+u\leqslant v\leqslant 6-u} \left(\tfrac{v-u}{2}\right)^{2/3} ue^v \tfrac{1}{6}\left(\tfrac{v-u}{2}\right)^{-2/3}\ dudv = \tfrac{1}{6}\int_{0\leqslant u\leqslant 2,\ 2+u\leqslant v\leqslant 6-u} ue^v\ dudv \\
&\overset{RF}{=} \tfrac{1}{6}\int_0^2\int_{2+u}^{6-u} ue^v\ dv\ du = \tfrac{1}{6}\int_0^2 u\int_{2+u}^{6-u} e^v\ dv\ du = \tfrac{1}{6}\int_0^2 u\left[e^v\right]_{v=2+u}^{v=6-u}\ du \\
&= \tfrac{1}{6}\int_0^2 u\left(e^{6-u}-e^{2+u}\right)\ du = \tfrac{1}{6}\left(e^6\int_0^2 ue^{-u}\ du - e^2\int_0^2 ue^u\ du\right) \\
&= \tfrac{1}{6}\left(e^6\left(\left[-ue^{-u}\right]_{u=0}^{u=2}+\int_0^2 e^{-u}\ du\right) - e^2\left(\left[ue^u\right]_{u=0}^{u=2}-\int_0^2 e^u\ du\right)\right) \\
&= \tfrac{1}{6}\left(e^6\left(-2e^{-2}-(e^{-2}-1)\right)-e^2\left(2e^2-(e^2-1)\right)\right) \\
&= \tfrac{e^2}{6}\left(-2e^2+e^4-1\right). \quad \square
\end{aligned}
$$

Exercise 29. In order $f = u + iv$ be holomorphic, we need that u, v are both $\mathbb{R}$−differentiable (and certainly v it is), and they verify the CR equations,

$$
\begin{cases} \partial_x u = \partial_y v, \\ \partial_y u = -\partial_x v. \end{cases}
$$

Thus we have to look for an $\mathbb{R}$−differentiable u such that

$$
\begin{cases} \partial_x u = 3y^2 - 3x^2 + 4x, \\ \partial_y u = -(-6xy + 4y - 1). \end{cases}
$$

From the first equation we get,

$$
u(x,y) = \int (3y^2 - 3x^2 + 4x)\ dx + k(y) = 3y^2x - x^3 + 2x^2 + k(y).
$$

Plugging this into the second equation we have

$$
6xy + k'(y) = 6xy - 4y + 1, \quad \Longleftrightarrow \quad k'(y) = -4y + 1, \quad \Longleftrightarrow \quad k(y) = -2y^2 + y + k,\ k \in \mathbb{R}.
$$

Thus, all the possible u that verify the CR eqns together with v are

$$
u(x,y) = 3y^2x - x^3 + 2x^2 - 2y^2 + y + k.
$$

Since such u are clearly $\mathbb{R}$−differentiable, $f = u + iv$ is $\mathbb{C}$−differentiable (holomorphic) on $\mathbb{R}^2$.

To determine the analytical expression for f as a function of complex variable $z = x + iy$, we may notice that

$$\begin{aligned} f &= u + iv = 3y^2x - x^3 + 2x^2 - 2y^2 + y + k + i\left(y^3 - 3x^2y + 4xy - x\right) \\ &= -i\underbrace{(x+iy)}_{z} + 2\underbrace{(x^2 - y^2 + i2xy)}_{z^2} - \underbrace{\left(x^3 - iy^3 - 3y^2x + i3x^2y\right)}_{z^3} + k \\ &= -z^3 + 2z^2 - iz + k. \quad \square \end{aligned}$$

Exercise 30. See notes for definitions and characterizations.

Let's focus on the resuire property. We first notice that is $\partial S = \emptyset$, ∂S is closed. We assume then that $\partial S \neq \emptyset$. To verify that ∂S is closed, we use the Cantor characterization. Let $(\vec{x}_n) \subset \partial S$ be such that $\vec{x}_n \longrightarrow \vec{x} \in \mathbb{R}^d$. We prove that $\vec{x} \in \partial S$. Fix $r > 0$. Since $\vec{x}_n \longrightarrow \vec{x}$, we have that for $n \geqslant N$ $\|\vec{x}_n - \vec{x}\| \leqslant \frac{r}{2}$. Now, since $\vec{x}_n \in \partial S$,

$$B(\vec{x}_n, r/2] \cap S \neq \emptyset, \ \wedge B(\vec{x}_n, r/2] \cap S^c \neq \emptyset.$$

Since $\|\vec{x}_n - \vec{x}\| \leqslant \frac{r}{2}$, we have that

$$B(\vec{x}_n, r/2] \subset B(\vec{x}, r],$$

therefore

$$B(\vec{x}, r] \cap S \supset B(\vec{x}_n, r/2] \cap S \neq \emptyset,$$

and, similarly, $B(\vec{x}, r] \cap S^c \neq \emptyset$. We conclude that $\vec{x} \in \partial S$, thus ∂S is closed. $\square$

Exercise 31. First of all let $z \neq 0$. Setting $w = \frac{1}{z}$, we have to solve

$$\sinh w = 0, \iff \frac{e^w - e^{-w}}{2} = 0, \iff e^{2w} = 1, \iff 2w = \log|1| + i(0 + k2\pi) = ik2\pi, \ k \in \mathbb{Z}.$$

Thus

$$\frac{1}{z} = w = ik\pi, \iff z = \frac{1}{ik\pi} = \frac{-i}{k\pi} = \frac{i}{k\pi}, \ k \in \mathbb{Z}\backslash\{0\}. \quad \square$$

Exercise 32. The problem asks to determine

$$\min/\max_{(x,y,z)\in D} \sqrt{(x-1)^2 + (y-2)^2 + (z+3)^2}.$$

Previous problem has the same min/max points (if any) of

$$\min/\max_{(x,y,z)\in D} \left((x-1)^2 + (y-2)^2 + (z+3)^2\right),$$

which is the problem we solve here.

We start discussing existence. D is certainly a closed set (defined by an equality of a continuous function). Let's see if D is also bounded. Since no condition on z is given, it means that if $(x, y, z_0) \in D$ then $(x, y, z) \in D$ for every $z \in \mathbb{R}$. In paricular $(x, x, z) \in D$ for every $x, z \in \mathbb{R}$. We deduce that D is unbounded. Thus, D is not compact. The function $f(x, y, z) = \|(x - 1, y - 2, z + 3)\|^2$ is clearly continuous, and since

$$\lim_{(x,y,z)\to\infty_3} f = +\infty,$$

we conclude that f has no maximum on D but it has global minimum on D.

To determine the minimum, we wish to apply the Lagrange multipliers' theorem. To this aim, we need first to check if D is the zero set of a submersion on D itself. Now, $D = \{g = 0\}$ where $g = (x-y)^2+(x-y)$, and g is a submersion on D iff $\nabla g \neq \vec{0}$ on D. We have

$$\nabla g = (2(x-y)-1, -2(x-y+1, 0) = \vec{0}, \iff 2(x-y)-1 = 0, \iff x-y = \frac{1}{2}.$$

However, if $x - y = \frac{1}{2}$ we easily see that the condition characterizing D is not fulfilled. Thus, $\nabla g \neq 0$ always. Thus, in particular, g is a submersion on D. Therefore, according to Lagrange multipliers' theorem, at $(x, y, z) \in D$ min point for f,

$$\nabla f = \lambda \nabla g, \iff \mathrm{rk}\begin{bmatrix} \nabla f \\ \\ \nabla g \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2(x-1) & 2(y-2) & 2(z+3) \\ \\ 2(x-y)-1 & -2(x-y)+1 & 0 \end{bmatrix} < 2.$$

This happens iff all 2×2 sub-determinants vanish, that is

$$\begin{cases} (1-2(x-y))\,(x+y-3) = 0, \\ 2(z+3)(2(x-y)-1) = 0, \\ 2(z+3)(1-2(x-y)) = 0. \end{cases}$$

The first equation yields to the alternative $x - y = \frac{1}{2}$, and plugging this into the other two equations we get identities $0 = 0$. Thus, we get points $(x, x - \frac{1}{2}, z)$. Now these points belong to D iff $\frac{1}{4} - \frac{1}{2} = 0$, which is false.

In the second case, $x + y = 3$, and plugging this into the other two equations we get $z = -3$, thus points $(x, 3-x, -3)$. Now,

$$(x, 3-x, -3) \in D, \iff (2x-3)^2-(2x-3) = 0, \iff (2x-3)(2x-4) = 0, \iff x = \frac{3}{2}, \vee x = 2.$$

We get points $(\frac{3}{2}, \frac{3}{2}, -3)$ and $(2, 1, -3)$. Since $f(\frac{3}{2}, \frac{3}{2}, -3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and $f(2, 1, -3) = 1 + 1 = 2$, we see that the points od D at minimum distance to $(1, 2, -3)$ is $(\frac{3}{2}, \frac{3}{2}, -3)$. □

Exercise 33. i) D is closed because it is defined by large inequalities. It is not open because $D \neq \emptyset, \mathbb{R}^3$. It is unbounded since $(n, n, \frac{1}{\cosh(2n^2)}) \in D$ for every $n \in \mathbb{N}$, therefore it is not compact.

ii) We have

$$\lambda_3(D) = \int_D 1\, dxdydz \overset{RF}{=} \int_{\mathbb{R}^2} \left(\int_0^{1/\cosh(x^2+y^2)} dz \right) dxdy = \int_{\mathbb{R}^2} \frac{1}{\cosh(x^2+y^2)}\, dxdy.$$

By introducing polar coordinates,

$$\lambda_3(D) = \int_{\rho \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi} \frac{1}{\cosh \rho^2} \rho\, d\rho d\theta = 2\pi \int_0^{+\infty} \frac{\rho}{\cosh \rho^2}\, d\rho.$$

Notice that

$$\frac{\rho}{\cosh \rho^2} = \frac{2\rho}{e^{\rho^2} + e^{-\rho^2}} = \frac{2\rho e^{\rho^2}}{1 + e^{2\rho^2}} = \partial_\rho \arctan(e^{\rho^2}),$$

thus

$$\lambda_3(D) = 2\pi \left[\arctan(e^{\rho^2})\right]_{\rho=0}^{\rho=+\infty} = 2\pi\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi^2}{2}.$$

iii) Proceeding as in ii), we have

$$I_\alpha := \int_D e^{\alpha(x^2+y^2)}\, dxdydz \overset{RF}{=} \int_{\mathbb{R}^2} \left(\int_0^{1/\cosh(x^2+y^2)} e^{\alpha(x^2+y^2)}\, dz \right) dxdy = \int_{\mathbb{R}^2} \frac{e^{\alpha(x^2+y^2)}}{\cosh(x^2+y^2)}\, dxdy.$$

Changing vars to polar coords,

$$I_\alpha = \int_{\rho\geqslant 0,\ 0\leqslant\theta\leqslant 2\pi} \frac{e^{\alpha\rho^2}}{\cosh\rho^2} \rho\, d\rho d\theta = 2\pi \int_0^{+\infty} \frac{2\rho e^{(\alpha+1)\rho^2}}{1+e^{2\rho^2}}\, d\rho.$$

Notice that

$$\frac{2\rho e^{(\alpha+1)\rho^2}}{1+e^{2\rho^2}} \sim_{+\infty} 2\rho \frac{e^{(\alpha+1)\rho^2}}{e^{2\rho^2}} = 2\rho e^{(\alpha-1)\rho^2}$$

and

$$\exists \int_0^{+\infty} \rho e^{(\alpha-1)\rho^2}\, d\rho \iff \alpha - 1 < 0, \iff \alpha < 1. \qquad \square$$

Exercise 34. i) In order $f = u + iv$ be $\mathbb{C}$−differentiable on $\mathbb{C}$ we need 1. that u, v are $\mathbb{R}$ differentiable on $\mathbb{R}^2$ (which is true, being u, v polynomials) and 2. u, v fulfil the CR equations, namely

$$\begin{cases} \partial_x u \equiv \partial_y v, \\ \\ \partial_y u \equiv -\partial_x v, \end{cases} \iff \begin{cases} 3x^2 + ay^2 \equiv bx^2 - 3y^2, \\ \\ 2axy \equiv -2bxy, \end{cases} \iff b = 3,\ a = -3.$$

ii) We have

$$f = (x^3 - 3xy^2) + i(3x^2y - y^3) = (x+iy)^3 = z^3. \qquad \square$$

Exercise 35. i) To prove that $\phi(t) := E(y(t), y'(t))$ is constant we show that the derivative of ϕ w.r.t. t vanishes. According to the total derivative formula, we have

$$\phi'(t) = \frac{d}{dt} E(y, y') = \partial_y E(y, y') y' + \partial_v E(y, y') y''.$$

Now,

$$E(y, v) = \frac{1}{2} m v^2 - f(y), \implies \partial_y E = -f'(y) = -F(y),\ \ \partial_v E = mv,$$

thus

$$\phi'(t) = -F(y)y' + my'y'' = y' \underbrace{(my'' - F(y))}_{=0\ by\ eqn} \equiv 0.$$

Therefore

$$E(y, y') \equiv k, \iff \frac{1}{2} m(y')^2 - f(y) \equiv k, \iff (y')^2 = \frac{2}{m}(f(y)+k), \iff y' = \pm\sqrt{\frac{2}{m}(f(y)+k)}.$$

The last one is a separable variables equation.

ii) If $m = 1$ and $F(y) = -2y - 3y^2$, then $f(y) = \int F(y)' dy = \int(-2y - 3y^2) = -y^2 - y^3$. Therefore

$$y' = \pm\sqrt{2(k - y^2 - y^3)},$$

where $E(y, y') \equiv k$. In particular, $E(y(0), y'(0)) = k$, and since $y(0) = -2$, $y'(0) = \sqrt{8}$ we have

$$E(-2, \sqrt{8}) = \frac{1}{2}(\sqrt{8})^2 - (-(-2)^2 - (-2)^3) = 4 - (-4 + 8) = 0.$$

Thus $k = 0$ and y solves the equation

$$y' = \pm\sqrt{-2(y^3 + y^2)} = \pm\sqrt{-2y^2(y + 1)} = \pm\sqrt{2}y\sqrt{-y - 1}.$$

Since at $t = 0$ we have $y'(0) = \sqrt{8} > 0$, $y(0) = -2 < 0$ the previous equation is

$$y' = \sqrt{2}y\sqrt{-y - 1}.$$

We can now solve this by separation of variables once we notice that y is not a constant solution. We have

$$\int \frac{y'}{y\sqrt{-y - 1}}\, dt = -\int \sqrt{2}\, dt = -\sqrt{2}t + c.$$

We have

$$\int \frac{y'}{y\sqrt{-y-1}}\, dt \overset{u=y(t),\ du=y'(t)\ dt}{=} \int \frac{1}{u\sqrt{-u-1}}\, du \overset{v=\sqrt{-u-1},\ u=-1-v^2,\ du=-2v\ dv}{=} \int \frac{1}{(-1-v^2)v}(-2v)\, dv$$

$$= 2\int \frac{1}{1+v^2}\, dv = 2\arctan v = 2\arctan\sqrt{-y - 1}.$$

Therefore

$$2\arctan\sqrt{-y - 1} = -\sqrt{2}t + c.$$

For $t = 0$ we have

$$2\arctan\sqrt{1} = c, \quad \Longleftrightarrow \quad c = \frac{\pi}{2}.$$

We conclude that

$$2\arctan\sqrt{-y - 1} = -\sqrt{2}t + \frac{\pi}{2}, \quad \Longleftrightarrow \quad \sqrt{-y - 1} = \tan\left(-\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right), \quad \Longleftrightarrow \quad y(t) = -1 - \tan^2\left(-\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right). \quad \square$$

Exercise 36. i) We have a second order linear equation

$$y'' + 9y = 6\sin(3t).$$

The homogeneous equation associated to this is $y'' + 9y = 0$, whose characteristic equation is $\lambda^2 + 9 = 0$, that is $\lambda = \pm i3$. The fundamental system of solutions for the homogeneous equation is then $w_1(t) = \sin(3t)$, $w_2(t) = \cos(3t)$, whose wronskian is

$$W(t) = \det\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det\begin{bmatrix} \sin(3t) & \cos(3t) \\ 3\cos(3t) & -3\sin(3t) \end{bmatrix} = -3\left(\sin^2(3t) + \cos^2(3t)\right) = -3.$$

Therefore, the general solution for the original equation is

$$y(t) = \left(c_1 - \int \frac{w_2}{W} 6\sin(3t)\ dt\right) w_1 + \left(c_2 + \int \frac{w_1}{W} 6\sin(3t)\ dt\right) w_2.$$

We have

$$6\int \tfrac{w_2}{W}\sin(3t)\ dt = 6\int \tfrac{\cos(3t)}{-3}\sin(3t)\ dt = -\int \sin(6t)\ dt = \tfrac{1}{6}\cos(6t),$$

$$6\int \tfrac{w_1}{W}\sin(3t)\ dt = 6\int \tfrac{\sin(3t)}{-3}\sin(3t)\ dt = -2\int \sin^2(3t)\ dt.$$

Now

$$\begin{aligned}\int \sin^2(3t)\ dt &= \int (\sin(3t))\left(-\tfrac{\cos(3t)}{3}\right)'\ dt = -\tfrac{1}{3}\sin(3t)\cos(3t) + \int \cos^2(3t)\ dt \\ &= -\tfrac{1}{6}\sin(6t) + \int 1 - \sin^2(3t)\ dt = -\tfrac{1}{6}\sin(6t) + t - \int \sin^2(3t)\ dt,\end{aligned}$$

thus

$$\int \sin^2(3t)\ dt = \frac{1}{2}\left(t - \frac{\sin(6t)}{6}\right).$$

In conclusion,

$$y(t) = \left(c_1 - \frac{\cos(6t)}{6}\right)\sin(3t) + \left(c_2 - t + \frac{\sin(6t)}{6}\right)\cos(3t), \quad c_1, c_2 \in \mathbb{R}.$$

ii) Imposing $y(0) = 0$ we get

$$c_2 = 0,$$

thus

$$y(t) = \left(c_1 - \frac{\cos(6t)}{6}\right)\sin(3t) - \left(t - \frac{\sin(6t)}{6}\right)\cos(3t).$$

Computing $y'(t)$ we have

$$y'(t) = \sin(6t)\sin(3t) + \left(c_1 - \frac{\cos(6t)}{6}\right)3\cos(3t) - (1 - \cos(6t))\cos(3t) + \left(t - \frac{\sin(6t)}{6}\right)3\sin(3t),$$

and, by imposing $y'(0) = 0$ we get

$$3\left(c_1 - \frac{1}{6}\right) = 0, \quad \Longleftrightarrow \quad c_1 = \frac{1}{6}.$$

The solution of the CP is then

$$y(t) = \frac{1}{6}(1 - \cos(6t))\sin(3t) - \left(t - \frac{\sin(6t)}{6}\right)\cos(3t).$$

iii) We may write the general solution in the form

$$y(t) = \underbrace{\left(c_1 - \frac{\cos(6t)}{6}\right)\sin(3t) + \left(c_2 + \frac{\sin(6t)}{6}\right)\cos(3t)}_{bounded} - \underbrace{t\cos(3t)}_{unbounded},$$

and since the unbounded component is independent of c_1, c_2 we deduce that all the solutions are unbounded for $t \longrightarrow \pm\infty$. □

Exercise 37. i) D is closed being defined by large inequalities involving continuous functions of (x, y). It is not open since $D \neq \emptyset, \mathbb{R}^2$. It is bounded because $x \geqslant 0$ and from $0 \leqslant y \leqslant 1 - x$, in particular $1 - x \geqslant 0$, that is $x \leqslant 1$, so $0 \leqslant x \leqslant 1$ and, at same time, $0 \leqslant y \leqslant 1 - x \leqslant 1$. Thus $0 \leqslant x, y \leqslant 1$ and this implies that D is bounded. Since D is closed and bounded it is also compact.

ii) Since f is clearly continuous on D and D is compact, f admits both global min/max on D. To determine min/max points, we may argue in the following way. If $(x, y) \in D$ is a min/max point for f then

- either $(x, y) \in \text{Int } D$
- or $(x, y) \in D \backslash \text{Int } D = \partial D$.

In the first case, since

$$\partial_x f = 3y + 2xy + y^2, \quad \partial_y f = 3x + x^2 + 2xy$$

so $\partial_x f, \partial_y f \in \mathscr{C}(D)$, f is then differentiable on D, according to Fermat theorem, at min/max points

$$\nabla f(x, y) = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} 3y + 2xy + y^2 = 0, \\ \\ 3x + x^2 + 2xy = 0. \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y(3 + 2x + y) = 0, \\ \\ x(3 + 2y + x) = 0. \end{cases}$$

The first equation leads to the alternative $y = 0$ or $3 + 2x + y = 0$. In the first case, the second equation becomes $x(3 + x) = 0$. whose solutions are $x = 0$ and $x = -3$. This produces points $(0, 0)$ and $(-3, 0)$. In any case these do not belong to Int D. In the second case, $y = -2x - 3$, from the second equation we obtain $x(-3 - 3x) = 0$, that is $x = 0$ or $x = -1$. This yields points $(0, -3), (-1, -1) \notin D$. In conclusion, no stationary point for f is in the interior of D.

Thus, min/max points for f are on $\partial D = A \cup B \cup C$ where $A = \{(0, y) \ : \ 0 \leqslant y \leqslant 1\}$, $B = \{(x, 0) \ : \ 0 \leqslant x \leqslant 1\}$ and, finally, $C = \{(x, 1 - x) \ : \ 0 \leqslant x \leqslant 1\}$. On A we have

$$f(0, y) \equiv 0,$$

thus every point is min/max point for f on A. On B, similarly, we have $f(x, 0) \equiv 0$, thus every point of B is at same time min/max for f on B. Finally, on C

$$f(x, 1 - x) = 3x(1 - x) + x^2(1 - x) + x(1 - x)^2 = 3x - 3x^2 + x^2 - x^3 + x - 2x^2 + x^3 = -4x^2 + 4x =: g(x).$$

Let's determine min/max points for g with $x \in [0, 1]$. We have $g'(x) = -8x + 4 \geqslant 0$ iff $x \leqslant \frac{1}{2}$. Thus $x = \frac{1}{2}$ is max point for g and $x = 0, 1$ are min points for g. This means that

- $\left(\frac{1}{2}, \frac{1}{2}\right)$ is max point for f on C
- $(0, 1), (1, 0)$ are min points for f on C.

We can now draw the conclusion:

- for minimum, candidates are points $(x, 0)$, $(0, y)$ with $0 \leqslant x, y \leqslant 1$ where $f = 0$. All these are min points for f on D;
- for maximum, candidates are points $\left(\frac{1}{2}, \frac{1}{2}\right)$ (where $f = 1$) and $(x, 0)$ and $(0, y)$ with $0 \leqslant x, y \leqslant 1$ (where $f = 0$). Thus, the max point is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Exercise 38. i) Let $\vec{F} = (\phi, \psi)$. In order $\vec{F}$ be irrotational on D we need

$$\partial_y \phi \equiv \partial_x \psi, \text{ on } D.$$

We have

$$\partial_y \phi = \frac{b(x^2+y^2)^2 - (ax+by)2(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{b(x^2+y^2) - 4y(ax+by)}{(x^2+y^2)^2} = \frac{bx^2 - 4axy - 3by^2}{(x^2+y^2)^2},$$

$$\partial_x \psi = \frac{c(x^2+y^2)^2 - (cx+dy)2(x^2+y^2)2x}{(x^2+y^2)^4} = \frac{c(x^2+y^2) - 4x(cx+dy)}{(x^2+y^2)^2} = \frac{-3cx^2 - 4dxy + cy^2}{(x^2+y^2)^2}.$$

Hence,

$$\partial_y \phi \equiv \partial_x \psi, \iff bx^2 - 4axy - 3by^2 \equiv -3cx^2 - 4dxy + cy^2, \iff \begin{cases} b = -3c, \\ a = d, \\ -3b = c \end{cases}$$

from which $b = c = 0$ and $a = d \in \mathbb{R}$. Thus

$$\vec{F} = \left(\frac{ax}{(x^2+y^2)^2}, \frac{ay}{(x^2+y^2)^2} \right), \ \forall (x, y) \in D.$$

ii) Necessary condition to be conservative is that $\vec{F}$ be irrotational, thus $\vec{F}$ is given as at the end of i). Now, such $\vec{F}$ is conservative iff $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = \frac{ax}{(x^2+y^2)^2}, \\ \partial_y f = \frac{ay}{(x^2+y^2)^2}. \end{cases}$$

From the first equation

$$f(x, y) = \int \frac{ax}{(x^2+y^2)^2} \, dx + k(y) = \frac{a}{2} \int \partial_x - (x^2+y^2)^{-1} \, dx + k(y) = -\frac{a}{2}(x^2+y^2)^{-1} + k(y).$$

Plugging this into the second equation we have

$$\partial_y f = \frac{ay}{(x^2+y^2)^2}, \iff \frac{ay}{(x^2+y^2)^2} + k'(y) = \frac{ay}{(x^2+y^2)^2}, \iff k'(y) = 0, \iff . \, .k(y) = k \in \mathbb{R}.$$

Thus, $\vec{F}$ is conservative with potentials

$$f(x, y) = -\frac{a}{2}(x^2+y^2)^{-1} + k, \ k \in \mathbb{R}.$$

iii) By previous discussion, when $(a, b, c, d) = (2, 0, 0, 2)$, field $\vec{F}$ is conservative. Thus

$$\int_\gamma \vec{F} = f(0, 2) - f(1, 0) = -\frac{1}{4} - (-1) = \frac{3}{4}. \quad \square$$

Exercise 39. i) Since $x^2 + z^2$ is invariant by rotations around the $y-$axis, D is invariant by rotations around such axis. We can draw any section containing the y axis, for instance $D \cap \{x = 0\}$ (section of D in plane yz). We have

$$D \cap \{x = 0\} = \{(0, y, z) \ : \ 1 - z^2 \geqslant y \leqslant \sqrt{1 - z^2}\}.$$

Figure:

ii) Notice that

$$\begin{aligned}\lambda_3(D) &= \int_D 1\,dxdydz \overset{RF}{=} \int_{1-(x^2+z^2)\leqslant\sqrt{1-(x^2+y^2)}} \left(\int_{1-(x^2+z^2)}^{\sqrt{1-(x^2+z^2)}} 1\,dy\right) dxdz \\ &= \int_{1-(x^2+z^2)\leqslant\sqrt{1-(x^2+z^2)}} \left(\sqrt{1-(x^2+z^2)} - (1-(x^2+z^2))\right)\,dxdz \\ &\overset{pol.\ coords}{=} \int_{1-\rho^2\leqslant\sqrt{1-\rho^2},\ 0\leqslant\theta\leqslant 2\pi} \rho\left(\sqrt{1-\rho^2}-(1-\rho^2)\right)\,d\rho d\theta \\ &\overset{RF}{=} 2\pi\int_{1-\rho^2\leqslant\sqrt{1-\rho^2}} \rho\left(\sqrt{1-\rho^2}-(1-\rho^2)\right)\,d\rho.\end{aligned}$$

Now, $1-\rho^2 \leqslant \sqrt{1-\rho^2}$ iff (being $1-\rho^2 \geqslant 0$ for the root), $\sqrt{1-\rho^2}\leqslant 1$ always true, the condition on ρ is $\rho^2 \leqslant 1$, that is $0\leqslant\rho\leqslant 1$. In conclusion,

$$\begin{aligned}\lambda_3(D) &= 2\pi\int_0^1 \rho\left(\sqrt{1-\rho^2}-(1-\rho^2)\right)\,d\rho = 2\pi\int_0^1 \rho(1-\rho^2)^{1/2}-\rho+\rho^3\,d\rho \\ &= 2\pi\left(\left[-\tfrac{1}{3}(1-\rho^2)^{3/2}\right]_{\rho=0}^{\rho=1} - \left[\tfrac{\rho^2}{2}\right]_{\rho=0}^{\rho=1} + \left[\tfrac{\rho^4}{4}\right]_{\rho=0}^{\rho=1}\right) = 2\pi\left(\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{4}\right) = \tfrac{\pi}{6}. \quad \square\end{aligned}$$

Exercise 40. See notes for CR equations and connection with $\mathbb{C}-$differentiability.

i) If $f = u+iv$ with, for example, u constant, then, by the CR eqns,

$$\begin{cases} 0 \equiv \partial_x u \equiv \partial_y v, \\ 0 = \partial_y u \equiv -\partial_x v, \end{cases} \Longrightarrow \begin{cases} \partial_x v \equiv 0, \\ \partial_y v \equiv 0. \end{cases}$$

From this it follows that v is constant.

iii) Suppose now that $f = u+iv$ be $\mathbb{C}-$differentiable and such that $|f| = \sqrt{u^2+v^2} = k$ or, equivalently, $u^2+v^2 \equiv k^2$. If $k=0$ the conclusion is trivial. Assume $k\neq 0$. By computing ∂_x we have

$$2u\partial_x u + 2v\partial_x v \equiv 0,$$

and because of CR equations

$$u\partial_x u - v\partial_y u = 0.$$

Similarly, computing ∂_y

$$2u\partial_y u + 2v\partial_y v = 0, \iff u\partial_y u + v\partial_x u = 0.$$

Multiplying the first relation by $\partial_x u$ and the second by $\partial_y u$ we obtain

$$u(\partial_x u)^2 \equiv v\partial_y u\partial_x u = -u(\partial_y u)^2, \iff u\left((\partial_x u)^2+(\partial_y u)^2\right)\equiv 0. \iff u^2\left((\partial_x u)^2+(\partial_y u)^2\right)\equiv 0.$$

Similarly,

$$v^2\left((\partial_x v)^2+(\partial_y v)^2\right)\equiv 0.$$

By CR eqns, $(\partial_x u)^2 + (\partial_y u)^2 \equiv (\partial_x v)^2 + (\partial_y v)^2$, thus summing up the two previous relations we get

$$(u^2+v^2)\left((\partial_x u)^2 + (\partial_y u)^2\right) \equiv 0, \iff k^2\left((\partial_x u)^2 + (\partial_y u)^2\right) \equiv 0, \iff (\partial_x u)^2 + (\partial_y u)^2 \equiv 0,$$

which means $\partial_x u \equiv \partial_y u \equiv 0$. Thus u is constant and we can now conclude by ii). □

Exercise 41. i) We have a separable variables equation. Solutions are either constant or obtained by separation of variables. In the first case, $y \equiv C$ is a solution iff $C(C^2+1) = 0$, that is $C = 0$. Other solution are obtained by separation of variables:

$$y' = y(y^2+1), \iff \frac{y'}{y(y^2+1)} = 1, \iff \int \frac{y'}{y(y^2+1)}\,dt = t + k.$$

Now,

$$\int \frac{y'}{y(y^2+1)}\,dt \overset{u=y(t),\ du=y'(t)\ dt}{=} \int \frac{1}{u(u^2+1)}\,du.$$

According to Hermite decomposition,

$$\frac{1}{u(u^2+1)} = \frac{A}{u} + \frac{Bu+C}{u^2+1}$$

from which $A = 1$, $B = -1$ and $C = 0$. Therefore

$$\int \frac{1}{u(u^2+1)}\,du = \log|u| - \frac{1}{2}\log(u^2+1) = \log\frac{|u|}{\sqrt{u^2+1}}.$$

Thus we have

$$\log\frac{|y|}{\sqrt{y^2+1}} = t + k,$$

that is

$$\frac{|y|}{\sqrt{y^2+1}} = ke^t, \iff \frac{y^2}{y^2+1} = ke^{2t},\ (k>0) \iff y^2 = \frac{ke^{2t}}{1-ke^{2t}}, \iff y = \pm\sqrt{\frac{ke^{2t}}{1-ke^{2t}}}.$$

ii) The solution for which $y(0) = 1$ cannot be a constant solution. Since $y(0) = 1$, we have

$$y(t) = \sqrt{\frac{ke^{2t}}{1-ke^{2t}}},$$

and $y(0) = 1$ means $\sqrt{\frac{k}{1-k}} = 1$, that is $k = \frac{1}{2}$. □

Exercise 42. i) Let $(g_1, g_2) := (x^2+y^2-1, x+y+z-1)$ in such a way $D = \{g_1 = 0,\ g_2 = 0\}$. To check that (g_1, g_2) is a submersion on D we have to verify that

$$\operatorname{rk}\begin{bmatrix} g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rk}\begin{bmatrix} 2x & 2y & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2,\ \forall (x,y,z) \in D.$$

Now, rank is < 2 iff the two gradients are linearly dependent. This is manifestly impossible because of their third component.

ii) D is closed being defined by equalities involving continuous functions. D is also bounded: indeed, by first equation we have $x^2, y^2 \leqslant 1$, thus $-1 \leqslant x, y \leqslant 1$, and by the secon

$$-1 \geqslant z = 1 - (x + y) \leqslant 3,$$

thus $z^2 \leqslant 9$ and $x^2 + y^2 + z^2 \leqslant 11$.

iii) Function f is continuous on D compact: existence of min/max is ensured by Weierstrass thm. To determine these points, we apply Lagrange multipliers thm. By i), D fulfils the assumption of the thm. Thus, at (x, y, z) min/max point for f on D we must have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \text{rk} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \text{rk} \begin{bmatrix} 2x + y - 1 & 2y + x + z - 1 & y \\ 2x & 2y & 0 \\ 1 & 1 & 1 \end{bmatrix} < 3,$$

that is iff the determinant of previous matrix vanishes. We get the condition

$$2y(x - y) + 2(y(2x + y - 1) - x(2y + x + z - 1)) = 0,$$

from which, simplifying,

$$y(y - x) + (y^2 - y - x^2 + x - xz) = 0.$$

Since we are looking for solutions $(x, y, z) \in D$, we must have $z = 1 - x - y$, and plugging this into previous equation yields,

$$y\,(2y - 1) = 0, \iff y = 0, \vee y = \frac{1}{2}.$$

Thus we get points $(x, 0, 1-x)$ and $(x, \frac{1}{2}, \frac{1}{2}-x)$, to which we have still to impose the condition $x^2+y^2 = 1$. In the first case $x^2 + 0^2 = 1$, thus $x = \pm 1$, that is points $(\pm 1, 0, \mp 1)$ (two points). In the second case, $x^2 + \frac{1}{4} = 1$, thus $x^2 = \frac{3}{4}$ and $x = \pm\frac{\sqrt{3}}{2}$, that is points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1+\sqrt{3}}{2}\right)$. We have

- $f(1, 0, -1) = 0$
- $f(-1, 0, 1) = 2$
- $f\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} - 2)$
- $f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1+\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}(\sqrt{3} + 2)$

From this we see that $(-1, 0, 1)$ is max point, $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1-\sqrt{3}}{2}\right)$ is min point. □

Exercise 43. i) D is closed, being defined by large inequalities involving continuous functions. Let's check that D is bounded (hence compact). Denoting by $\rho = \sqrt{x^2 + y^2} = \|(x, y)\|$ we have

$$(x, y) \in D, \implies \rho^2 \leqslant 2\rho\cos\theta - \rho = \rho(2\cos\theta - 1), \iff \rho \leqslant 2\cos\theta - 1 \leqslant 1.$$

Therefore, D is bounded. In particular, D cannot be be open: only $\emptyset, \mathbb{R}^2$ are both open and closed, and $(0, 0) \in D$ (thus $D \neq \emptyset$), and D is bounded, thus $D \subsetneq \mathbb{R}^2$.

ii) The area of D is

$$\lambda_2(D) = \int_D 1\,dxdy = \int_{x^2+y^2 \leqslant 2x - \sqrt{x^2+y^2}} 1\,dxdy \overset{pol\ coords}{=} \int_{\rho \leqslant 2\cos\theta - 1} \rho\,d\rho d\theta.$$

Now, notice that since $\rho \geqslant 0$, this imposes $2\cos\theta - 1 \geqslant 0$, that is $\cos\theta \geqslant \frac{1}{2}$. In one period this means $-\frac{\pi}{3} \leqslant \theta \leqslant \frac{\pi}{3}$. Thus

$$\begin{aligned}\lambda_2(D) &= \int_{\rho\leqslant 2\cos\theta-1,\ -\frac{\pi}{3}\leqslant\theta\leqslant\frac{\pi}{3}} \rho\, d\rho d\theta \overset{RF}{=} \int_{-\pi/3}^{\pi/3}\int_0^{2\cos\theta-1} \rho\, d\rho\, d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta-1)^2\, d\theta \\ &= \frac{1}{2}\left(\frac{2\pi}{3} - 4\int_{-\pi/3}^{\pi/3}\cos\theta\, d\theta + 4\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\, d\theta\right) \\ &= \frac{\pi}{3} - 2\sqrt{3} + 2\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\, d\theta.\end{aligned}$$

About this last integral we have

$$\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\, d\theta = \int_{-\pi/3}^{\pi/3}(\cos\theta)(\sin\theta)'\, d\theta = [\sin\theta\cos\theta]_{\theta=-\pi/3}^{\theta=\pi/3} + \int_0^{2\pi}(\sin\theta)^2\, d\theta = \frac{\sqrt{3}}{2} - \int_{-\pi/3}^{\pi/3}(\cos\theta)^2\, d\theta,$$

from which $\int_{-\pi/3}^{\pi/3}(\cos\theta)^2\, d\theta = \frac{\sqrt{3}}{4}$. We conclude that $\lambda_2(D) = \frac{\pi}{3} - \frac{3\sqrt{3}}{2}$. □

Exercise 44. i) In order $f = u‘ + iv$ be $\mathbb{C}-$differentiable on $\mathbb{C}$, we need u, v $\mathbb{R}-$differentiable on $\mathbb{R}^2$ and fulfilling the CR equations. About u it is clear that, being $\partial_x u, \partial_y u \in \mathscr{C}(\mathbb{R}^2)$, u is $\mathbb{R}-$differentiable on $\mathbb{R}^2$ by the differentiability test. Thus, we look for a v differentiable such that

$$\begin{cases} \partial_x u \equiv \partial_y v, \\ \partial_y u = -\partial_x v, \end{cases} \iff \begin{cases} \partial_x v = -\partial_y u = -(-20x^3y + 20xy^3), \\ \partial_y v = \partial_x u = 5x^4 - 30x^2y^2 + 5y^4. \end{cases}$$

From first equation,

$$v(x, y) = \int 20x^3y - 20xy^3\, dx + k(y) = 5x^4y - 10x^2y^3 + k(y),$$

and plugging this into the second equation we have

$$5x^4 - 30x^2y^2 + k'(y) = 5x^4 - 30x^2y^2 + 5y^4, \iff k'(y) = 5y^4, \iff k(y) = y^5 + k,$$

where k is now a constant. Thus, the v that fulfils CR eqns together with u is

$$v(x, y) = 5x^4y - 10x^2y^3 + 5y^4 + k,$$

and since this is also differentiable (being $\partial_x v, \partial_y v \in \mathscr{C}(\mathbb{R}^2)$), we conclude that $f = u + iv$ is $\mathbb{C}-$differentiable.

ii) We have

$$f = \left(x^5 - 10x^3y^2 + 5xy^4\right) + i\left(5x^4y - 10x^2y^3 + 5y^4 + k\right)$$

Noticed that, for $z = x + iy$,

$$z^5 = (x + iy)^5 = x^5 + i5x^4y - 10x^3y^2 - i10x^2y^3 + 5xy^4 + iy^5$$

thus $f = z^5 + ik$, $k \in \mathbb{R}$. □

Exercise 45. See notes for definitions. We aim to prove that $f^{-1}(S)$ is open if S it is. Suppose this is false. There exists then a point $x \in f^{-1}(S)$ for which

$$\nexists B(x, r] \subset f^{-1}(S).$$

This means that:

$$\forall r > 0,\ B(x,r] \cap f^{-1}(S)^c \neq \varnothing.$$

Taking $r = \frac{1}{n}$

$$\forall n \in \mathbb{N},\ n \geqslant 1,\ \exists x_n \in B(x, 1/n] \cap f^{-1}(S)^c.$$

This means that $\|x_n - x\| \leqslant \frac{1}{n}$, thus $x_n \longrightarrow x$. By continuity, $f(x_n) \longrightarrow f(x)$. Furthermore, by construction of (x_n), we have that $x_n \in f^{-1}(S)^c$, that is $f(x_n) \notin S$ for every n. However, since $f(x) \in S$ (recall that $x \in f^{-1}(S)$), and S is supposed to be open,

$$\exists B(f(x), \rho] \subset S.$$

And since $f(x_n) \longrightarrow f(x)$, we have that

$$\exists N\ :\ f(x_n) \in B(f(x), \rho] \subset S,\ \forall n \geqslant N,$$

which is in contradiction with the construction of (x_n). We deduce that the initial assumption must be false, that is $f^{-1}(S)$ is open. □