* Class of partial recursive functions $R$

Least rich class of functions i.e. least class of functions

- Including the BASIC FUNCTIONS
- Closed under
  1. COMPOSITION
  2. PRIMITIVE RECURSION
  3. UNBOUNDED MINIMALISATION

Theorem: $R = C$

Proof

$(R \subseteq C)$ $C$ is rich

$(C \subseteq R)$

Let $f: \mathbb{N}^k \rightarrow \mathbb{N}$ be a function in $C$

and let $P$ a URM-program for $f$

Define

$$
\begin{align*}
C_p^t &: \mathbb{N}^{k+2} \rightarrow \mathbb{N} \\
C_p^t (\bar{x}, t) &= \text{content of register $R_1$ after $t$ steps of $P(\bar{x})$}
\end{align*}
$$

$$
\begin{align*}
J_p &: \mathbb{N}^{k+1} \rightarrow \mathbb{N} \\
J_p (\bar{x}, t) &= \begin{cases} 
\text{instruction to be executed after $t$ steps of } P(\bar{x}) \\
0 & \text{if } P(\bar{x}) \text{ halts in } t \text{ or fewer steps}
\end{cases}
\end{align*}
$$

Then

$$f(\bar{x}) = C_p^t (\bar{x}, \mu t. J_p (\bar{x}, t))$$

We conclude by proving $C_p^t, J_p \in R$
program P (std form) for f

\[ P_1 \rightarrow I_1 \]
\[ P_2 \rightarrow I_2 \]
\[ \vdots \]
\[ P_n \rightarrow I_n \]

memory

\[
\begin{array}{cccc}
\pi_1 & \pi_2 & \pi_3 & \ldots & \pi_m & 0 & \ldots & 0 \\
\end{array}
\]

\[
C = \prod_{i=1}^{n} \pi_i = \prod_{i=1}^{m} \pi_i \\
\]
\[
\pi_i = (C)_i 
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 & 5 & \ldots & 0 & \ldots & 0 \\
\end{array}
\]

\[
C = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \\
= 2^2 \cdot 3 \cdot 5^2 = 20 
\]

\[
\begin{array}{l}
C_p : \mathbb{N}^{k+1} \rightarrow \mathbb{N} \\
C_p (\bar{z}, t) = \text{content of memory after } t \text{ steps of } P(\bar{z}) \\
\end{array}
\]

\[
\begin{array}{l}
J_p : \mathbb{N}^{k+1} \rightarrow \mathbb{N} \\
J_p (\bar{z}, t) = \begin{cases} 1 & \text{if } P(\bar{z}) \text{ halts in } t \text{ or fewer steps} \\
0 & \text{otherwise} \end{cases} \\
\end{array}
\]

we define \( J_p, C_p \) by primitive recursion

\[
\begin{array}{l}
C_p (\bar{z}, 0) = \prod_{i=1}^{k} \pi_i \\
J_p (\bar{z}, 0) = 1 \\
\end{array}
\]

recursion cases

we define \( C_p (\bar{z}, t+1) \)
\( J_p (\bar{z}, t+1) \)

using

\[
C_p (\bar{z}, t) = C \\
J_p (\bar{z}, t) = J 
\]
\[
C_p(\bar{z}, t+1) = \begin{cases} 
q_t(\rho_m^{(c)m}, c) & \text{if } 1 \leq j \leq E(P) \land I_j = Z(m) \\
p_m \cdot c & \text{if } 1 \leq j \leq E(P) \land I_j = S(m) \\
p_m^{(c)m} \cdot q_t(\rho_m^{(c)m}, c) & \text{if } 1 \leq j \leq E(P) \land I_j = T(m,m) \\
c & \text{otherwise}
\end{cases}
\]

\[
J_p(\bar{z}, t) = \begin{cases} 
\bar{z} + 1 & \text{if } 1 \leq j < E(P) \land I_j = S(m), Z(m), T(m,m) \\
u & \text{if } 1 \leq j \leq E(P) \land I_j = J(m,m,u) \land (c)_m = (c)_m \\
0 & \text{otherwise}
\end{cases}
\]

Hence \( J_p C_p \in \mathbb{R} \)

and thus

\[
f(\bar{z}) = \left( C_p(\bar{z}, \mu t.J_p(\bar{z}, t)) \right)_1
\]

therefore \( f \in \mathbb{R} \)
**Primitive Recursive Functions**

$PR = \text{least class of functions which}
\rightarrow \text{includes the basic functions}
\rightarrow \text{closed under}

1. composition
2. primitive recursion $\iff$ for loop
3. minimisation $\iff$ while loop

$\begin{align*}
PR & \subseteq \mathbb{N} & R \cap \operatorname{Tot} \\
\subseteq & \mathbb{N} & \subseteq \mathbb{N}\\
C_{\forall} & \subseteq \omega & C_{\forall}, \text{while}
\end{align*}$

**Ackermann’s Function**

$\psi : \mathbb{N}^2 \to \mathbb{N}$

$$
\begin{cases}
\psi(0, y) = y + 1 \\
\psi(x + 1, 0) = \psi(x, 1) \\
\psi(x + 1, y + 1) = \psi(x, \psi(x + 1, y))
\end{cases}
$$

$(\mathbb{N}^2, \leq_{\text{lex}})

(x, y) \leq_{\text{lex}} (x', y')$ if

- $x < x'$ or
- $(x = x')$ and $(y \leq y')$

$$(1000, 1000000) \leq_{\text{lex}} (1001, 0)$$

$$(1000, 1000000) >_{\text{lex}} (1000, 0)$$

$f : \mathbb{Z} \to \mathbb{Z}$

$f(z) = \begin{cases}
0 & z \geq 0 \\
f(z - 1) & z < 0
\end{cases}$

$f(-1)$

f(-2)

f(-3)$
* Partially ordered set (poset)

\[(D, \leq)\]

- reflexive \(x \leq x\)
- antisymmetric \(x \leq y \text{ and } y \leq x \Rightarrow x = y\)
- transitive \(x \leq y \text{ and } y \leq z \Rightarrow x \leq z\)

* Well-founded posets

\((D, \leq)\) is well-founded if \(\forall X \in D\) \(X\) has a minimal element

\[D = \{ (\text{pear}, m), (\text{apple}, m) \mid m \in \mathbb{N} \}\]

\[(x, y) \leq (x', y') \text{ if } (x = x') \text{ and } (y \leq y')\]

\(\mathbb{Z}\) well-founded? NO

\(\mathbb{N}\) well-founded? YES

Note: \((D, \leq)\) well-founded if and only if there is no infinite descending chain in \(D\)

\[x \in \{ \text{pear}, \text{apple} \}, y \in \mathbb{N}\]

\[
\text{let } x \in \mathbb{N}^2 \setminus \emptyset \quad x_0 = \min \left\{ x \mid \exists y. (x, y) \in X \right\}
\]

\[
y_0 = \min \left\{ x \mid (x_0, y) \in X \right\}
\]

\[
(x_0, y_0) = \min \ X
\]
\* Induction \hspace{1cm} P(m) \hspace{1cm} m \in \mathbb{N}

\[ P(0) \text{ and assuming } P(m) \text{ you can deduce } P(m+1) \]

\[ \Downarrow \]

P(m) holds for all m

- A binary tree with height m has at most \( 2^{m+1} - 1 \) modes

  \( m = 0 \) \hspace{1cm} \text{number of modes} = 1 \leq 2^{0+1} - 1 = 2 - 1 = 1 \]

  \( m \rightarrow m+1 \)

- Complete induction

  to prove that P(m) holds for all m \in \mathbb{N}

  show

  \[ \text{for all } m, \text{ assuming } P(m') \text{ for all } m' < m \text{ then } P(m) \]

- Well-founded Induction

  \((D, \leq)\) well-founded order

  \[ P(x) \text{ property over } D \]

  if for all \( d \in D \), assuming \( \forall d' < d \ P(d') \)

  \[ \Downarrow \]

  \[ \forall d \in D \ P(d) \]
1. \( \psi \) is total

\[
\forall (x,y) \in \mathbb{N}^2 \quad \psi (x,y) \downarrow
\]

Proceed by well-founded induction on \((\mathbb{N}^2, \leq_{\text{lex}})\)

**Proof**

Let \((x,y) \in \mathbb{N}^2\), assume \(\forall (x',y') \prec_{\text{lex}} (x,y) \quad \psi (x',y') \downarrow\)

We want to show \(\psi (x,y) \downarrow\)

3 cases

- \((x=0)\) \quad \psi (x,y) = \psi (0,y) = y+1 \downarrow

- \((x>0, y=0)\) \quad \psi (x,0) = \psi \left( x - 1, 1 \right) \downarrow

\[
(x - 1, 1) <_{\text{lex}} (x,y) \quad \text{hence} \quad \psi (x - 1, 1) \downarrow \text{ by ind. hyp.}
\]

- \((x>0, y>0)\) \quad \psi (x,y) = \psi (x-1, \varphi (x,y-1)) = \psi (x-1, y) \downarrow

\[
(x,y) <_{\text{lex}} (x,y) \quad \Rightarrow \quad \psi (x,y) \downarrow \text{ by hyp.}
\]

\[
\begin{align*}
(0,0) & \quad (0,1) & \quad (0,2) & \quad \ldots \\
& \quad (1,0) & \quad (1,1) & \quad (1,2) & \quad \ldots \\
& \quad (2,0) & \quad (2,1) & \quad (2,2) & \quad \ldots
\end{align*}
\]

\[
(\mathbb{N}^2, \leq_{\text{lex}})
\]

2. \( \psi \in \mathbb{R} = \mathbb{C} \)

\[
\psi (1,1) = \psi (0, \varphi (1,0)) = \psi (0,2) = 3
\]

\[
\psi (0,1) = \frac{3}{2}
\]

\((1,1,3) \quad (0,2,3) \quad (1,0,2) \quad (0,1,2)\).
valid set of triples: informally

\((x, y, z) \in \mathbb{N}^3\) \quad \rightarrow \quad \exists \ z = \psi(x, y)

\quad \rightarrow \quad S \text{ contains all triples needed to compute } \psi(x, y)

formally \quad S \subseteq \mathbb{N}^3 \quad \text{valid if}

1. \quad (0, y, z) \in S \quad \Rightarrow \quad z = y + 1
2. \quad (x + 1, 0, z) \in S \quad \Rightarrow \quad (x, 1, z) \in S
3. \quad (x + 1, y + 1, z) \in S \quad \Rightarrow \quad \exists u. \quad (x + 1, y, u) \in S \quad \land \quad (x, u, z) \in S

you can prove that \(\psi(x, y, z) \in \mathbb{N}^3\)

\(\psi(x, y) = z \quad \text{iff} \quad \exists S \subseteq \mathbb{N}^2 \quad \text{a valid finite set of triples}

\text{s.t.} \quad (x, y, z) \in S

then

\[
\psi(x, y) = \mu (S, z), \quad \left( S \subseteq \mathbb{N}^3 \quad \text{valid finite set of triples} \right)
\]

\text{encode as a number}

\[
S = \{(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_m, y_m, z_m)\}
\]

\[
\prod \left( \prod (x_i, y_i, z_i), \ldots, \prod (x_0, y_0, z_0) \right)
\]

\[
K_1 \quad \ldots \quad K_m
\]

\[
\prod_{i=1}^{m} p_i^{K_i}
\]

\[\therefore \psi \in \Pi_0 = \mathbb{C} \]
\( \psi \notin \text{PR} \)

\[
\begin{align*}
\text{successor} & \\
\text{x + y} & \quad \text{successor} \\
x + 0 &= x \\
x + (y+1) &= (x+y) + 1 \\
\text{x \times y} & \\
x \times 0 &= 0 \\
x \times (y+1) &= (x \times y) + x \\
\text{x}^y & \\
x^0 &= 1 \\
x^{y+1} &= (x^y) \times x \\
\end{align*}
\]

\text{nesting primitive recursion}

\[
\begin{align*}
\psi(0, y) &= y + 1 \\
\psi(x+1, 0) &= \psi(x, 1) \\
\psi(x+1, y+1) &= \psi(x, \psi(x+1, y))
\end{align*}
\]

\text{consider \( x \) as a "fixed" parameter}

\[
\psi(x, y) = \psi_x(y)
\]

\[
\begin{align*}
\psi_{x+1}(y) &= \psi_x(\psi_{x+2}(y-1)) \\
&= \psi_x(\psi_x(\psi_{x+1}(y-2))) \\
&= \ldots \\
&= \psi_x \psi_x \ldots \psi_x \psi_{x+1}(0) \\
&= \psi_{x+1}(1)
\end{align*}
\]

\text{roughly: increasing \( x \) to \( x+1 \) requires iterating the function \( \psi_x \) which increases the number of nested primitive recursion}
the full function would require infinitely many
nested primitive recursions

Some more ideas...

concretely:

\[ \psi_0(y) = y + 1 \]
\[ \psi_1(y) = \psi_0^{y+1}(1) = y + 2 \]
\[ \psi_2(y) = \psi_2^{y+1}(1) = 2(y+1) + 1 = 2y + 3 \approx 2y \]
\[ \psi_3(y) = \psi_2^{y+1}(1) \approx 2^{2^{2^y}} \]
\[ \psi_4(y) = \psi_3^{y+1}(1) \approx 2^{2^{2^{2^y}}} \]

\[ \psi_0(1) = 2 \]
\[ \psi_1(1) = 5 \]
\[ \psi_2(1) = 13 \]
\[ \psi_3(1) \approx 2^{16} \]
\[ \psi_4(2) \approx 2^{2^{16}} \approx 10^{400} \]

ONE CAN PROVE: Given a function \( f : \mathbb{N} \rightarrow \mathbb{N} \in \mathbb{PR} \) and a program \( P \) computing \( f \) using only "for-loops" (primitive recursion)
if \( J \) is the maximum level of nesting of for-loops

\[ f(\infty) < \psi_{J+1}(\max\{x,y\}) \]

Now, assume \( \psi \in \mathbb{PR} \), let \( J \) be the level of nesting of for-loops (of primitive recursive defs)
for computing \( \psi \)

\[ \psi(x,y) < \psi_{J+1}(\max\{x,y\}) \]
Let \( x = y = j + 1 \)

\[
\psi(j+1, j+1) < \psi_{j+1}(j+1) = \psi(j+1, j+1)
\]

\( \text{contradiction} \)

\[ \Rightarrow \psi \not\in \mathbb{R} \]