giovedì 19 ottobre 2023 12:29

Elliptic curves have dim 1

In dimension > 1 we have abelian variety

An abelian variety over $\mathbb{C}$ a complete connected variety endowed with an operation

$$+ : A \times A \longrightarrow A$$
$$(P_1, P_2) \longmapsto P_1 + P_2$$

an inverse

$$- : A \longrightarrow A$$
$$P \longmapsto -P$$

a neutral object 0 such that $(A(\mathbb{C}), +, -, 0)$ is a commutative group

Consider a rank $2g$ lattice Λ in $\mathbb{C}^g$, that is

$$\Lambda = \gamma_1 \mathbb{Z} + \gamma_2 \mathbb{Z} + \_\_ + \gamma_{2g} \mathbb{Z}$$

with $\gamma_1, , \gamma_{2g} \in \mathbb{C}$ LI

(for elliptic curves $g = 1$)

A theta function on $\mathbb{C}^g$ is an holomorphic function

$$\Theta : \mathbb{C}^g \longrightarrow \mathbb{C}$$

which satisfy a function equation

$$\frac{\Theta(z+\lambda)}{\Theta(z)} = \alpha(\lambda)\, e^{\pi H(z,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}$$

where $H(\_,\_)$ is an hermitian form on $\mathbb{C}^g \times \mathbb{C}^g$ satisfying $\mathrm{Im}(H(\lambda,\lambda')) \in \mathbb{Z}$ $\forall \lambda, \lambda' \in \Lambda$

and $\alpha : \Lambda \longrightarrow \{ \xi \in \mathbb{C} \mid |\xi| = 1 \}$ satisfying

$\alpha(\lambda + \lambda') = \alpha(\lambda)\alpha(\lambda') e^{i\pi \mathrm{Im} H(\lambda, \lambda')}$

It exists d theta functions such that

$$\begin{aligned} \exp_A : \mathbb{C}^g &\longrightarrow A(\mathbb{C}) \subseteq \mathbb{P}^d \\ z &\longmapsto [\theta_0(z) : \theta_1(z) : \dots : \theta_d(z)] \end{aligned}$$

whose kernel is Λ. In particular

$$A(\mathbb{C}) \cong \mathbb{C}^g / \Lambda$$

$$\begin{aligned} \exp_{\mathcal{E}} : \mathbb{C} &\longrightarrow \mathcal{E}(\mathbb{C}) \subseteq \mathbb{P}^2 \\ z &\longmapsto [\wp(z) : \wp'(z) : 1] \end{aligned}$$

The g-dimensional $\mathbb{C}$ vector space of differential forms of <u>first kind</u> is generated by $\omega_1, \dots, \omega_g$ such that

$$\exp_A^*(\omega_i) = dz_i \quad \text{for } i = 1, \dots, g.$$

The g-dimensional $\mathbb{C}$ vector space of differential forms of <u>second kind</u> is generated by $\eta_1, \dots, \eta_g$ such that

$$\exp_A^*(\eta_i) = dh_i(z)$$

where $h_i(z) = \frac{d}{dz} \log(\theta(z)) = \frac{1}{\theta(z)} \frac{d}{dz_i} \theta(z)$

with $\theta(z)$ a theta function on $\mathbb{C}^g$

The De Rham realization of A, $T_{dR}(A)$ is the $2g$-dimensional $\mathbb{C}$ vector space generated by $\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g$

The Hodge realization of A, $T_H(A)$ is the $2g$-dimensional $\mathbb{Q}$-vector space generated by Λ : $T_H(A) = \Lambda \otimes \mathbb{Q}$

The infinite dimensional $\mathbb{C}$-vector space of differential forms of the third kind is generated by

$$F_a(z) = \frac{\theta(z+a)}{\theta(z)\,\theta(a)} \, e^{-\sum_{i=1}^{\infty} h_i(a) z_i} \qquad \forall a \in \mathbb{C}^g \setminus \Lambda$$

The ℓ-adic realization of A, $T_\ell(A)$, $\varprojlim_{n} A[\ell^n]$

Since $A(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$, $A[\ell] = (\mathbb{C}^g/\Lambda)$

$$= \mathbb{Z}/\ell\mathbb{Z} \times \dots \times \mathbb{Z}/\ell\mathbb{Z}$$

Elliptic curve	Abelian Variety
1	g
$\Lambda = \alpha\mathbb{Z} + \beta\mathbb{Z}$	$\Lambda = \gamma_1\mathbb{Z} + \dots + \gamma_{2g}\mathbb{Z}$
$\mathcal{E}(\mathbb{C}) = \mathbb{C}/\Lambda$	$A(\mathbb{C}) = \mathbb{C}^g/\Lambda$
$\omega = \frac{dx}{y} = dz$	$\omega_1 = dz_1, \dots \omega_g = dz_g$
$\eta = -x\frac{dx}{y} = d\zeta(z)$	$\eta_1, \dots, \eta_g$ with $\eta_i = dh_i(z)$

$$\eta_i = d h_i(z)$$

$$\exp_{\mathcal{E}}(z) = [\wp(z) : \wp'(z) : 1] \qquad \exp_A(z) = [\theta_0(z) : \dots : \theta_d(z)]$$

$$\zeta(z) \qquad h_1(z), \dots, h_g(z)$$

$$h_i(z) = \frac{d}{dz_i} \log \theta(z)$$

$$\sigma(z) \qquad \theta(z)$$

$$f_a(z) = \frac{\sigma(z+a)}{\sigma(z)\,\sigma(a)}\, e^{-\zeta(a) z} \qquad F_q(z) = \frac{\theta(z+q)}{\theta(z)\,\theta(q)}\, e^{-\sum_{i=1}^{g} h_i(q) z_i}$$

$$\begin{tikzcd} \log_{\mathcal{E}} : \mathcal{E}(\mathbb{C}) \arrow[r] & \mathbb{C} \arrow[l, bend right, "\exp_{\mathcal{E}}"'] \end{tikzcd} \qquad \begin{tikzcd} \log_A : A(\mathbb{C}) \arrow[r] & \mathbb{C}^g \arrow[l, bend right, "\exp_A"'] \end{tikzcd}$$

$$P \longrightarrow \int_0^P \omega \qquad P \longmapsto \left(\int_0^P \omega_1, \dots, \int_0^P \omega_g \right)$$

Grothendieck period conjecture

$A / \overline{\mathbb{Q}}$ then

$$\underbrace{\operatorname{trdeg}_{\mathbb{Q}} \overline{\mathbb{Q}}(\text{periods}(A))}_{\text{OK}} = \underbrace{\dim_{\mathbb{Q}} G_{mot}(A)}_{?}$$

periods of A are the entries of the matrix which represent the isomorphism between $T_{dR}(A) \simeq \mathrm{Hom}(T_H(A), \mathbb{C})$

$$\omega \longmapsto \left[\gamma \mapsto \int_\gamma \omega \right]$$

$$\mathcal{E}: \quad \begin{pmatrix} \int_\gamma \omega & \int_\gamma \eta \\ \int_\beta \omega & \int_\beta \eta \end{pmatrix} = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} \qquad 2\times 2$$

$$A: \quad 2g\times 2g \quad \begin{pmatrix} \int_{\gamma_1}\omega_1 & \int_{\gamma_1}\omega_2 & \dots & \int_{\gamma_1}\omega_g & \int_{\gamma_1}\eta_1 & \dots & \int_{\gamma_1}\eta_g \\ \vdots \\ \int_{\gamma_{2g}}\omega_1 & \int_{\gamma_{2g}}\omega_2 & & \int_{\gamma_{2g}}\omega_g & \int_{\gamma_{2g}}\eta_1 & & \int_{\gamma_{2g}}\eta_g \end{pmatrix}$$

$$\Pi = [\mathbb{Z} \xrightarrow{u} \mathcal{E}] \qquad u(1) = P \in \mathcal{E}(\mathbb{C}) \qquad p = \int_0^P \omega = \log_{\mathcal{E}}(P) \in \mathbb{C}$$

$$\begin{pmatrix} 1 & \int_0^P \omega & \int_0^P \eta \\ 0 & \omega_1 & \eta_1 \\ 0 & \omega_2 & \eta_2 \end{pmatrix} = \begin{pmatrix} 1 & p & \zeta(p) \\ 0 & \omega_1 & \eta_1 \\ 0 & \omega_2 & \eta_2 \end{pmatrix}$$

$$\Pi = [\mathbb{Z} \xrightarrow{u} A] \qquad u(1) = P \in A(\mathbb{C})$$

$$p = \left(\int_0^P \omega_1, \dots \int_0^P \omega_g\right) \in \mathbb{C}^g$$

$$\int_0^P \eta_1, \text{—}, \int_0^P \eta_g \qquad \zeta(p)$$

$$\begin{pmatrix} 1 & \int_0^P \omega_1 & \int_0^P \omega_2 & \dots & \int_0^P \eta_1 & & \int_0^P \eta_g \\ & \int_{\gamma_1}\omega_1 & \int_{\gamma_1}\omega_2 & & \int_{\gamma_1}\eta_1 & \dots & \int_{\gamma_1}\eta_g \end{pmatrix}$$

$$
\begin{pmatrix}
\int_{\gamma_1} \omega_1 & \int_{\gamma_1} \omega_2 & \int_{\gamma_1} \eta_1 & \cdots & \int_{\gamma_1} \eta_g \\
\vdots & & & & \\
\int_{2g} \omega_1 & \int_{2g} \omega_2 & \int_{\gamma_{2g}} \eta_1 & & \int_{\gamma_{2g}} \eta_g
\end{pmatrix}
$$

$$
\int_0^P \eta_1 = \int_0^P d h_1(z) = \int_0^P \left(\frac{1}{\theta(z)} \frac{d}{dz_1} \theta(z) \right)
$$