Knowledge Representation and Learning
Final exam

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Exercise 1 (6 points) *Consider the following formulaiton of the "Wolf, goat and cabbage problem": A farmer with a wolf, a goat, and a cabbage must cross a river by boat. The boat can carry only the farmer and a single item. If left unattended together, the wolf would eat the goat, or the goat would eat the cabbage. How can they cross the river without anything being eaten?*

1. *Suggest a minimal set of propositinal variables with the relative intutive meaning that can be used to specify the puzzle;*
2. *use the propositional variables defined before to describe the initial situation, some intermediate situation, and the final situation of the puzzle;*
3. *with the defined propositional variables write a set of formulas Γ that are satisfied by the safe situations and not satisfied by the unsafe ones.*

Solution

1. We consider the following set of propositional variable with the associated intuitive meaning:

$$\begin{aligned} W &: \text{The wolf is on the west side of the river} \\ G &: \text{The goat is on the west side of the river} \\ C &: \text{The cabbage is on the west side of the river} \\ F &: \text{The farmer is on the west side of the river} \end{aligned}$$

   With this interpretation and with the assumption that the river has two sides, namely, a west side and an east side, we have that $\neg W$ means that the walf is on the east side of the river, and similar interpretations can be given to $\neg G$, $\neg C$ and $\neg F$.

2. In the initial situation all the participants are one the same side of the river. Without loss of generality we can assume that they are on the west side. Therefore the initial state can be formalized by the formula

$$W \wedge G \wedge C \wedge F$$

   An intermediate situation could be any situation where there are some items on one part and some on the other, E.g.,

$$W \wedge \neg G \wedge \neg C \wedge F$$

   denotes the situation in which the walf and the farmer are on the west side, and the goat and the cabbage are on the east side of the river. The final situation is when everybody is on the east side. Therefore the final situation is represented by the formula

$$\neg W \wedge \neg G \wedge \neg C \wedge \neg F$$

3. a dangerous situation happens when there is the wolf and the goat on the same side and the farmer is not attending because it is on the other side. Similarly a dangerous situation is when the cabbage and the goat are on the same side and the farmer is on the other side Therefore we can use the following formulas to denote the two types of dangerous situations

$$(W \leftrightarrow G) \wedge (G \leftrightarrow \neg F) \tag{1}$$

$$(C \leftrightarrow G) \wedge (G \leftrightarrow \neg F) \tag{2}$$

The first formulas states that a dangerous situation happens if the walf and the goat are on the same side ($W \leftrightarrow G$) and the farmer is on the other side ($G \leftrightarrow \neg F$). A similar intuitive meaning can be associated to the other formula. Since a situation is dangerous if either one or both the above formulas are true, the total set of dangerous situation is formalized by the disjunction of the tow formulas. To check if we have done the correct encoding consider the truth table of the disjunciton of the two formulas:

C	F	G	W	((W	↔	G)	∧	(G	↔	¬	F))	∨	((C	↔	G)	∧	(G	↔	¬	F))
T	T	T	T	T	T	T	F	T	F	F	T	F	T	T	T	F	T	F	F	T
T	T	T	F	F	F	T	F	T	F	F	T	F	T	T	T	F	T	F	F	T
T	T	F	T	T	F	F	F	F	T	F	T	F	T	F	F	F	F	T	F	T
T	T	F	F	F	T	F	T	F	T	F	T	T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T	T	T	F	T	T	T	T	T	T	T	T	F
T	F	T	F	F	F	T	F	T	T	T	F	T	T	T	T	T	T	T	T	F
T	F	F	T	T	F	F	F	F	F	T	F	F	T	F	F	F	F	F	T	F
T	F	F	F	F	T	F	F	F	F	T	F	F	T	F	F	F	F	F	T	F
F	T	T	T	T	T	T	F	T	F	F	T	F	F	F	T	F	T	F	F	T
F	T	T	F	F	F	T	F	T	F	F	T	F	F	F	T	F	T	F	F	T
F	T	F	T	T	F	F	F	F	T	F	T	T	F	T	F	T	F	T	F	T
F	T	F	F	F	T	F	T	F	T	F	T	T	F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T	T	T	T	F	T	F	F	T	F	T	T	T	F
F	F	T	F	F	F	T	F	T	T	T	F	F	F	F	T	F	T	T	T	F
F	F	F	T	T	F	F	F	F	F	T	F	F	F	T	F	F	F	F	T	F
F	F	F	F	F	T	F	F	F	F	T	F	F	F	T	F	F	F	F	T	F

From the truth table one can see that the dangerous situation are those in which the walf and the goat are not on the same side of the farmer, and those in which that cabbage and the goat are not on the same side of the farmer.

The safe situations are those which are not dangerous. THerefore they are formalized by the negation of the dangerous formula i.e.,

$$\neg((W \leftrightarrow G) \wedge (G \leftrightarrow \neg F)) \wedge \neg((C \leftrightarrow G) \wedge (G \leftrightarrow \neg F))$$

$\square$

Exercise 2 (3 points) *Provide the Tseitin's Transformation of the following formula:*

$$(A \to B) \vee (C \vee \neg D) \to ((A \to B) \to (C \wedge \neg D))$$

Solution In the tseitin's tranformation we introduce one propositinal variable for every subformula that is not a literal. The formula of the exercise has the following subformulas

$$
\begin{aligned}
&(A \to B) \vee (C \vee \neg D) \to ((A \to B) \to (C \wedge \neg D)) \\
&(A \to B) \vee (C \vee \neg D) \\
&(A \to B) \\
&(C \vee \neg D) \\
&((A \to B) \to (C \wedge \neg D)) \\
&(C \wedge \neg D))
\end{aligned}
$$

We therefore introduce 6 propositional variables $x_1, \ldots, x_6$, which are associated to the following subformulas

$$
\begin{aligned}
&x_1 : (A \to B) \vee (C \vee \neg D) \to ((A \to B) \to (C \wedge \neg D)) \\
&x_2 : (A \to B) \vee (C \vee \neg D) \\
&x_3 : (A \to B) \\
&x_4 : (C \vee \neg D) \\
&x_5 : ((A \to B) \to (C \wedge \neg D)) \\
&x_6 : (C \wedge \neg D))
\end{aligned}
$$

We then add the following interdependencies that formalizes the structure of the formula

$$
\begin{aligned}
&x_1 \leftrightarrow (x_2 \to x_3) \\
&x_2 \leftrightarrow (x_4 \vee x_5) \\
&x_4 \leftrightarrow (A \to B) \\
&x_5 \leftrightarrow (C \vee \neg D) \\
&x_3 \leftrightarrow (x_4 \to x_6) \\
&x_6 \leftrightarrow (C \wedge \neg D)
\end{aligned}
$$

Their transformation in CNF is the following:

$$
\begin{aligned}
x_1 \leftrightarrow (x_2 \to x_3) &: \{\neg x_1 \vee \neg x_2 \vee x_3, x_2 \vee x_1, \neg x_3 \vee x_1\} \\
x_2 \leftrightarrow (x_4 \vee x_5) &: \{\neg x_2 \vee x_4 \vee x_5, \neg x_4 \vee x_2, \neg x_4 \vee x_2\} \\
x_4 \leftrightarrow (A \to B) &: \{\neg x_4 \vee \neg A \vee B, A \vee x_4, \neg B \vee x_4\} \\
x_5 \leftrightarrow (C \vee \neg D) &: \{\neg x_5 \vee C \vee \neg D, \neg C \vee x_5, D \vee x_5\} \\
x_3 \leftrightarrow (x_4 \to x_6) &: \{\neg x_3 \vee \neg x_4 \vee x_6, x_4 \vee x_3, \neg x_6 \vee x_3\} \\
x_6 \leftrightarrow (C \wedge \neg D) &: \{\neg x_6 \vee C, \neg x_6 \vee \neg D, \neg V \vee D \vee x_6\}
\end{aligned}
$$

□

Exercise 3 (5 points) *Let $\mathcal{P}$ be a set of n propositinal variables and let $w : 2^{\mathcal{P}} \to \mathbb{R}$ be a generic weight function. Find a method that extracts a (possibly small) set of weighted clauses F such that w_F is equivalent to w.*

Solution Let $\mathcal{I}_1 \ldots, \mathcal{I}_n$ with $n = 2^{|\mathcal{P}|}$ be the ordering of the interpretations of $\mathcal{P}$ such that $w(\mathcal{I}_i) \leq w(\mathcal{I}_{i+1})$ for every $1 \leq i < n$. For every $i = 1 \ldots, n$ we construct a set of clauses $\mathcal{C}_i$ that are satisfied only by the first i-th interpretations. We do this by induction starting from $i = 1$

$$\mathcal{C}_1 = \{\{l\} \mid \mathcal{I}_1(l) = 1, l \in Lit(\mathcal{P})\}$$

Notice that the only interpretation that satisfies all the clauses in $\mathcal{C}_1$ is $\mathcal{I}_1$. Therefore we have that $\mathcal{I} \models \mathcal{C}_1$ if and only if $\mathcal{I} = \mathcal{I}_1$. Now suppose that $\mathcal{C}_i$ is a set of clauses such that $\mathcal{I} \models \mathcal{C}_i$ if and only if $\mathcal{I} = \mathcal{I}_1$ or $\mathcal{I} = \mathcal{I}_2$ or $\ldots \mathcal{I} = \mathcal{I}_i$, we can build $\mathcal{C}_{i+}$ by computing the set of clauses of the formula

$$\mathcal{C}_i \vee \bigwedge_{\substack{l \in Lit(\mathcal{P}) \\ \mathcal{I}_{i+1} \models l}} l$$

which is equal to the set of clauses:

$$\mathcal{C}_{i+1} = \{C \cup \{l\} \mid C \in \mathcal{C}_i,\ \mathcal{I}_{i+1} \models l,\ l \in Lit(\mathcal{P})\}$$

Let us associate to every $C \in \mathcal{C}_i$ a weight equal to -1 In this way we have that

$$\begin{aligned}
w(\mathcal{I}_n) &= -|\mathcal{C}_n| \\
w(\mathcal{I}_{n-1}) &= -|\mathcal{C}_n| - |C_n \setminus C_{n-1}| = |C_n \cup C_{n-1}| \\
\ldots & \\
w(\mathcal{I}_i) &= -|\mathcal{C}_n \cup \mathcal{C}_{n-1} \cup \cdots \cup \mathcal{C}_i| \\
\ldots & \\
w(\mathcal{I}_1) &= -|\mathcal{C}_n \cup \cdots \cup \mathcal{C}_1|
\end{aligned}$$

Notice that every $\mathcal{C}_i$ cannot be equal to $\mathcal{C}_j$ for $i \neq j$ otherwise they will be satisfied by the same set of models. This means that $|\mathcal{C}_n \cup \cdots \cup \mathcal{C}_i| < |\mathcal{C}_n \cup \cdots \cup \mathcal{C}_i \cup \mathcal{C}_{i+1}|$ from which we have that

$$w(\mathcal{I}_1) < w(\mathcal{I}_2) < \cdots < w(\mathcal{I}_n)$$

Let us see how this work for a simple case Suppose we have the following sequence of models, on the three propositions p, q, r

$$\begin{aligned}
\mathcal{I}_1 &= \{p, q, \neg r\} \\
\mathcal{I}_2 &= \{\neg p, \neg q, r\} \\
\mathcal{I}_3 &= \{p, \neg q, \neg r\} \\
\mathcal{I}_4 &= \{\neg p, q, \neg r\} \\
\mathcal{I}_5 &= \{p, q, r\} \\
\mathcal{I}_6 &= \{p, \neg q, r\} \\
\mathcal{I}_7 &= \{\neg p, \neg q, \neg r\} \\
\mathcal{I}_8 &= \{\neg p, q, r\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_1 &= \{\{p\}, \{q\}, \{\neg r\}\} \\
\mathcal{C}_2 &= \left\{ \begin{array}{c} \{p, \neg p\}, \{\neg p, q\}, \{\neg p, \neg r\}, \\ \{p, \neg q\}, \{q, \neg q\}, \{\neg q, \neg r\}, \\ \{p, r\}, \{q, r\}, \{\neg r, r\}\} \end{array} \right\}
\end{aligned}$$

Notice that the clauses that contains a literal and the its opposite are satisfied by all interpretation therefore they don't affect the ordering, therefore they can be removed. This implies that

$$\mathcal{C}_2 = \left\{ \begin{array}{c} \{\neg p, q\}, \{\neg p, \neg r\}, \\ \{p, \neg q\}, \{\neg q, \neg r\}, \\ \{p, r\}, \{q, r\} \end{array} \right\}$$

Then we proceed with the others by immediately simplifying the set of clauses by removing the valid clauses.

$$
\begin{aligned}
\mathcal{C}_3 &= \left\{ \begin{array}{c} \{p, \neg q\}, \{p, \neg q, \neg r\}, \\ \{p, r\}, \{p, q, r\}, \{\neg p, \neg q, \neg r\}, \\ \{\neg q, \neg r\}, \{p, \neg q, r\}, \\ \{\neg p, q, \neg r\}, \{\neg p, \neg r\}, \end{array} \right\} \\
\mathcal{C}_4 &= \left\{ \begin{array}{c} \{\neg p, \neg q, \neg r\}, \{p, q, r\}, \\ \{p, \neg q, \neg r\}, \\ \{\neg q, \neg r\}, \{\neg p, \neg r\} \end{array} \right\} \\
\mathcal{C}_5 &= \left\{ \begin{array}{c} \{p, q, r\}, \{p, \neg q, \neg r\}, \\ \{\neg p, q \neg r\} \end{array} \right\} \\
\mathcal{C}_6 &= \left\{ \ \{p, q, r\}, \{p, \neg q, \neg r\}, \ \right\} \\
\mathcal{C}_7 &= \left\{ \ \{p, \neg q, \neg r\} \ \right\} \\
\mathcal{C}_8 &= \{\}
\end{aligned}
$$

□

Exercise 4 (4 points) *Find the computational circuit for the weight model counting of the formula*

$$(A \vee B) \wedge (\neg C \rightarrow A)$$

for the generic weight function w_A, $w_{\neg A}$, w_B, $w_{\neg B}$, w_C, *and* $w_{\neg C}$.

Solution To find the computational tree of a formula, we first have to transform it in sd-DNNF (smooth deterministic decomposable negated normal form). We first rewrite $\rightarrow$ with $\neg$ and $\vee$

$$(A \vee B) \wedge (\neg\neg C \vee A)$$

Then we transform it in NNF.

$$(A \vee B) \wedge (C \vee A)$$

Since the two conjuncts contain a common variable, namely A, the formula is not decomposable (D). Therefore we have to apply Shannon's expansion on the common variable A. After the expansion we obtain the formula:

$$(A \wedge (\top \vee B) \wedge (C \vee \top)) \vee (\neg A \wedge (\bot \vee B) \wedge (C \vee \bot))$$

which is equivalent to:

$$A \vee (\neg A \wedge B \wedge C)$$

We have obtained a formula in d-DNNF, which is not smooth (s), since the two disjunct do not contains the same set of propositional variables. To smoothen it, we have to add $(p \vee \neg p)$ for every missing propositional variable in a dijunct. We thaerefore obtain the formula

$$(A \wedge (B \vee \neg B) \wedge (C \vee \neg C)) \vee (\neg A \wedge B \wedge C)$$

The formula is now in sd-DNNF and can be transformed in the computational circuit by replacing each literal with the corresponding weight $\wedge$ with the product and $\vee$ with sum. We obtain:

$$(w_A \cdot (w_B + w_{\neg B}) \cdot (w_C + w_{\neg C})) + (w_{\neg A} \cdot w_B \cdot w_C)$$

□

Exercise 5 (4 point) *For each of the following formula say if it is valid, or non valid. If it is not valid provide a counter-model (i.e., a model in which they are false)*

1. $(\forall x A(x) \to \forall y B(y)) \to \forall x(A(x) \to B(x))$
2. $\forall x \exists y(A(x) \wedge B(y)) \to \exists y \forall x(A(x) \wedge B(y))$ *p*

Solution

1. The formula $(\forall x A(x) \to \forall y B(y)) \to \forall x(A(x) \to B(x))$ is not valid. Ineed we can build a counter model where the premise of the implication $\forall x A(x) \to \forall y B(y)$ is true and the consquence $\forall x(A(x) \to B(x))$ is false. To make $\forall x(A(x) \to B(x))$ false, it is enough to have an element of the domain that is in $\mathcal{I}(A)$ but not in $\mathcal{I}(B)$, Suppose that the interpretation domain is $\{1,2\}$ then we can impose $1 \in \mathcal{I}(A)$ and $1 \notin \mathcal{I}(B)$ In this way we have that $\mathcal{I} \models A(x)[x \leftarrow 1]$ and $\mathcal{I} \not\models B(x)[x \leftarrow 1]$, and therefore $\mathcal{I} \not\models A(x) \to B(x)[x \leftarrow 1]$, and finally $\mathcal{I} \not\models \forall x(A(x) \to B(x))$. To make $\forall x A(x) \to \forall y B(y)$ true we have two possibilities, either to make $\forall y B(y)$ true or to make $\forall x A(x)$ false. The first obtion is not viable, since we have that $B(1)$ is false. So let's take the second option and have $\forall x A(x)$ false. This can be done by making $A(2)$ false. In summary, we have the following interpretation $\mathcal{I}$ defined as follows:

$$\begin{aligned}\Delta^{\mathcal{I}} &= \{1,2\}\\ \mathcal{I}(A) &= \{1\}\\ \mathcal{I}(B) &= \emptyset\end{aligned}$$

   We have that $\mathcal{I} \models \forall x A(x) \to \forall y B(y)$ since $\mathcal{I} \not\models \forall x A(x)$, and that $\mathcal{I} \not\models \forall x(A(x) \to; B(x))$ since $\mathcal{I} \not\models A(x) \to B(x)[x \leftarrow 1]$, and therefore $\mathcal{I} \not\models \forall x(A(x) \to B(x))$.

2. $\forall x \exists y(A(x) \wedge B(y)) \to \exists y \forall x(A(x) \wedge B(y))$ instead is valid. This can be proved directly or by using the following equivalences

$$\forall x(A \wedge B(x)) \leftrightarrow A \wedge \forall x B(x) \tag{3}$$
$$\exists x(A \wedge B(x)) \leftrightarrow A \wedge \exists x B(x) \tag{4}$$
$$\tag{5}$$

   when x does not occour free in A. Let us prove by using the above equivalence.

$$\begin{aligned}
&\forall x \exists y(A(x) \wedge B(y)) && \\
&\forall x(A(x) \wedge \exists y B(y)) && \text{by (4) since } y \text{ is not free in } A(x)\\
&\forall x A(x) \wedge \exists y B(y) && \text{by (3) since } x \text{ is not free in } \exists y B(y)\\
&\exists y(\forall x(A(x) \wedge B(y)) && \text{by (4) since } y \text{ is not free in } \forall x A(x)\\
&\exists y \forall x(A(x) \wedge B(y)) && \text{by (3) since } x \text{ is not free in } B(y)
\end{aligned}$$

□

Exercise 6 (4 points) *Transform the following formula in prenex Skolemized conjunctive normal form:*

$$\forall x \forall y(R(x,y) \leftrightarrow \exists z(R(x,z) \wedge R(z,y))$$

Solution Let us first translate in prenex normal form, by rewriting $\leftrightarrow$ in terms of $\vee$ and $\neg$

$$\begin{aligned}
\forall x \forall y(\neg R(x,y) \vee \exists z(R(x,z) \wedge R(z,y)) \wedge \\
\forall x \forall y(\neg \exists z(R(x,z) \wedge R(z,y)) \vee R(x,y))
\end{aligned}$$

Then we push the $\neg$ close to the atoms, obtaining

$$\begin{aligned}
\forall x \forall y(\neg R(x,y) \vee \exists z(R(x,z) \wedge R(z,y)) \wedge \\
\forall x \forall y \forall z(\neg R(x,z) \vee \neg R(z,y) \vee R(x,y))
\end{aligned}$$

We remove the $\exists z$ quantifier by scolemization with $f(x,y)$ obtaining

$$\begin{aligned}
\forall x \forall y(\neg R(x,y) \vee (R(x,f(x,y)) \wedge R(f(x,y),y)) \wedge \\
\forall x \forall y \forall z(\neg R(x,z) \vee \neg R(z,y) \vee R(x,y))
\end{aligned}$$

Not that the formula is in prenex normal form we can transform it in clausal form by removing the quantifiers, and expanding in clauses. We obtain the following three clauses:

$$\begin{aligned}
\{\neg R(x,y), R(x,f(x,y))\} \\
\{\neg R(x,y), R(f(x,y),y))\} \\
\{\neg R(x,z), \neg R(z,y), R(x,y)\}
\end{aligned}$$

$\square$

Exercise 7 (4 points) *Use the formula for first order model counting of universally quantified formulas to count the models of*

$$\forall xy((A(x) \leftrightarrow A(y)) \rightarrow R(x,y))$$

Solution The set of 1-types are

$$\begin{aligned}
1(x) &= A(x) \wedge R(x,x) \\
2(x) &= A(x) \wedge \neg R(x,x) \\
3(x) &= \neg A(x) \wedge R(x,x) \\
4(x) &= \neg A(x) \wedge \neg R(x,x)
\end{aligned}$$

The 2-tables are:

$$\begin{aligned}
1(x,y) &= R(x,y) \wedge R(y,x) \\
2(x,y) &= R(x,y) \wedge \neg R(y,x) \\
3(x,y) &= \neg R(x,y) \wedge R(y,x) \\
4(x,y) &= \neg R(x,y) \wedge \neg R(y,x)
\end{aligned}$$

Let us compute $\phi(x,x) \wedge \phi(y,y) \wedge \phi(x,y) \wedge \phi(y,x)$

$$\begin{aligned}
(A(x) \leftrightarrow A(x)) \rightarrow R(x,x) \wedge \\
(A(y) \leftrightarrow A(y)) \rightarrow R(y,y) \wedge \\
(A(x) \leftrightarrow A(y)) \rightarrow R(x,y) \wedge \\
(A(y) \leftrightarrow A(x)) \rightarrow R(y,x)
\end{aligned}$$

which is equivalent to

$$\phi(x,y) = R(x,x) \wedge R(y,y) \wedge ((A(x) \leftrightarrow A(y)) \rightarrow R(x,y) \wedge R(y,x))$$

Let us now compute the values of n_{ij} for the pairs of 1-types i and j with $i \leq j$

i	j	l	$\phi(x,y)$	n_{ij}
$1(x)$	$1(y)$	$1(x,y)$	1	$n_{11}=1$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	
$1(x)$	$2(y)$	$1(x,y)$	0	$n_{12}=0$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	
$1(x)$	$3(y)$	$1(x,y)$	1	$n_{13}=4$
		$2(x,y)$	1	
		$3(x,y)$	1	
		$4(x,y)$	1	
$1(x)$	$4(y)$	$1(x,y)$	0	$n_{14}=0$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	
$2(x)$	$2(y)$	$1(x,y)$	0	$n_{22}=0$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	

i	j	l	$\phi(x,y)$	n_{ij}
$2(x)$	$3(y)$	$1(x,y)$	0	$n_{23}=0$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	
$2(x)$	$4(y)$	$1(x,y)$	0	$n_{24}=0$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	
$3(x)$	$3(y)$	$1(x,y)$	1	$n_{33}=1$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	
$3(x)$	$4(y)$	$1(x,y)$	0	$n_{34}=0$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	
$4(x)$	$4(y)$	$1(x,y)$	0	$n_{44}=0$
		$2(x,y)$	0	
		$3(x,y)$	0	
		$4(x,y)$	0	

The general formula for the model counting of universally quantified formula with four 1-types is the following:

$$\sum_{k_1+k_2+k_3+k_4=n} \binom{n}{k_1,k_2,k_3,k_4} \prod_{1\leq i\leq j\leq 4} n_{ij}^{\boldsymbol{k}(i,j)} \tag{6}$$

where $\boldsymbol{k}(i,j) = k_i k_j$ if $i \neq j$ and $\frac{k_i(k_j-1)}{2}$ is $i=j$. $\square$ Since $n_{ij}=0$ when i and j are 2 or 4, we have that n_{ij} with $i=2,4$ or $j=2,4$ must be equal to 0. Therefore the general formula becomes:

$$\sum_{k_1+k_3=n} \binom{n}{k_1} n_{11}^{\frac{k_1(k_1-1)}{2}} n_{13}^{k_1 k_3} n_{33}^{\frac{k_3(k_3-1)}{2}} \tag{7}$$

By replacing the values of n_{ij} we obtain

$$\sum_{k_1=0}^{n} \binom{n}{k_1} 4^{k_1(n-k_1)} \tag{8}$$