martedì 14 novembre 2023 10:55

Giovedì 14 dicembre

ore 16:30 - 18:30

aula 0D Vallisneri

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$x, y \longmapsto x \cdot y = (x_1 \ldots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}$$

$T \subseteq \mathbb{R}^n$

$$T^\perp = \{ v \in \mathbb{R}^n \mid v \cdot t = 0 \quad \forall t \in T \}$$

$$= \{ v \in \mathbb{R}^n \mid v \perp t \quad \forall t \in T \}$$

$$= \{ v \in \mathbb{R}^n \mid v \cdot t_i = 0 \quad \text{per } i = 1, \ldots, k$$

con $\{t_1, \ldots, t_k\}$ base di T $\}$

<u>Es</u> Sia $T = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^4$

Calcolare $T^\perp = \{ v \in \mathbb{R}^4 \mid v \cdot t = 0 \quad \forall t \in T \}$

Cerco una base di T.

$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ sono generatori di T

A casa verifico che sono LI

$$\Rightarrow \left\{ t_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, t_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, t_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ è base di } T$$

e $\dim T = 3$

$$T^{\perp} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot t_1 = 0, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot t_2 = 0, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot t_3 = 0 \right\}$$

$$0 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot t_1 = (x_1\, x_2\, x_3\, x_4) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 1 + x_4 \cdot 0$$

$$= x_1 + x_3 = 0$$

$$0 = t_2 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot t_2 = (x_1\, x_2\, x_3\, x_4) \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= x_2 + x_3 = 0$$

$$0 = t_3 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (0\ 0\ 1\ 1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \cdot x_1 + 0 \cdot x_2 + 1x_3 + 1x_4$$

$$= x_3 + x_4 = 0$$

$$= x_3 + x_4 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in T^\perp \iff \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_4 = -x_3 \end{cases}$$

$$\iff \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \\ -x_3 \end{pmatrix} = -x_3 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$T^\perp = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 6 \\ 6 \\ -6 \\ 6 \end{pmatrix} \right\rangle$$

$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ è base $\Rightarrow \dim T^\perp = 1$

<u>Oss</u> $T \cap T^\perp = \{0_{\mathbb{R}^n}\} \Rightarrow \dim T \cap T^\perp = 0$

$\dim T = 3$, $\dim T^\perp = 1$

$$\underbrace{\dim T + T^\perp}_{4} = \underbrace{\dim T}_{3} + \underbrace{\dim T^\perp}_{1} - \underbrace{\dim T \cap T^\perp}_{0}$$

$T + T^\perp \subseteq \mathbb{R}^4$, $\dim T + T^\perp = 4$, $\dim \mathbb{R}^4 = 4$

$\Rightarrow \quad T + T^\perp = \mathbb{R}^4$

Inoltre $T \cap T^\perp = \{0_{\mathbb{R}^4}\} \quad \Rightarrow \quad T \oplus T^\perp = \mathbb{R}^4$

Teorema

Sia $T \subseteq \mathbb{R}^n$ e $\dim T = m$

Allora $\mathbb{R}^n = T \oplus T^\perp$ e $\dim T^\perp = n - m$

dim Sia $\mathcal{B} = \{t_1, \dots, t_m\}$ base ON di T, cioè

base
① $t_1, \dots, t_m$ sono generatori
② $t_1, \dots, t_m$ sono LI

ON
③ $\|t_i\| = 1 \iff t_i \cdot t_i = 1 \quad \forall i$
$\iff t_i$ versori $\forall i$
④ $t_i \perp t_j \quad \forall i \neq j \iff t_i \cdot t_j = 0 \quad \forall i \neq j$

Completo $\mathcal{B}$ in una base $\mathcal{C} = \{t_1, \dots, t_m, u_{m+1}, \dots, u_n\}$ di $\mathbb{R}^n$

Applico Gram Schmidt a $\mathcal{C}$ ed ottengo

$\mathcal{C}' = \{t_1, \dots, t_m, v_{m+1}, \dots, v_n\}$

$\mathcal{C}' = \{ t_1, \dots, t_m, v_{m+1}, \dots, v_n \}$

base ON di $\mathbb{R}^n$

$T^\perp = \{ w \in \mathbb{R}^n \mid w \cdot t_i = 0 \quad \text{per } i=1, \dots, m \}$

$$w \in \mathbb{R}^n \Rightarrow w = \alpha_1 t_1 + \dots + \alpha_m t_m + \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n$$

poiché $w \in \mathbb{R}^n$ e $\mathcal{C}'$ è base di $\mathbb{R}^n$

$$0 = w \cdot t_1 = (\alpha_1 t_1 + \dots + \alpha_m t_m + \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n) \cdot t_1$$

$$= \alpha_1 \underbrace{t_1 \cdot t_1}_{1 \circledast} + \alpha_2 \underbrace{t_2 \cdot t_1}_{0 \circledast} + \dots + \alpha_m \underbrace{t_m \cdot t_1}_{0 \circledast} + \alpha_{m+1} \underbrace{v_{m+1} \cdot t_1}_{0 \circledast} + \dots + \alpha_n \underbrace{v_n \cdot t_1}_{0 \circledast}$$

$\circledast$ poiché $\mathcal{C}' = \{ t_1, \dots, t_m, v_{m+1}, \dots, v_n \}$ è base ON

$0 = w \cdot t_1 = \alpha_1$

$0 = w \cdot t_2 = \alpha_2$

$0 = w \cdot t_m = \alpha_m$

$w \in T^\perp \Leftrightarrow w \cdot t_i = 0 \quad \text{per } i=1, \dots, m$

$\Leftrightarrow \quad \alpha_i = 0 \quad$ per $i = 1, \dots, m$

$$w = \underbrace{\alpha_1}_{0} t_1 + \dots + \underbrace{\alpha_m}_{0} t_m + \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n$$

$$= \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n$$

$$\in \langle v_{m+1}, \dots, v_n \rangle$$

$T^\perp = \langle v_{m+1}, \dots, v_n \rangle$ e $\dim T^\perp = n - m$

Poiché $T \cap T^\perp = \{0_{\mathbb{R}}\}$, $\dim T \cap T^\perp = 0$

Per la formula di Grassmann

$$\underbrace{\dim T + T^\perp}_{n} = \underbrace{\dim T}_{m} + \underbrace{\dim T^\perp}_{n-m} - \underbrace{\dim T \cap T^\perp}_{0}$$

$T + T^\perp \subseteq \mathbb{R}^n$, $\dim T + T^\perp = n = \dim \mathbb{R}^n$

$\Rightarrow \quad T + T^\perp = \mathbb{R}^n$

$T \cap T^\perp = \{0_{\mathbb{R}^n}\} \quad \Rightarrow \quad T \oplus T^\perp = \mathbb{R}^n$

<u>Es</u> Calcolare $T^\perp$ con $T = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ sono LI e generatori di T

$\Rightarrow \left\{ \underbrace{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}_{t_1}, \underbrace{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}}_{t_2} \right\}$ è base di T e $\dim T = 2$

Dalla teoria sappiamo che $T \oplus T^\perp = \mathbb{R}^4$

$\Rightarrow \dim T^\perp = \dim \mathbb{R}^4 - \dim T = 4 - 2 = 2$

$$T^\perp = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \cdot t_1 = 0 = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \cdot t_2 \right\}$$

$$0 = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \cdot \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = (x_1\, x_2\, x_3\, x_4) \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = x_1 + x_2$$

$$0 = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \cdot \begin{pmatrix}0\\0\\1\\1\end{pmatrix} = (x_1\, x_2\, x_3\, x_4) \begin{pmatrix}0\\0\\1\\1\end{pmatrix} = x_3 + x_4$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in T^\perp \iff \begin{cases} x_1 + x_2 = 0 \\ x_3 + x_4 = 0 \end{cases}$$

$$\iff \begin{cases} x_2 = -x_1 \\ x_4 = -x_3 \end{cases}$$

$$\iff \begin{pmatrix}x_1\\-x_1\\x_3\\-x_3\end{pmatrix} = x_1 \begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + x_3 \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

$\begin{pmatrix}1\\-1\end{pmatrix} \quad \begin{pmatrix}0\\0\end{pmatrix}$

$$\iff \quad T^\perp = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

Dalla teoria so che $\dim T^\perp = 2$

I due generatori $T^\perp$ sono L I e dunque

$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ è base di $T^\perp$

PROIEZIONE ORTOGONALE

Poiché $T \oplus T^\perp = \mathbb{R}^n$, ogni vettore di $\mathbb{R}^n$ si scrive <u>in modo unico</u> come

$$v = v_T + v_{T^\perp}$$

con $v_T \in T$ e $v_{T^\perp} \in T^\perp$

Allora esiste (ed è ben definita) la funzione <u>proiezione ortogonale su T</u>

$$pr_T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$v_T + v_{T^\perp} = v \longmapsto pr_T(v) = v_T$$

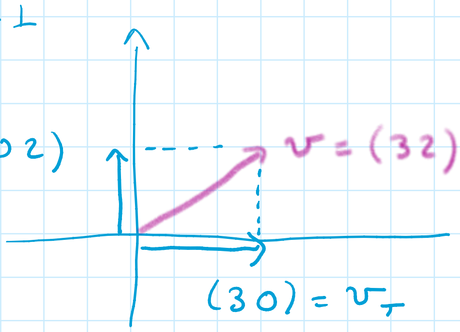
v_T è la proiezione di v su T

$v_{T^\perp}$ è la proiezione di v su $T^\perp$

Oss

$\mathbb{R}^2$

$<\begin{pmatrix}0\\1\end{pmatrix}> = T^\perp$



$v_{T^\perp} = (0\ 2)$

$v = (3\ 2) = (3\ 0) + (0\ 2) = v_T + v_{T^\perp}$

$T = <\begin{pmatrix}1\\0\end{pmatrix}>$

$(3\ 0) = v_T$

Es Determinare la proiezione ortogonale di $u = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, $u' = \begin{pmatrix}1\\-1\\2\end{pmatrix}$ e $u'' = \begin{pmatrix}1\\1\\0\end{pmatrix}$

su $T = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \mid \underline{x+y=0},\ \underline{2y+z=0} \right\} \subseteq \mathbb{R}^3$

$T, T^\perp \subseteq \mathbb{R}^3 \Rightarrow \dim T \text{ e } \dim T^\perp \leq 3$

Calcoliamo una base di T

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \in T \iff \begin{cases} x+y=0 \\ 2y+z=0 \end{cases} \iff \begin{cases} x=-y \\ z=-2y \end{cases}$$

$$\iff \begin{pmatrix}-y\\y\\-2y\end{pmatrix} = y\begin{pmatrix}-1\\1\\-2\end{pmatrix}$$

$T = <\begin{pmatrix}-1\\1\\-2\end{pmatrix}>$ e $\dim T = 1$ e $\left\{\begin{pmatrix}-1\\1\\-2\end{pmatrix}\right\}$ è base di T

Dalla teoria sappiamo che $\mathbb{R}^3 = T \oplus T^\perp$

$\Rightarrow \dim T^\perp = \dim \mathbb{R}^3 - \dim T = 3 - 1 = 2$

$$T^\perp = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\-2\end{pmatrix} = 0 \right\}$$

$$0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = (x\ y\ z) \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = -x + y - 2z = 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in T^{\perp} \iff y = x + 2z$$

$$\iff \begin{pmatrix} x \\ x+2z \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$T^{\perp} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\rangle$ so che $\dim T^{\perp} = 2$

$\Rightarrow$ i due generatori $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ e $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ sono una base di $T^{\perp}$

$T = \left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle \qquad T^{\perp} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

$u' = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \in T \qquad u' = u'_T + u'_{T^\perp}$ con $u'_{T^\perp} = 0$

$= u' + 0 \qquad u'_T = u'$

$\Rightarrow pr_T(u') = u'$

$u'' = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in T^{\perp} \qquad u'' \overset{\text{unica}}{=} u''_T + u''_{T^\perp}$ con $u''_T = 0$

$= 0 + u'' \qquad u''_T = u''$

$\Rightarrow pr_T(u'') = 0$

$u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = u_T + u_{T^\perp}$

u_T = proiezione di u su T

$= (u \cdot v)\, v$ con $\{v\}$ base ON di T

Oss Se $\dim T = 2$ e $\{v_1, v_2\}$ base ON di T

allora u_T = proiezione di u su T

$= (u \cdot v_1)\, v_1 + (u \cdot v_2)\, v_2$

Devo cercare base ON di T

Una base di T è $\left\{ \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$

$$\left\| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}} = \sqrt{(1\ -1\ 2) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}$$

$$= \sqrt{1 + 1 + 4} = \sqrt{6}$$

Una base ON di T è $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$

(ho applicato Gram Schmidt)

$$u_T = \left[u \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right] \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{6} \left[\underbrace{(1\ 0\ -1) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{1 \cdot 1 + 0(-1) + (-1)2 = -1} \right] \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{6} (-1) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{6} \quad (-1) \quad \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

Per cercare la proiezione ortogonale di v su T

① Cerco base ON di T. Sarà $\{t_1, \dots, t_m\}$

② $P_T(v) = \underbrace{[v \cdot t_1]}_{\in \mathbb{R}} t_1 + \underbrace{[v \cdot t_2]}_{\in \mathbb{R}} t_2 + \dots + \underbrace{[v \cdot t_m]}_{\in \mathbb{R}} t_m$

Proprietà della proiezione ortogonale su $T \subseteq \mathbb{R}^n$

$$pr_T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\underbrace{v_T + v_{T^\perp} = v}_{\text{decomposizione unica}} \longmapsto v_T$$

decomposizione unica

poiché $T \oplus T^\perp = \mathbb{R}^n$

① pr_T è lineare

② $\operatorname{Ker} pr_T = T^\perp = V_0 = \{ v \mid pr_T(v) = 0 \}$

Se $v \in T^\perp$ $\quad v = v_T + v_{T^\perp} = 0 + v \quad$ cioè $v_{T^\perp} = v$

$\Rightarrow \quad pr_T(v) = 0 = v_T$ $\qquad\qquad v_T = 0$

③ $\operatorname{Im} pr_T = T$

È chiaro che $\operatorname{Im} pr_T \subseteq T$

Verifichiamo che $T \subseteq \operatorname{Im} pr_T$:

$w \in T \quad \Rightarrow \quad w = \underbrace{w}_{w_T} + \underbrace{0}_{w_{T^\perp}}$

$w = pr_T(w) \in \operatorname{Im} pr_T$

④ $\operatorname{Im} pr_T = T = V_1 = \{ v \mid pr_T(v) = v \}$

⑤ pr_T è diagonalizzabile.

$\mathbb{R}^n = T \oplus T^\perp = V_1 \oplus V_0$

$\mathbb{R}^n$ è somma diretta degli autospazi di pr_T

$\Rightarrow pr_T$ è diagonalizzabile

<u>Oss</u> $\quad T = V_1 \quad$ e $\quad T^\perp = V_0$

$\Rightarrow \quad V_1 \perp V_0$