Sia $\mathcal{B} = \{ \underline{x}_1, \dots, \underline{x}_n \}$ una base di $\mathbb{R}^n$.

① $\mathcal{B}$ è detta <u>ORTOGONALE</u> $\Leftrightarrow$

$\underline{x}_i \perp \underline{x}_j$ per $i \neq j$ $\Leftrightarrow$

$\underline{x}_i \cdot \underline{x}_j = 0_{\mathbb{R}}$ per $i \neq j$

② $\mathcal{B}$ è detta <u>ORTO NORMALE</u> $\Leftrightarrow$

$\underline{x}_i \perp \underline{x}_j$ per $i \neq j$ e $\| \underline{x}_i \| = 1 \ \forall i$ $\Leftrightarrow$

$\underline{x}_i \cdot \underline{x}_j = 0_{\mathbb{R}}$ per $i \neq j$ e $\underline{x}_i \cdot \underline{x}_i = 1 \ \forall i$

<u>Proposizione</u>

Sia $\mathcal{B} = \{ \underline{x}_1, \dots, \underline{x}_n \}$ una base ON di $\mathbb{R}^n$

Allora $\forall v \in \mathbb{R}^n$

$$v = \underset{\substack{\uparrow \\ \mathbb{R}}}{\underline{(v \cdot \underline{x}_1)}} \underline{x}_1 + \dots + \underset{\substack{\uparrow \\ \mathbb{R}}}{\underline{(v \cdot \underline{x}_n)}} \underline{x}_n = \begin{pmatrix} v \cdot \underline{x}_1 \\ v \cdot \underline{x}_2 \\ \vdots \\ v \cdot \underline{x}_n \end{pmatrix}_{\mathcal{B}}$$

<u>dim</u> $\forall v \in \mathbb{R}^n \ \exists \lambda_i \in \mathbb{R}$ tali che

$$v = \lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \dots + \lambda_n \underline{x}_n = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}_{\mathcal{B}}$$

(poiché $\mathcal{B}$ è una base di $\mathbb{R}^n$)

$$v \cdot \underline{x}_1 = (\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \dots + \lambda_n \underline{x}_n) \cdot \underline{x}_1$$

② e ③

② e ③

$$= \lambda_1 \underbrace{\underline{x}_1 \cdot \underline{x}_1}_{1} + \lambda_2 \underbrace{\underline{x}_2 \cdot \underline{x}_1}_{0} + \dots + \lambda_n \underbrace{\underline{x}_n \cdot \underline{x}_1}_{0}$$

poiché $\|\underline{x}_1\| = 1$ poiché $\underline{x}_2 \perp \underline{x}_1$ poiché $\underline{x}_n \perp \underline{x}_1$

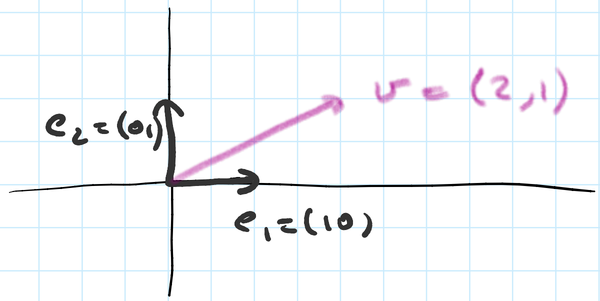
$\mathcal{B} = \{\underline{x}_1, \dots, \underline{x}_n\}$ è base ON

$$= \lambda_1$$

Idem si dimostra che $\lambda_i = v \cdot \underline{x}_i$

$$\Rightarrow \quad v = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ | \\ \lambda_n \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} v \cdot \underline{x}_1 \\ | \\ v \cdot \underline{x}_n \end{pmatrix}_{\mathcal{B}}$$

Es



$\mathcal{E} = \{e_1, e_2\}$ è ON

$$v = (2, 1) = 2 e_1 + 1 e_2$$

$$v \cdot e_1 = (2\ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \cdot 1 + 1 \cdot 0 = 2$$

$$v \cdot e_2 = (2\ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \cdot 0 + 1 \cdot 1 = 1$$

$$v = (v \cdot e_1)\, e_1 + (v \cdot e_2)\, e_2$$

Proposizione. $(n \leq m)$

Sia $\{\underline{x}_1, \dots, \underline{x}_n\}$ base **ON** di T, T sottosp. vett di $\mathbb{R}^m$

Il vettore $(v \cdot \underline{x}_1)\underline{x}_1 + \dots + (v \cdot \underline{x}_n)\underline{x}_n$

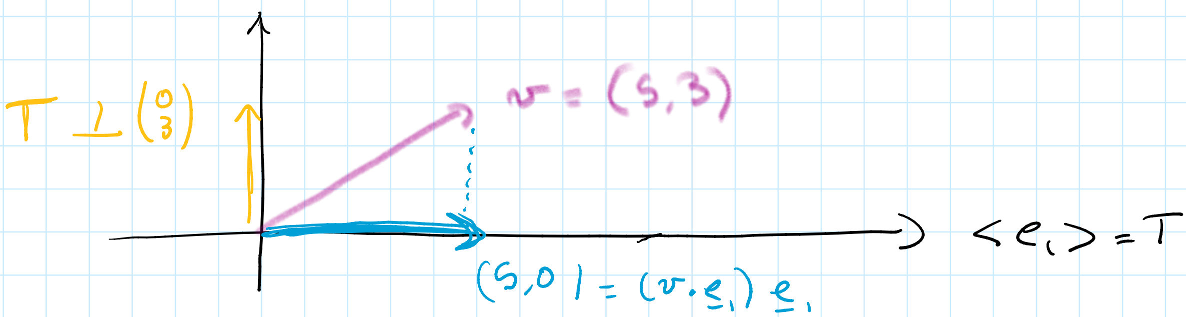
è LA PROIEZIONE ORTOGONALE di $v \in \mathbb{R}^m$ su

$T = \langle \underline{x}_1, \dots, \underline{x}_n \rangle$

Inoltre $v - (v \cdot \underline{x}_1)\underline{x}_1 - \dots - (v \cdot \underline{x}_n)\underline{x}_n$

è ORTOGONALE a $T = \langle \underline{x}_1, \dots, \underline{x}_n \rangle$

Es



$\mathbb{R}^2$, $\mathcal{B} = \{ \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$ è base ON

$$(v \cdot \underline{e}_1)\, \underline{e}_1 = \text{proiezione ortogonale di } v \text{ su } \langle e_1 \rangle$$

$$= \left[(5,3) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (5 \cdot 1 + 3 \cdot 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$v - (v \cdot \underline{e}_1) \cdot \underline{e}_1$ è ortogonale a $T = \langle \underline{e}_1 \rangle$

$$= \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\,\underline{e}_2$$

PROCEDIMENTO DI GRAM-SCHMIDT

(mi permette di ottenere una base ON a partire da una qualsiasi base)

Sia T un sottosp. vettoriale di $\mathbb{R}^n$, $T \neq \{0_{\mathbb{R}^n}\}$ e $\dim T = m \leq n$

Sia $\mathcal{B} = \{\underline{t}_1, \dots, \underline{t}_m\}$ una base di T.

Allora esiste una base $\mathcal{V} = \{\underline{v}_1, \dots, \underline{v}_m\}$ di T tale che

① $\mathcal{V}$ è ON

② $\langle \underline{v}_1 \rangle = \langle \underline{t}_1 \rangle$, $\langle \underline{v}_1, \underline{v}_2 \rangle = \langle \underline{t}_1, \underline{t}_2 \rangle$

.... $\langle \underline{v}_1, \dots, \underline{v}_h \rangle = \langle \underline{t}_1, \dots, \underline{t}_h \rangle$ con $h \leq m$

$$\underline{v}_1 = \frac{\underline{t}_1}{\|\underline{t}_1\|}$$

$$\underline{v}_2 = \frac{\underline{t}_2 - (\underline{t}_2 \cdot \underline{v}_1)\,\underline{v}_1}{\|\underline{t}_2 - (\underline{t}_2 \cdot \underline{v}_1)\,\underline{v}_1\|}$$

$\perp v_1$

$$\underline{v}_3 = \frac{\underline{t}_3 - (\underline{t}_3 \cdot \underline{v}_1)\,\underline{v}_1 - (\underline{t}_3 \cdot \underline{v}_2)\,\underline{v}_2}{\|\underline{t}_3 - (\underline{t}_3 \cdot \underline{v}_1)\,\underline{v}_1 - (\underline{t}_3 \cdot \underline{v}_2)\,\underline{v}_2\|}$$

$\perp v_1$ e v_2

$$\underline{v}_m = \frac{\underline{t}_m - \sum_{i=1}^{m-1} (\underline{t}_m \cdot \underline{v}_i)\, \underline{v}_i}{\left\| \underline{t}_m - \sum_{i=1}^{m-1} (\underline{t}_m \cdot \underline{v}_i)\, \underline{v}_i \right\|}$$

$\underline{Es}$ Determinare una base ON dalla base di $T \subseteq \mathbb{R}^4$ data da $\left\{ \underline{t}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \underline{t}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \underline{t}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\{ \underline{t}_1, \underline{t}_2, \underline{t}_3 \}$ non è ON poiché

$$\|\underline{t}_1\| = \sqrt{\underline{t}_1 \cdot \underline{t}_1} = \sqrt{(1\,0\,1\,0) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} = \sqrt{1\cdot 1 + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0} = \sqrt{2} \neq 1$$

cioè $\underline{t}_1$ non è versore

Oppure $\underline{t}_2$ non è ortogonale a $\underline{t}_3$:

$$\underline{t}_2 \cdot \underline{t}_3 = (0\,1\,1\,0) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 0 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$$

A partire da $\{ \underline{t}_1, \underline{t}_2, \underline{t}_3 \}$ ricaviamo una base ON applicando Gram-Schmidt.

$$\underline{v}_1 = \frac{\underline{t}_1}{\|\underline{t}_1\|}$$

$\|\underline{t}_1\|$

$$\|\underline{t}_1\| = \sqrt{\underline{t}_1 \cdot \underline{t}_1} = \sqrt{(1\,0\,1\,0)\begin{pmatrix}1\\0\\1\\0\end{pmatrix}} = \sqrt{1\cdot1+0\cdot0+1\cdot1+0\cdot0}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$$\underline{v}_1 = \frac{1}{\sqrt{2}}\,\underline{t}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{pmatrix}$$

$$\underline{v}_2 = \frac{\underline{t}_2 - (\underline{t}_2\cdot\underline{v}_1)\,\underline{v}_1}{\|\underline{t}_2 - (\underline{t}_2\cdot\underline{v}_1)\,\underline{v}_1\|}$$

$$\underline{t}_2 - (\underline{t}_2\cdot\underline{v}_1)\,\underline{v}_1 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \left[\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{1}{2}\left[(0\,1\,1\,0)\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right]\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{1}{2}\left[\underbrace{0\cdot1+1\cdot0+1\cdot1+0\cdot0}_{1}\right]\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1\\1-1/2\\0\end{pmatrix} = \begin{pmatrix}-1/2\\1\\1/2\\0\end{pmatrix}$$

$$\left\|\begin{pmatrix}-1/2\\1\\1/2\\0\end{pmatrix}\right\| = \sqrt{\begin{pmatrix}-1/2\\1\\1/2\\0\end{pmatrix}\cdot\begin{pmatrix}-1/2\\1\\1/2\\0\end{pmatrix}}$$

$$= \sqrt{\begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \\ 0 \end{pmatrix}}$$

$$= \sqrt{\frac{1}{4} + 1 + \frac{1}{4} + 0} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$$

$$\underline{v}_2 = \frac{2}{\sqrt{6}} \begin{pmatrix} -1/2 \\ 1 \\ 1/2 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{v}_3 = \frac{\underline{t}_3 - (\underline{t}_3 \cdot \underline{v}_1)\,\underline{v}_1 - (\underline{t}_3 \cdot \underline{v}_2)\,\underline{v}_2}{\left\| \underline{t}_3 - (\underline{t}_3 \cdot \underline{v}_1)\,\underline{v}_1 - (\underline{t}_3 \cdot \underline{v}_2)\,\underline{v}_2 \right\|}$$

$$\underline{t}_3 - (\underline{t}_3 \cdot \underline{v}_1)\,\underline{v}_1 - (\underline{t}_3 \cdot \underline{v}_2)\,\underline{v}_2 =$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right] \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \left[\begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6} \left[\begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right] \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \left[\underbrace{0 + 0 + 1 + 0}_{1} \right] \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6} \left[\underbrace{0 + 0 + 1 + 0}_{1} \right] \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} -1/2 + 1/6 \\ -2/6 \\ 1 - 1/2 - 1/6 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/6 \\ -2/6 \\ 2/6 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\left\| \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix}}$$

$$= \sqrt{\begin{pmatrix} -1/3 & -1/3 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix}}$$

$$= \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + 1}$$

$$= \sqrt{\frac{12}{9}} = \frac{1}{3}\sqrt{12}$$

$$\underline{v}_3 = \frac{3}{\sqrt{12}} \begin{pmatrix} -1/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$

La ON di T ottenuta a partire da $\{\underline{t}_1, \underline{t}_2, \underline{t}_3\}$

La ON di T ottenuta a partire da $\{\underline{t}_1, \underline{t}_2, \underline{t}_3\}$

è $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 3 \end{pmatrix} \right\}$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{6}} \left[(1\,0\,1\,0) \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right]$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{6}} \left[\underbrace{1 \cdot (-1) + 0 \cdot 2 + 1 \cdot 1 + 0 \cdot 0}_{\underbrace{-1+1}_{0}} \right]$$

$$= 0 .$$

<u>Def</u> Sia $T \subseteq \mathbb{R}^n$ sottosp. vettoriale

IL COMPLEMENTO ORTOGONALE DI T

è

$$T^\perp = \{ v \in \mathbb{R}^n \mid v \cdot t = 0_{\mathbb{R}} \quad \forall t \in T \}$$

$$= \{ v \in \mathbb{R}^n \mid v \perp t \quad \forall t \in T \}$$

$$= \{ v \in \mathbb{R}^n \mid v \perp T \}$$

Se $\dim T = k$ e $\{ t_1, \dots, t_k \}$ base di T allora

$$T^\perp = \{ v \in \mathbb{R}^n \mid v \cdot t_i = 0 \quad \text{per } i=1,\dots,k \}$$

Proprietà del complemento ortogonale

① $T = \{0_{\mathbb{R}^n}\} \implies T^\perp = \mathbb{R}^n$

② $T = \mathbb{R}^n \implies T = \{0_{\mathbb{R}^n}\}$

③ $T^\perp$ è un sottospazio vettoriale di $\mathbb{R}^n$

dim 1 $\quad \forall\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$$x \cdot 0_{\mathbb{R}^n} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= (x_1 \dots x_n) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= x_1 \cdot 0 + x_2 \cdot 0 + \dots + x_n 0 = 0$$

$\implies x \cdot 0_{\mathbb{R}^n} = 0 \quad \forall\, x \in \mathbb{R}^n$

$\implies x \perp 0_{\mathbb{R}^n} \quad \forall\, x \in \mathbb{R}^n$

$$\implies \{0_{\mathbb{R}^n}\}^\perp = \{ x \in \mathbb{R}^n \mid x \perp 0_{\mathbb{R}^n} \} = \mathbb{R}^n$$

$$= \{ x \in \mathbb{R}^n \mid x \cdot 0_{\mathbb{R}^n} = 0 \} = \mathbb{R}^n$$

dim 2 Sia $\{\underline{t}_1, \dots, \underline{t}_n\}$ base ON di $\mathbb{R}^n$

$\forall w \in \mathbb{R}^n = T \quad w = \lambda_1 \underline{t}_1 + \dots + \lambda_n \underline{t}_n$

$$w \in \underset{\mathbb{R}^n}{\overset{\cap\!|}{T^\perp}} \iff w \cdot \underline{t}_i = 0 \quad \text{per } i = 1, \dots, n$$

$$0 = w \cdot \underline{t}_1 = (\lambda_1 \underline{t}_1 + \dots + \lambda_n \underline{t}_n) \cdot \underline{t}_1$$

$$= \lambda_1 \underbrace{\underline{t}_1 \cdot \underline{t}_1}_{1} + \lambda_2 \underbrace{\underline{t}_2 \cdot \underline{t}_1}_{0} + \dots + \lambda_n \underbrace{\underline{t}_n \cdot \underline{t}_1}_{0}$$

$\|\underline{t}_1\| = 1$ $\underline{t}_2 \perp \underline{t}_1$ $\underline{t}_n \perp \underline{t}_1$

$\{\underline{t}_1, \dots, \underline{t}_n\}$ è base ON

$$= \lambda_1$$

$$0 = w \cdot \underline{t}_k = \lambda_k \qquad \text{per } k = 1, \dots, n$$

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

$$\Rightarrow w = \lambda_1 \underline{t}_1 + \dots + \lambda_n \underline{t}_n = 0_{\mathbb{R}^n}$$

Dunque $T^\perp = (\mathbb{R}^n)^\perp = \{0_{\mathbb{R}^n}\}$

dim 3 $T^\perp$ è sotto sp. vettoriale $\iff$

$T^\perp$ è chiuso rispetto alla somma di vettori e

$T^\perp$ è chiuso rispetto alla moltiplicazione

$T^\perp$ è chiuso rispetto alla moltiplicazione per uno scalare.

$$v_1, v_2 \in T^\perp = \{ w \in \mathbb{R}^n \mid w \cdot t = 0 \ \forall t \in T \}$$

$$\Rightarrow \quad v_1 \cdot t \overset{*}{=} 0_{\mathbb{R}} \qquad v_2 \cdot t \overset{*}{=} 0_{\mathbb{R}} \qquad \forall t \in T$$

Domanda $v_1 + v_2 \in T^\perp$?

$$(v_1 + v_2) \cdot t \overset{2}{=} v_1 \cdot t + v_2 \cdot t \qquad \forall t \in T$$

$$\overset{* \; *}{=} 0_{\mathbb{R}} + 0_{\mathbb{R}} = 0_{\mathbb{R}}$$

$$\Rightarrow \quad v_1 + v_2 \in T^\perp$$

Sia $\lambda \in \mathbb{R}$, $v \in T^\perp$ cioè $v \cdot t \overset{*}{=} 0_{\mathbb{R}} \ \forall t \in T$

Domand $\lambda v \in T^\perp$?

$$(\lambda v) \cdot t \overset{3}{=} \lambda (v \cdot t) \overset{*}{=} \lambda \, 0_{\mathbb{R}} = 0_{\mathbb{R}}$$

$$\Rightarrow \quad \lambda v \in T^\perp .$$

Proprietà $\quad T \subseteq \mathbb{R}^n$

① $T \cap T^\perp = \{ 0_{\mathbb{R}^n} \}$ $\quad (\Rightarrow \; T \oplus T^\perp)$

② $(T^\perp)^\perp = T$

② $(T^{\perp})^{\perp} = T$

③ $T_1 \subseteq T_2 \Rightarrow T_1^{\perp} \supseteq T_2^{\perp}$

④ $(T_1 + T_2)^{\perp} = T_1^{\perp} \cap T_2^{\perp}$

⑤ $(T_1 \cap T_2)^{\perp} = T_1^{\perp} + T_2^{\perp}$.