martedì 14 novembre 2023 10:54

PRODOTTO SCALARE

Def IL PRODOTTO SCALARE (STANDARD) in $\mathbb{R}^n$ è l'applicazione

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \longmapsto \underline{x} \cdot \underline{y}$$

$$\underline{x} \cdot \underline{y} \overset{\text{def}}{=} \sum_{i=1}^{n} x_i y_i = (x_1 \dots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Es $\underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$\underline{x} \cdot \underline{y} = (1\ 2) \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1 \cdot 2 + 2 \cdot 0 = 2 \in \mathbb{R}$$

PROPRIETÀ DEL PRODOTTO SCALARE

1) $\underline{x} \cdot \underline{y} = \underline{y} \cdot \underline{x}$

② $(\underline{x}+\underline{y})\cdot\underline{z} = \underline{x}\cdot\underline{z} + \underline{y}\cdot\underline{z}$

③ $(\lambda\underline{x})\cdot\underline{y} = \lambda(\underline{x}\cdot\underline{y}) = \underline{x}\cdot(\lambda\underline{y})$

④ $\underline{x}\cdot\underline{x} \geq 0$

$\underline{x}\cdot\underline{x} = 0_{\mathbb{R}} \iff \underline{x} = 0_{\mathbb{R}^n}$

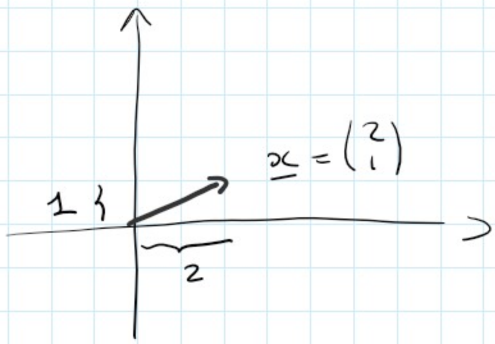
Def LA NORMA di $\underline{x} \in \mathbb{R}^n$ è

$$\|\underline{x}\| = \sqrt{\underline{x}\cdot\underline{x}} \in \mathbb{R}$$

Oss ④ $\Rightarrow$ $\underline{x}\cdot\underline{x} \geq 0 \Rightarrow \|\underline{x}\| = \sqrt{\underline{x}\cdot\underline{x}}$ è ben definita

Oss ④ $\Rightarrow \|\underline{x}\| = 0_{\mathbb{R}} \iff \underline{x}\cdot\underline{x} = 0 \iff \underline{x} = 0_{\mathbb{R}^n}$

Es



$$\|\underline{x}\| = \sqrt{\underline{x}\cdot\underline{x}} = \sqrt{(2\ 1)\begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \sqrt{4+1} = \sqrt{5}$$

La funzione norma è

$$\|\_\| : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\underline{x} \longmapsto \|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}} = \text{lunghezza di } \underline{x}$$

PROPRIETÀ DELLA NORMA

1. $\|\underline{x}\| = 0_{\mathbb{R}} \iff \underline{x} = 0_{\mathbb{R}^n}$

2. $\|\alpha \underline{x}\| = \underbrace{|\alpha|}_{\text{valore assoluto di } \alpha \in \mathbb{R}} \|\underline{x}\| \qquad \forall\ \underline{x} \in \mathbb{R}^n \text{ e } \alpha \in \mathbb{R}$

__Def__ I __VERSORI__ sono i vettori $\underline{x} \in \mathbb{R}^n$ di norma 1
cioè $\|\underline{x}\| = 1$

__Es__ In $\mathbb{R}^2$ la base canonica è $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$\|e_1\| = \sqrt{e_1 \cdot e_1} = \sqrt{(1 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \sqrt{1+0} = \sqrt{1} = 1$$

$$\|e_2\| = \sqrt{e_2 \cdot e_2} = \sqrt{(0 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \sqrt{0+1} = \sqrt{1} = 1$$

Tutti i vettori delle basi canoniche degli $\mathbb{R}^n$ sono versori

Def La NORMALIZZAZIONE di $\underline{x} \in \mathbb{R}^n$ è il versore $\underbrace{\frac{1}{\|\underline{x}\|}}_{\in \mathbb{R}} \underbrace{\underline{x}}_{\in \mathbb{R}^n} \in \mathbb{R}^n$

Es $\underline{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad \|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}} = \sqrt{(2\,0)\begin{pmatrix} 2 \\ 0 \end{pmatrix}} = \sqrt{4+0}$

$= \sqrt{4} = 2$

$$\frac{1}{\|\underline{x}\|}\underline{x} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/2 \\ 0/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left\| \frac{1}{\|\underline{x}\|}\underline{x} \right\| = \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \sqrt{(1\,0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \sqrt{1+0} = 1$$

DISUGUAGLIANZA DI CAUCHY-SCHWARZ

$\forall\, \underline{x}, \underline{y} \in \mathbb{R}^n \qquad \underbrace{|\underline{x} \cdot \underline{y}|}_{\in \mathbb{R}} \leq \|\underline{x}\|\,\|\underline{y}\|$

dim Supponiamo che $\underline{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0_{\mathbb{R}^n}$

e $\underline{y} = \begin{pmatrix} y_1 \\ y_n \end{pmatrix} \in \mathbb{R}^n$

$$\underline{x} \cdot \underline{y} = (0 \dots 0)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = 0y_1 + \dots + 0y_n = 0$$

$$|\underline{x} \cdot \underline{y}| = |0| = 0$$

$\|\underline{x}\| \; \|\underline{y}\| = 0 \; \|\underline{y}\| = 0$

$\Downarrow$

$\boxed{\underline{x} = 0_{\mathbb{R}^n} \iff \|\underline{x}\| = 0}$

$$|\underline{x} \cdot \underline{y}| = 0 \leq 0 \leq \|\underline{x}\| \; \|\underline{y}\|$$

Se $\underline{x} = 0_{\mathbb{R}^n}$ la disuguaglianza è vera.

Posso supporre che $\underline{x} \neq 0_{\mathbb{R}^n}$, cioè $\underline{x} \cdot \underline{x} \neq 0_{\mathbb{R}}$

$\forall \lambda \in \mathbb{R}$ abbiamo

$$0 \underset{④}{\leq} (\underline{y} - \lambda \underline{x}) \cdot (\underline{y} - \lambda \underline{x}) =$$

②

$$= \underline{y} \cdot (\underline{y} - \lambda \underline{x}) - \lambda \underline{x} \cdot (\underline{y} - \lambda \underline{x})$$

② + ③

$$= \underline{y} \cdot \underline{y} - \lambda \underline{y} \cdot \underline{x} - \lambda \underline{x} \cdot \underline{y} + \lambda^2 \underline{x} \cdot \underline{x}$$

①

$$= \underline{y} \cdot \underline{y} - \lambda \underline{x} \cdot \underline{y} - \lambda \underline{x} \cdot \underline{y} + \lambda^2 \underline{x} \cdot \underline{x}$$

$$= \underline{y} \cdot \underline{y} - 2\lambda \underline{x} \cdot \underline{y} + \lambda^2 \underline{x} \cdot \underline{x}$$

Pongo $\lambda = \dfrac{\underline{x} \cdot \underline{y}}{\underline{x} \cdot \underline{x}} \in \mathbb{R}$ (Oss $\underline{x} \neq 0_{\mathbb{R}^n} \iff \underline{x} \cdot \underline{x} \neq 0_{\mathbb{R}}$)

$$= \underline{y}\cdot\underline{y} - 2\,\frac{\underline{x}\cdot\underline{y}}{\underline{x}\cdot\underline{x}}\,\underline{x}\cdot\underline{y} + \frac{(\underline{x}\cdot\underline{y})^2}{(\underline{x}\cdot\underline{x})^{\cancel{2}}}\,\cancel{\underline{x}\cdot\underline{x}}$$

$$= \underline{y}\cdot\underline{y} \underbrace{- 2\,\frac{(\underline{x}\cdot\underline{y})^2}{\underline{x}\cdot\underline{x}} + \frac{(\underline{x}\cdot\underline{y})^2}{\underline{x}\cdot\underline{x}}}$$

$$= \underline{y}\cdot\underline{y} - \frac{(\underline{x}\cdot\underline{y})^2}{\underline{x}\cdot\underline{x}}$$

$$= \|\underline{y}\|^2 - \frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{x}\|^2} \qquad \|\underline{x}\| = \sqrt{\underline{x}\cdot\underline{x}}$$

$$0 \leq \|\underline{y}\|^2 - \frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{x}\|^2}$$

$$\frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{x}\|^2} \leq \|\underline{y}\|^2$$

$$(\underline{x}\cdot\underline{y})^2 \leq \|\underline{x}\|^2\,\|\underline{y}\|^2$$

$$|\underline{x}\cdot\underline{y}| \leq \|\underline{x}\|\,\|\underline{y}\|.$$

DISUGUAGLIANZA TRIANGOLARE

$\forall\, \underline{x}, \underline{y} \in \mathbb{R}^n \qquad \|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

$\forall \underline{x}, \underline{y} \in \mathbb{R}^n \qquad \|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

dim $\quad \|\underline{x}+\underline{y}\|^2 \overset{\text{def}}{=} (\underline{x}+\underline{y}) \cdot (\underline{x}+\underline{y})$

$$\overset{②}{=} \underline{x} \cdot (\underline{x}+\underline{y}) + \underline{y} \cdot (\underline{x}+\underline{y})$$

$$\overset{②}{=} \underline{x}\cdot\underline{x} + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{x} + \underline{y}\cdot\underline{y}$$

$$\overset{①}{=} \underline{x}\cdot\underline{x} + \underline{x}\cdot\underline{y} + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{y}$$

$$= \|\underline{x}\|^2 + 2\,\underline{x}\cdot\underline{y} + \|\underline{y}\|^2$$

$\underline{x}\cdot\underline{y} \in \mathbb{R}$

$$\leq \|\underline{x}\|^2 + 2\,|\underline{x}\cdot\underline{y}| + \|\underline{y}\|^2$$

Per la desiguagianza di Cauchy Schwarz

$$\leq \|\underline{x}\|^2 + 2\,\|\underline{x}\|\,\|\underline{y}\| + \|\underline{y}\|^2$$

$$= (\|\underline{x}\| + \|\underline{y}\|)^2$$

$$\|\underline{x}+\underline{y}\|^2 \leq (\|\underline{x}\| + \|\underline{y}\|)^2$$

$$\Rightarrow \|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$$

Def Dati $\underline{x}, \underline{y} \in \mathbb{R}^n$, $\underline{x}, \underline{y} \neq 0_{\mathbb{R}^n}$

l' <u>ANGOLO</u> tra essi è $\phi \in [0, \pi]$

vale che

$$\cos\phi = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \, \|\underline{y}\|}$$

<u>Es</u> Calcolare l'angolo tra $\underline{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ e $\underline{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\underline{x} \cdot \underline{y} = \underline{x}^T \underline{y} = (1\ 1\ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 = 1$$

$$\|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}} = \sqrt{\underline{x}^T \underline{x}} = \sqrt{(1\ 1\ 0) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} = \sqrt{1+1+0} = \sqrt{2}$$

$$\|\underline{y}\| = \sqrt{\underline{y} \cdot \underline{y}} = \sqrt{(0\ 1\ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} = \sqrt{0+1+0} = \sqrt{1} = 1$$

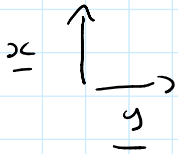
$$\cos\phi = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \, \|\underline{y}\|} = \frac{1}{\sqrt{2} \cdot 1} = \frac{1}{\sqrt{2}}$$

<u>Def</u> $\underline{x}, \underline{y} \in \mathbb{R}^n$ sono ORTOGONALI tra loro,

si scrive $\underline{x} \perp \underline{y}$, se $\underline{x} \cdot \underline{y} = 0_{\mathbb{R}}$

<u>Oss</u> $\underline{x} \cdot \underline{y} = 0_{\mathbb{R}} \Rightarrow \cos\widehat{\underline{x}\underline{y}} = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \, \|\underline{y}\|} = 0_{\mathbb{R}}$

$$\Rightarrow \widehat{\underline{x}\underline{y}} = \frac{\pi}{2} = 90^\circ$$



Def Sia T un sottospazio vettoriale di $\mathbb{R}^n$ e $\underline{x} \in \mathbb{R}^n$

Si dice che

$\underline{x}$ è ortogonale a T $\iff$ $\underline{x} \perp T$

se $\underline{x}$ è ortogonale a tutti i vettori di T, cioè

$$\underline{x} \cdot \underline{t} = 0_{\mathbb{R}} \qquad \forall\, \underline{t} \in T$$

Def Siano S e T due sottospazi vettoriali di $\mathbb{R}^n$

S e T sono ORTOGONALI $\iff$ $S \perp T$

se $\forall\, \underline{s} \in S$ e $\forall\, \underline{t} \in T$ $\quad \underline{s} \cdot \underline{t} = 0_{\mathbb{R}}$

Proposizione

Sia T un sottosp. vettoriale di $\mathbb{R}^n$, con base $\{\underline{t}_1, \dots, \underline{t}_k\}$ $(k \leq n \;,\; k = \dim T)$ Sia $\underline{x} \in \mathbb{R}^n$

$$\underline{x} \perp T \iff \underline{x} \perp \underline{t}_i \quad \text{per } i = 1, \dots, k$$

$$\iff \underline{x} \cdot \underline{t}_i = 0_{\mathbb{R}} \quad \text{per } i = 1, \dots, k$$

dim $\boxed{\underline{x} \perp T \implies \underline{x} \perp \underline{t}_i \quad \forall\, i = 1, \dots, k}$

$$\underline{x} \perp T \overset{\text{def}}{\iff} \underline{x} \perp \underline{t} \qquad \forall\, \underline{t} \in T$$

$$\Longleftrightarrow \quad \underline{x} \cdot \underline{t} = 0_{\mathbb{R}} \qquad \forall\, \underline{t} \in T$$

$\underline{t}_1, \dots, \underline{t}_k \in T$

$$\Longrightarrow \quad \underline{x} \cdot \underline{t}_i = 0_{\mathbb{R}} \quad \text{per } i = 1, \dots, k$$

$$\Longleftrightarrow \quad \underline{x} \perp \underline{t}_i \quad \text{per } i = 1, \dots, k$$

$$\boxed{\underline{x} \perp \underline{t}_i \quad \text{per } i = 1, \dots, k \quad \Longrightarrow \quad \underline{x} \perp T}$$

$\{\underline{t}_1, \dots, \underline{t}_k\}$ base di T $\Longrightarrow$ $\forall\, \underline{t} \in T$ $\quad \underline{t} = x_1 \underline{t}_1 + \dots + x_k \underline{t}_k$

$$\underline{x} \cdot \underline{t} = \underline{x} \cdot (x_1 \underline{t}_1 + \dots + x_k \underline{t}_k)$$

$$\overset{(2)+(3)}{=} x_1 \underbrace{\underline{x} \cdot \underline{t}_1}_{\substack{0_{\mathbb{R}} \\ \underline{x} \perp \underline{t}_1}} + x_2 \underbrace{\underline{x} \cdot \underline{t}_2}_{\substack{0_{\mathbb{R}} \\ \underline{x} \perp \underline{t}_2}} + \dots + x_k \underbrace{\underline{x} \cdot \underline{t}_k}_{\substack{0_{\mathbb{R}} \\ \underline{x} \perp \underline{t}_k}}$$

$$= x_1 \cdot 0_{\mathbb{R}} + x_2 \cdot 0_{\mathbb{R}} + \dots + x_k \, 0_{\mathbb{R}} = 0_{\mathbb{R}}$$

$\forall\, \underline{t} \in T \quad \underline{x} \cdot \underline{t} = 0_{\mathbb{R}}$ cioè $\underline{x} \perp T$

Es $\quad \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad T = \left\langle \underline{t}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \underline{t}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$\underline{t}_1$ e $\underline{t}_2$ sono LI $\Longrightarrow$ $\{\underline{t}_1, \underline{t}_2\}$ è base T

$$\underline{x} \perp T \quad \Longleftrightarrow \quad \underline{x} \cdot \underline{t}_1 = 0_{\mathbb{R}} = \underline{x} \cdot \underline{t}_2$$

$\underline{x} \cdot \underline{t}_1 = (1\,1\,0)\begin{pmatrix}0\\0\\1\end{pmatrix} = 1\cdot 0 + 1\cdot 0 + 0\cdot 1 = 0_{\mathbb{R}}$

$\underline{x} \cdot \underline{t}_2 = (1\,1\,0)\begin{pmatrix}-1\\1\\0\end{pmatrix} = 1\,(-1) + 1\cdot 1 + 0\cdot 0 = -1+1 = 0_{\mathbb{R}}$

$\Rightarrow \quad \underline{x} \perp T$

Def Una base $\{\underline{x}_1, \ldots, \underline{x}_n\}$ di $\mathbb{R}^n$ è detta

① ORTOGONALE se $\underline{x}_i \cdot \underline{x}_j = 0_{\mathbb{R}}$ per $i \neq j$

$\Leftrightarrow$ se i vettori della base sono a due a due ortogonali.

$\Leftrightarrow$ se $\underline{x}_i \perp \underline{x}_j$ per $i \neq j$

② ORTONORMALI se $\underline{x}_i \cdot \underline{x}_j = 0_{\mathbb{R}}$ per $i \neq j$

e $\underline{x}_i \cdot \underline{x}_i = 1$ per $i = 1, \ldots, n$

$\Leftrightarrow \quad \underline{x}_i \perp \underline{x}_j$ per $i \neq j$ e $\|\underline{x}_i\| = 1 \ \forall i$

$\Leftrightarrow$ i vettori della base sono a due a due ortogonali e in più essi sono dei versori

Oss $\|\underline{x}_i\| = 1 = \sqrt{\underline{x}_i \cdot \underline{x}_i} \Leftrightarrow \underline{x}_i \cdot \underline{x}_i = 1$

Oss Tutte le basi canoniche di tutti gli $\mathbb{R}^n$ sono ortonormali

$\{\underline{x}_1, \dots, \underline{x}_n\}$ è ortonormale (ON) $\Leftrightarrow$

$$\underline{x}_i \cdot \underline{x}_j = \underbrace{\delta_{ij}}_{\text{SIMBOLO DI KRONEKER}} = \begin{cases} 1 & \text{se } i=j \\ 0_{\mathbb{R}} & \text{se } i \neq j \end{cases}$$