martedì 14 novembre 2023 10:56

DISTANZA in $\mathbb{A}^3(\mathbb{R})$

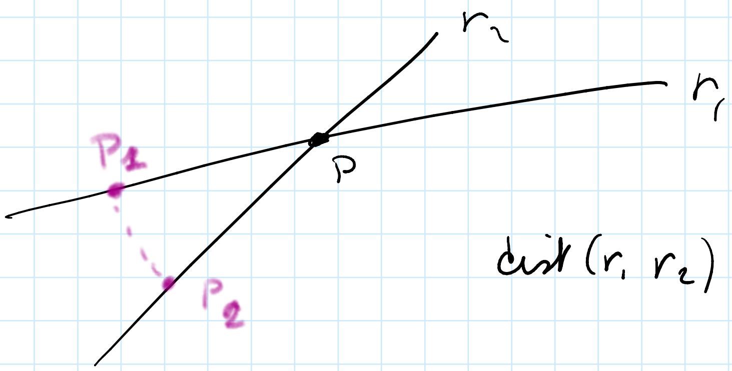
Def $\operatorname{dist}(P,Q) = \|\vec{PQ}\| = \|Q-P\|$

Def Siano $L_1 = P_1 + W_1$ e $L_2 = P_2 + W_2$ due sotto sottospazi affini di $\mathbb{A}^3(\mathbb{R})$ (cioè L_1, L_2 sono o punti, o rette, o piani)

$$\operatorname{dist}(L_1, L_2) = \min\left\{ \operatorname{dist}(Q_1,Q_2) \;\middle|\; \begin{array}{l} Q_1 \in L_1 \\ Q_2 \in L_2 \end{array} \right\}$$

Oss $\operatorname{dist}(P, L_1) = 0 \iff P \in L_1$

$\operatorname{dist}(L_1, L_2) = 0 \iff L_1 \cap L_2 \neq \emptyset$



$\operatorname{dist}(r_1, r_2) = \operatorname{dist}(P,P) = \|\vec{PP}\| = 0$

DISTANZA PUNTO PIANO

Sia $P = \begin{pmatrix} x_P \\ y_P \\ z_P \end{pmatrix}$ $\quad \pi : ax + by + cz + d = 0$

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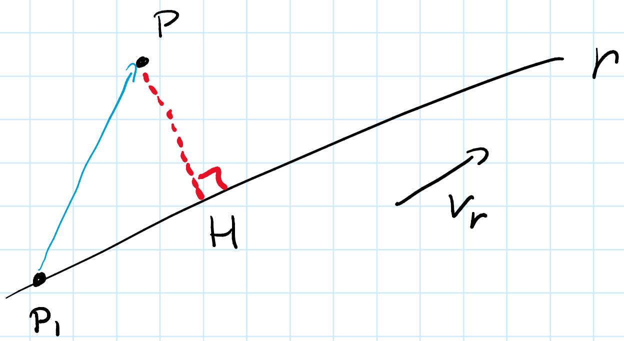
Se $P \in \pi$, $dist(P, \pi) = dist(P P) = \| \overrightarrow{PP} \| = 0$

Se $P \notin \pi$, $dist(P, \pi) \overset{def}{=} \min \{ dist(P, Q) \mid Q \in \pi \}$

$$= \frac{|a x_p + b y_p + c z_p + d|}{\sqrt{a^2 + b^2 + c^2}}$$

DISTANZA PUNTO RETTA



Se $P \in r$ $\quad dist(P, r) = 0$

Se $P \notin r$ $\quad dist(P, r) = dist(P, H) = \| \overrightarrow{PH} \|$

$$= \frac{\| \overrightarrow{PP_1} \times v_r \|}{\| v_r \|} \qquad (*)$$

Es: Calcoliamo la distanza di $P = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ dalla retta

$$r : \begin{cases} x = t \\ y = 1 + 2t \\ z = -1 + 3t \end{cases} \qquad r = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \langle \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{v_r} \rangle$$

$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad P = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in r$

v_r

$$v_r = \begin{pmatrix}1\\2\\3\end{pmatrix} \qquad P_1 = \begin{pmatrix}0\\1\\-1\end{pmatrix} \in r$$

$$\overrightarrow{PP_1} = P_1 - P = \begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\| v_r \| = \sqrt{\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}} = \sqrt{(1\,2\,3)\begin{pmatrix}1\\2\\3\end{pmatrix}} = \sqrt{1+4+9} = \sqrt{14}$$

$$\overrightarrow{PP_1} \times v_r = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{pmatrix} = e_1 \det\begin{pmatrix}1 & -1\\2 & 3\end{pmatrix} - e_2 \det\begin{pmatrix}0 & -1\\1 & 3\end{pmatrix}$$
$$+ e_3 \det\begin{pmatrix}0 & 1\\1 & 2\end{pmatrix}$$

$$= e_1 [3+2] - e_2 [+1] + e_3 [-1]$$

$$= (5, -1, -1) = 5 e_1 - e_2 - e_3$$

$$\operatorname{dist}(P, r) = \frac{\| \overrightarrow{PP_1} \times v_r \|}{\| v_r \|} = \frac{\| (5\,-1\,-1) \|}{\sqrt{14}}$$

$$\| (5\,-1\,-1) \| = \sqrt{(5\,-1\,-1) \cdot (5\,-1\,-1)} = \sqrt{(5\,-1\,-1)\begin{pmatrix}5\\-1\\-1\end{pmatrix}}$$
$$= \sqrt{25+1+1} = \sqrt{27}$$

$$\operatorname{dist}(P, r) = \frac{\sqrt{27}}{\sqrt{14}}$$

DISTANZA RETTA-RETTA in $\mathbb{A}^3(\mathbb{R})$

Siano r e s due rette

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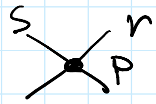
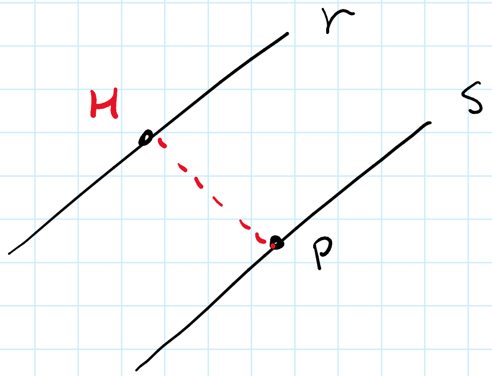
① r e s sono parallele e coincidenti, cioè $r=s$

$\Rightarrow \ dist(r,s)=0$

② r e s sono incidenti, cioè 

$\Rightarrow \ dist(r,s) = dist(P\,P) = 0$

③ r e s sono parallele, senza punti in comune

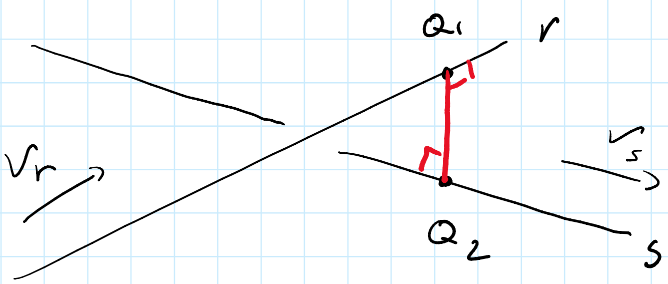


Fisso un punto P su S

e $dist(r,s) = dist(P,r)$

= distanza punto retta data da (*)

④ r e s sono sghembe:



Devo trovare $Q_1 \in r$ e $Q_2 \in s$ tale che

$$\begin{cases} \overrightarrow{Q_1Q_2} \perp r \\ \overrightarrow{Q_1Q_2} \perp s \end{cases} \iff \begin{cases} \overrightarrow{Q_1Q_2} \cdot v_r = 0 \\ \overrightarrow{Q_1Q_2} \cdot v_s = 0 \end{cases}$$

Q_1, Q_2 sono detti PUNTI DI MINIMA DISTANZA

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La retta passante per Q_1 e Q_2 è detta RETTA DI MINIMA DISTANZA

$$dist(r,s) = dist(Q_1\, Q_2) = \| \overrightarrow{Q_1 Q_2} \|$$

Es $r: \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + < \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} >$ $\quad s: \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + < \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} >$

$$r: \begin{cases} x = 1 + \alpha \\ y = 2 \\ z = 0 \end{cases} \qquad s: \begin{cases} x = 1 + \beta \\ y = 3 + \beta \\ z = 4 + \beta \end{cases}$$

Un generico punto di r è $P_r = \begin{pmatrix} 1+\alpha \\ 2 \\ 0 \end{pmatrix}$

Un generico punto di s è $P_s = \begin{pmatrix} 1+\beta \\ 3+\beta \\ 4+\beta \end{pmatrix}$

$$\overrightarrow{P_r P_s} = P_s - P_r = \begin{pmatrix} 1+\beta \\ 3+\beta \\ 4+\beta \end{pmatrix} - \begin{pmatrix} 1+\alpha \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta-\alpha \\ 1+\beta \\ 4+\beta \end{pmatrix}$$

Impongo l'ortogonalità di $\overrightarrow{P_r P_s}$ ad entrambe le rette r e s.

$$\overrightarrow{P_r P_s} \perp r \iff \overrightarrow{P_r P_s} \cdot v_r = 0$$

$$\iff \begin{pmatrix} \beta-\alpha \\ 1+\beta \\ 4+\beta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} \beta-\alpha & 1+\beta & 4+\beta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\iff \beta - \alpha = 0$$

$$\iff \beta = \alpha$$

$\overrightarrow{P_r P_s} \perp s \iff \overrightarrow{P_r P_s} \cdot v_s = 0$

$$\iff \begin{pmatrix} \beta-\alpha & 1+\beta & 4+\beta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\iff \underline{\beta} - \alpha + 1 + \underline{\beta} + 4 + \underline{\beta} = 0$$

$$\iff 3\beta - \alpha + 5 = 0$$

$$\begin{cases} \beta = \alpha \\ 3\beta - \alpha + 5 = 0 \end{cases} \qquad \begin{cases} \beta = \alpha \\ 3\alpha - \alpha + 5 = 0 \end{cases}$$

$$\begin{cases} \beta = \alpha \\ 2\alpha + 5 = 0 \end{cases} \qquad \begin{cases} \beta = \alpha \\ \alpha = \frac{-5}{2} \end{cases} \qquad \begin{cases} \beta = -\frac{5}{2} \\ \alpha = -\frac{5}{2} \end{cases}$$

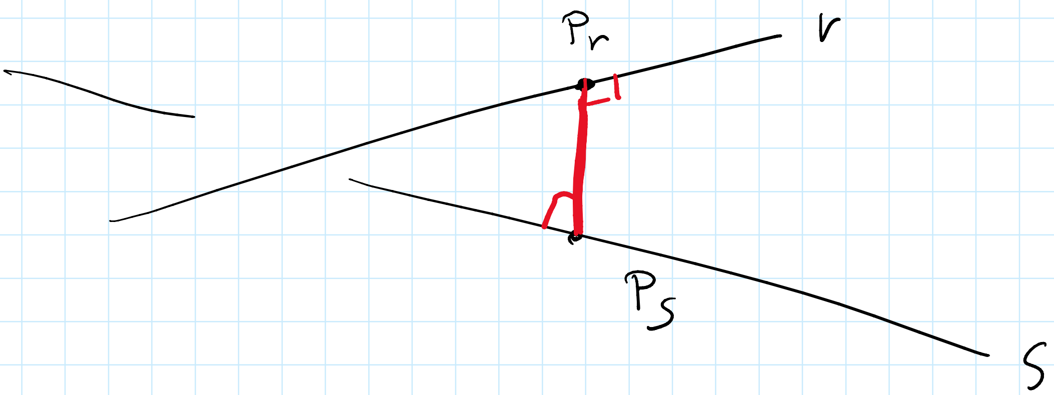
Metto $\beta = -\frac{5}{2}$ e $\alpha = -\frac{5}{2}$ in $P_r = \begin{pmatrix} 1+\alpha \\ 2 \\ 0 \end{pmatrix}$

e $P_s = \begin{pmatrix} 1+\beta \\ 3+\beta \\ 4+\beta \end{pmatrix}$

$$P_r = \begin{pmatrix} 1 - \frac{5}{2} \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 2 \\ 0 \end{pmatrix} \qquad P_s = \begin{pmatrix} 1 - \frac{5}{2} \\ 3 - \frac{5}{2} \\ 4 - \frac{5}{2} \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1/2 \\ 3/2 \end{pmatrix}$$

P_r — r

$4-\frac{3}{2}$ $\quad 3/2$



$P_r = \begin{pmatrix} -3/2 \\ 2 \\ 0 \end{pmatrix}$ e $P_s = \begin{pmatrix} -3/2 \\ 1/2 \\ 3/2 \end{pmatrix}$ sono i punti di minima distanza tra r e s.

$$\text{dist}(r,s) = \|\overrightarrow{P_r P_s}\| = \|P_s - P_r\|$$

$$\overrightarrow{P_r P_s} = P_s - P_r = \begin{pmatrix} -3/2 \\ 1/2 \\ 3/2 \end{pmatrix} - \begin{pmatrix} -3/2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/2 + 3/2 \\ 1/2 - 2 \\ 3/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix}$$

$$\text{dist}(r,s) = \left\| \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix} \right\| = \sqrt{\begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix}}$$

$$\overset{\text{def}}{=} \sqrt{\begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix}^T \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix}}$$

$$= \sqrt{\begin{pmatrix} 0 & -3/2 & 3/2 \end{pmatrix} \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix}}$$

$$= \sqrt{\begin{pmatrix} 0 & -3/2 & 3/2 \end{pmatrix} \begin{pmatrix} \\ -3/2 \\ 3/2 \end{pmatrix}}$$

$$= \sqrt{\frac{9}{4} + \frac{9}{4}}$$

$$= \sqrt{\frac{18}{4}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$$

La retta di minima distanza è

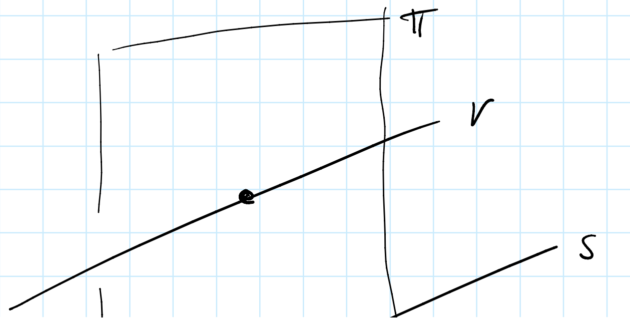
$$P_r + \langle \overrightarrow{P_r P_s} \rangle \quad \text{oppure} \quad P_s + \langle \overrightarrow{P_r P_s} \rangle$$

$$r : \begin{pmatrix} -3/2 \\ 2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix} \right\rangle \quad \text{oppure} \quad \begin{pmatrix} -3/2 \\ 1/2 \\ 3/2 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix} \right\rangle$$

$$\begin{cases} x = -3/2 \\ y = 2 - \frac{3}{2} t \\ z = \frac{3}{2} t \end{cases} \quad \text{oppure} \quad \begin{matrix} x = -3/2 \\ y = 1/2 - 3/2\, t \\ z = 3/2 + 3/2\, t \end{matrix}$$

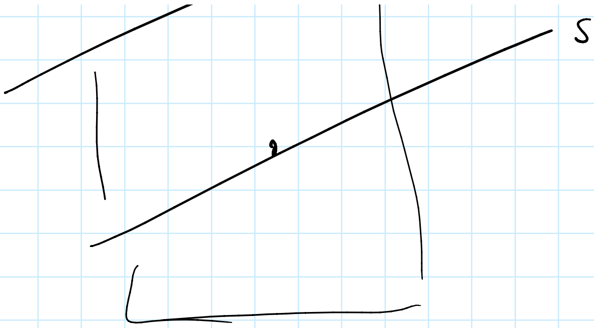
$$r : \begin{pmatrix} -3/2 \\ 2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix} \right\rangle = \begin{pmatrix} -3/2 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3/2 \\ 3/2 \end{pmatrix}$$

<u>Oss</u>



$\pi \perp r$

$\pi \perp s$



$\pi \cap r = P$

$\pi \cap s = Q$

$dist(r, s) = \| \vec{PQ} \|$

<u>Es</u>

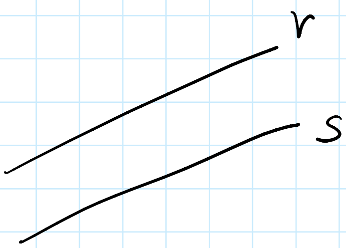
$r: P + \langle v_r \rangle \qquad P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad v_r = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$s: Q + \langle v_s \rangle \qquad Q = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad v_s = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

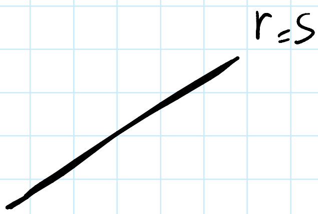
Determinare se possibile un piano π tale che $r \subseteq \pi$ e $s \subseteq \pi$

$\Longleftrightarrow$

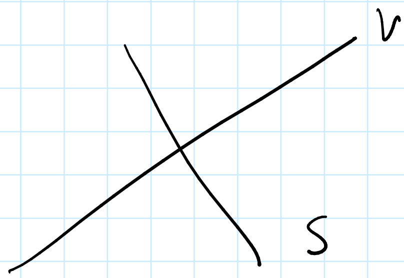
Determinare se r e s sono complanari



parallele senza punti in comune

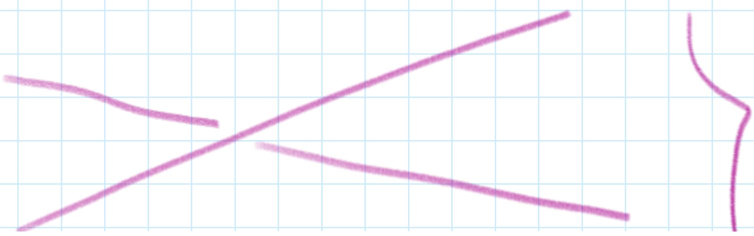


parallele e coincidenti

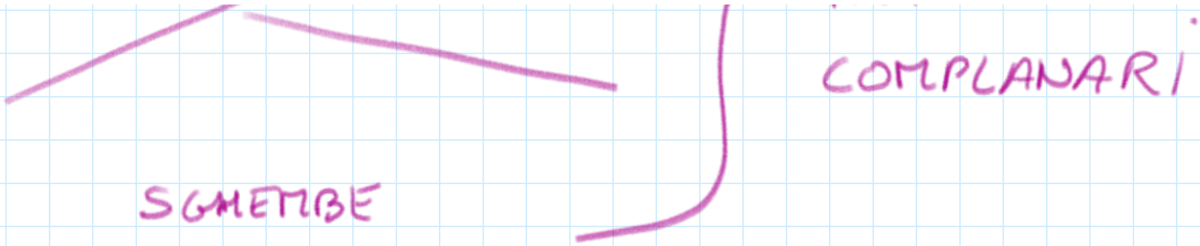


incidenti

COMPLANARI



NON COMPLANARI



$\exists$ un tale π $\iff$ r e s sono o parallele senza punti in comune o parallele coincidenti o incidenti

$$v_r = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_s = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \implies v_r \not\parallel v_s$$

$$\iff v_s \in \langle v_r \rangle$$

$$\iff v_s \text{ e } v_r \text{ sono LI}$$

r e s non sono parallele.

Verifichiamo se sono incidenti

r e s incidenti $\iff \exists \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in r \cap s$

$$\iff \underbrace{\begin{aligned} x &= 1 + \lambda \\ y &= 1 \\ z &= 0 - \lambda \end{aligned}}_{\text{eq. parametriche di } r} \quad \begin{aligned} &= \\ &= \\ &= \end{aligned} \quad \underbrace{\begin{aligned} &3 + 2t \\ &2 + t \\ &1 + t \end{aligned}}_{\text{eq. parametriche di } s} \qquad \text{per } \lambda, t \in \mathbb{R}$$

$$P + \langle v_r \rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \qquad Q + \langle v_s \rangle = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$P+\langle v_r\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \langle \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $\qquad$ $Q+\langle v_s\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}\rangle$

$$\begin{cases} 1+\lambda = 3+2r \\ 1 = 2+r \\ -\lambda = 1+r \end{cases} \Rightarrow r = 1-2 = -1 \quad \Rightarrow -\lambda = 0$$

Per $r=-1$ e $\lambda = 0$ verifichiamo se ho $1+\lambda = 3+2r$

$$1+0 = 3+2(-1) = 3-2 = 1 \quad \text{OK!}$$

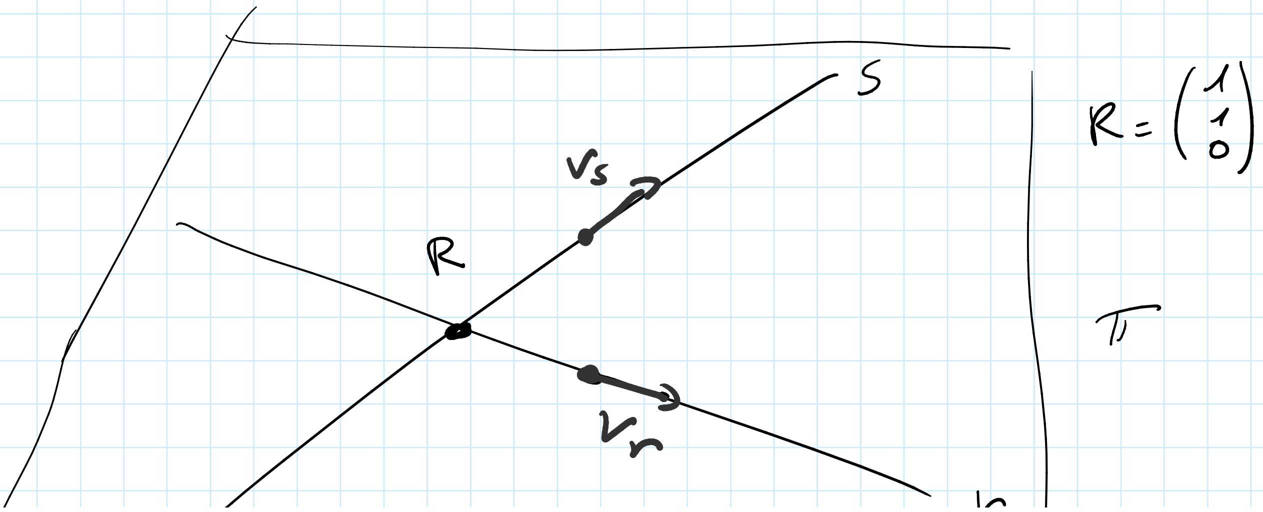
$$\lambda = 0 \Rightarrow \begin{cases} x = 1+\lambda = 1 \\ y = 1 \quad = 1 \\ z = -\lambda \quad = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in r$$

$$r = -1 \Rightarrow \begin{cases} x = 3+2r = 3-2 = 1 \\ y = 2+r = 2-1 = 1 \\ z = 1+r = 1-1 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in s$$

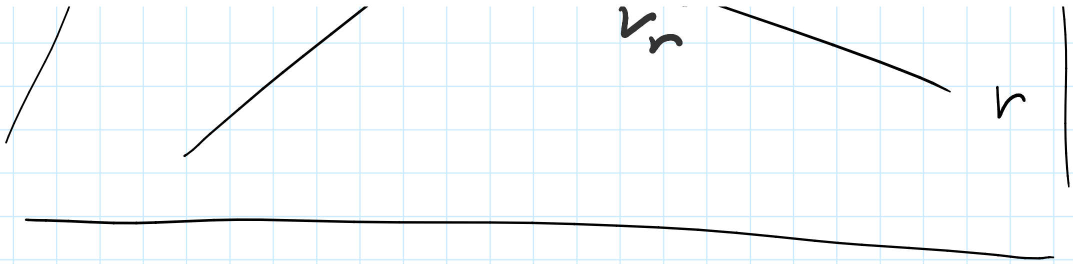
$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in r \cap s \Rightarrow$ r e s sono incidenti

$\Rightarrow$ r e s sono complanari

$\Rightarrow \exists\, \pi$ piano t.c. $\pi \supseteq r$ e $\pi \supseteq s$



$R = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$



$\pi : \quad R + \langle v_s, v_r \rangle$

$$\pi : \begin{pmatrix}1\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix} \right\rangle = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}1\\0\\-1\end{pmatrix} + \beta \begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1 + \lambda + 2\beta \\ 1 + \beta \\ -\lambda + \beta\end{pmatrix}$$

$$\begin{cases} x = 1 + \lambda + 2\beta \\ y = 1 + \beta \\ z = -\lambda + \beta \end{cases}$$

eq. parametriche di π

<u>Es</u> $\quad r : \begin{pmatrix}1\\0\\1\end{pmatrix} + \left\langle \begin{pmatrix}2\\1\\0\end{pmatrix} \right\rangle \quad$ Sia $v = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ vettore di $\mathbb{R}^3$

Determinare il piano π tale che la retta r sia contenuta in π e π sia parallelo a v.

Eq. parametriche $\begin{cases} x = 1 + 2\lambda + \mu \\ y = \lambda \\ z = 1 - \mu \end{cases}$

Eq. cartesiana. $x - 2y + z - 2 = 0$.