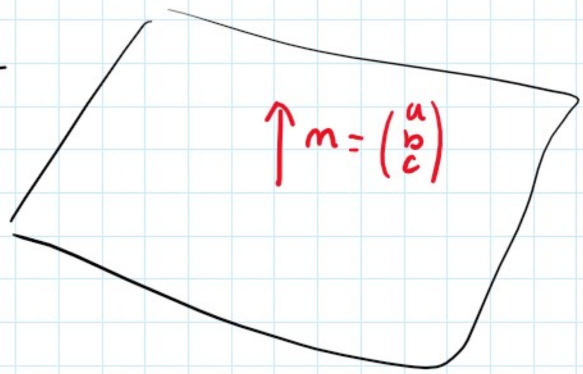
martedì 14 novembre 2023 10:56

Un piano in $\mathbb{A}^3(\mathbb{R})$ ha eq. cartesiana

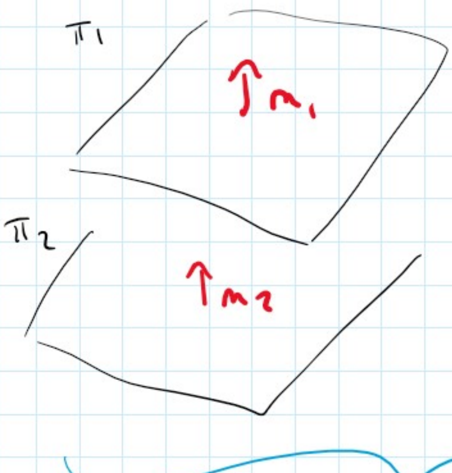
$$ax + by + cz = d$$

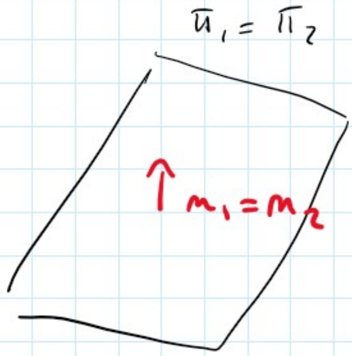


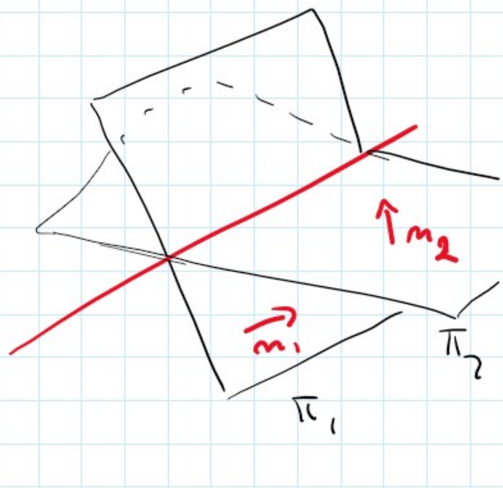
$n \perp \pi$

$\pi_1 \quad a_1 x + b_1 y + c_1 z = d_1$ $\qquad n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \perp \pi_1$

$\pi_2 \quad a_2 x + b_2 y + c_2 z = d_2$ $\qquad n_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \perp \pi_2$







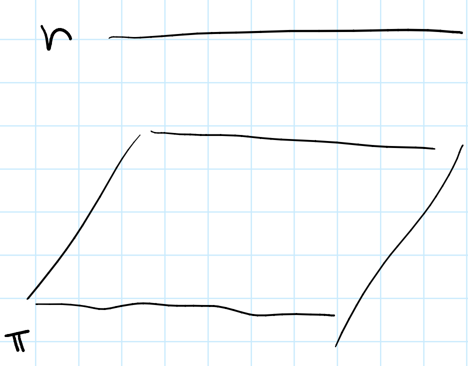
$$= rg \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 1$$

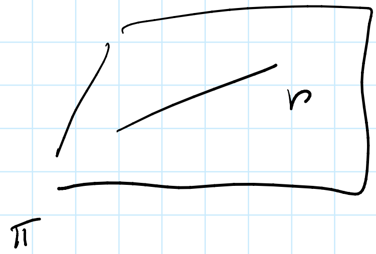
$$= rg \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

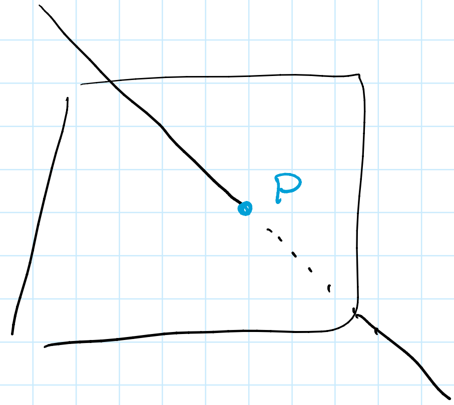
$$rg \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$$

$*$

POSIZIONE RECIPROCA DI RETTA E PIANO







$r \,//\, \pi$ senza punti in comune

$r \subseteq \pi$

$P = r \cap \pi$

Sia $r \begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$

Per ⊛ $\operatorname{rg}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$

$\pi \quad a_0 x + b_0 y + c_0 z = d_0$

Calcolare $r \cap \pi$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in r \cap \pi \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in Sol(\Sigma)$ con $\Sigma: \; A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = D$

dove $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_0 & b_0 & c_0 \end{pmatrix} \qquad D = \begin{pmatrix} d_1 \\ d_2 \\ d_0 \end{pmatrix}$

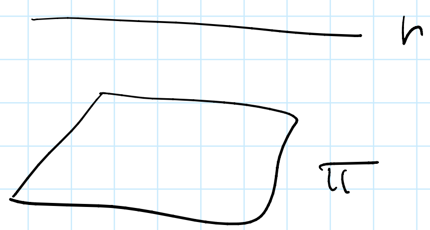
$\iff \begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases} \Big\} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in r$

$$\iff \left\{\begin{array}{ll} a_1x+b_1y+c_1z=d_1 & \rbrace \ \begin{pmatrix}x\\y\\z\end{pmatrix}\in r \\ a_2x+b_2y+c_2z=d_2 & \\ a_0x+b_0y+c_0z=d_0 & \rbrace \ \begin{pmatrix}x\\y\\z\end{pmatrix}\in \pi \end{array}\right.$$

Poiché $rg\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2 \implies rg(A) = rg\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_0 & b_0 & c_0 \end{pmatrix} \geq 2$

① $rg(A) = 2$

(1.1) $rg(A|D) = 3 \overset{RC}{\implies} \nexists$ soluzioni di Σ

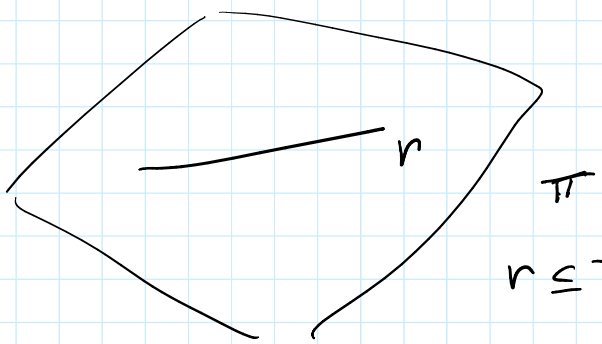


$\implies r \,//\, \pi$ senza punti in comune

(1.2) $rg(A|D) = 2 \overset{R.C.}{\implies} \exists\, \infty$ soluzioni di Σ che dipendono da

$$\underbrace{3}_{\substack{\text{numero delle incognite di } \Sigma \\ x,\,y,\,z}} - \underbrace{2}_{rg(A) = rg(A|D)} = 1 \text{ parametri}$$

$\implies r \cap \pi = r$



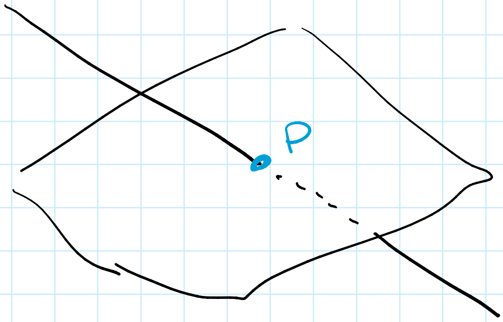
$r \subseteq \pi$

② $rg(A) = 3 \Rightarrow rg(A|D) = 3$

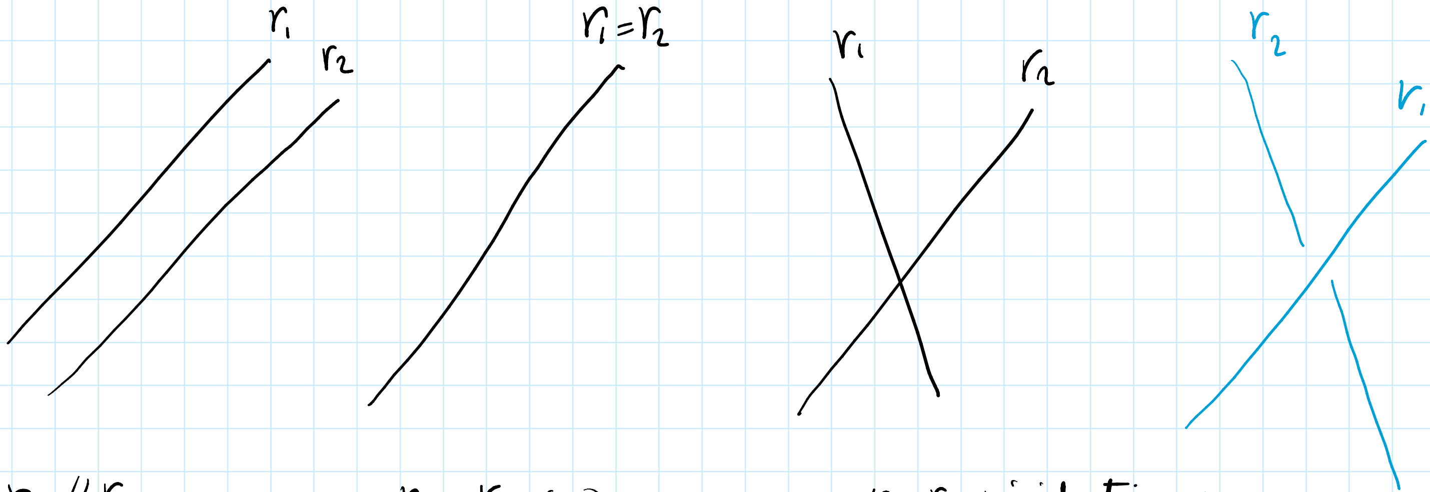
$\overset{RC}{\Rightarrow} \exists!$ soluzione di Σ

$\Rightarrow r \cap \pi = \{\text{punto}\}$

$= \{P\}$



POSIZIONE RECIPROCA DI 2 RETTE



$r_1 // r_2$

senza punti in comune

$r_1 = r_2 \iff$
r_1 e r_2 coincidenti $\iff$
$r_1 // r_2$ e hanno
∞ punti in comune

r_1 e r_2 incidenti
$\iff$
$r_1 \cap r_2 = \{\text{punto}\}$

r_1 e r_2
sghembe
$\iff$
non hanno punti in comune e $r_1 \not\parallel r_2$

COMPLANARI
$\iff$
$\exists$ un piano π che contiene r_1 e r_2

NON COMPLANARI
$\iff$
$\nexists$ un piano π
che contiene r_1 e r_2

che contiene r_1 e r_2

Siano $r \begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{cases}$

per ⊛ $rg\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}=2$

$s \begin{cases} a_3x+b_3y+c_3z=d_3 \\ a_4x+b_4y+c_4z=d_4 \end{cases}$

per ⊛ $rg\begin{pmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix}=2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in r \cap s \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in Sol(\Sigma)$ con $\Sigma: A\begin{pmatrix} x \\ y \\ z \end{pmatrix}=D$

dove $A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix}$ e $D=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$

Per ⊛ $rg(A) \geq 2$

① Se $rg(A)=2$

(1.1) $rg(A|D)=3 \overset{RC}{\Longrightarrow} \nexists$ soluzioni di Σ

$\Rightarrow r \parallel s$

$\iff$ parallele senza punti in comune.

(1.2) $rg(A|D)=2 \overset{R.C.}{\Longrightarrow} \exists\, \infty$ soluzioni di Σ

(1.2) $rg(A|D) = 2 \overset{R.C.}{\Longrightarrow} \exists\ \infty$ soluzione di Σ parametrizzate da $\underbrace{3}_{\substack{\text{incognite}\\ x,y,z}} - \underbrace{2}_{rg(A) = rg(A|D)} = 1$ parametri liberi

$\Rightarrow r \cap s =$ retta

$\Rightarrow r = s$

$\Leftrightarrow$ parallele con ∞ punti in comune.

② $rg(A) = 3$

($rg(A)$ non può essere 4 poiché in $\mathbb{R}^3$ ho al massimo 3 vettori LI)

(2.1) $rg(A|D) = 3 \overset{R.C.}{\Longrightarrow} \exists!$ soluzione

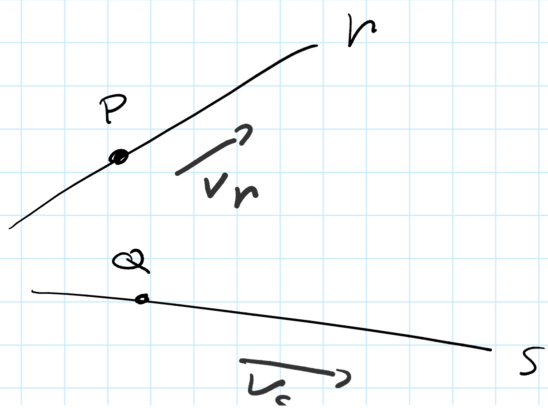
$\Rightarrow r \cap s = \{$punto$\}$

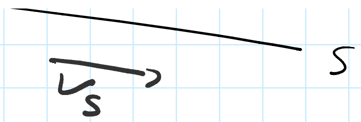
(2.2) $rg(A|D) = 4 \overset{R.C.}{\Longrightarrow} \nexists$ soluzioni

$\Rightarrow r$ e s sono sghembe.

Oss $r: P + \langle v_r \rangle$

$s: Q + \langle v_s \rangle$





(1) $r // s \iff v_r // v_s$

(2) r e s complanari $\iff$ r e s incidenti o
$//$ senza pt in comune o
$//$ e uguali

(3) r e s non complanari $\iff$ r e s sghembe

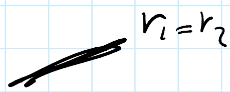
Es $r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle \qquad r_2 = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \left\langle \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} \right\rangle$

(1) Determinare la posizione reciproca di r_1 e r_2

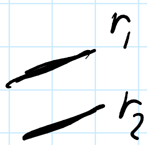
$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$

$$v_2 = -2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -2\, v_1 \iff v_1 // v_2$$
$$\iff v_1 \in \langle v_2 \rangle$$
$$\iff v_2 \in \langle v_1 \rangle$$

Ho 2 possibilità $\quad r_1 // r_2$ e $r_1 = r_2$



$r_1 // r_2$ e $r_1 \cap r_2 = \emptyset$



Per capire in che caso sono, basta controllare
se esiste $P \in r_1 \cap r_2$
se per $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in r_1 \quad \exists \lambda$ t.c. $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$

se per $\begin{pmatrix}1\\1\\0\end{pmatrix} \in r_1$ $\quad \exists \lambda$ t.c. $\underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\1\\-1\end{pmatrix} + \lambda \begin{pmatrix}-2\\-2\\2\end{pmatrix}}_{\begin{pmatrix}1\\1\\0\end{pmatrix} \in r_2 \,:\, \begin{pmatrix}3\\1\\-1\end{pmatrix} + \left\langle \begin{pmatrix}-2\\-2\\2\end{pmatrix} \right\rangle}$

se $\begin{pmatrix}1\\1\\0\end{pmatrix} \in r_1 \cap r_2$

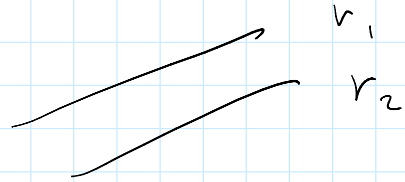
$$\begin{cases} 1 = 3 - 2\lambda \\ 1 = 1 - 2\lambda \\ 0 = -1 + 2\lambda \end{cases} \quad \Rightarrow \quad \lambda = \frac{1}{2}$$

$$\Rightarrow 1 = 1 - 2\,\frac{1}{2} = 1 - 1 = 0$$

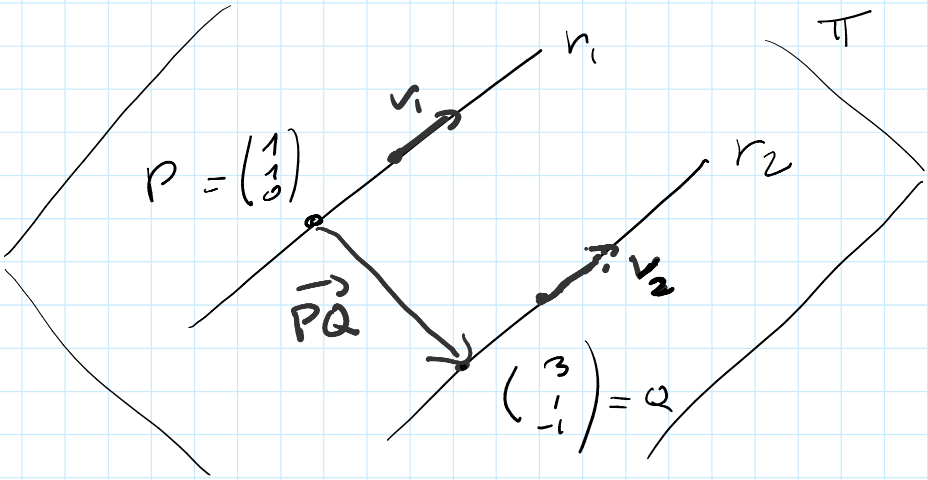
$$1 = 0 \quad \lightning$$

$\Rightarrow \;\nexists \lambda$ t.c. $\begin{pmatrix}1\\1\\0\end{pmatrix} \in r_2$

$\Rightarrow \; r_1 \,//\, r_2$ senza punti in comune.



② Determinare il piano π che contiene r_1 e r_2



$r_1 = \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{P} + \left\langle \underbrace{\begin{pmatrix}1\\1\\-1\end{pmatrix}}_{v_1} \right\rangle$

$r_2 = \underbrace{\begin{pmatrix}3\\1\\-1\end{pmatrix}}_{Q} + \left\langle \underbrace{\begin{pmatrix}-2\\-2\\2\end{pmatrix}}_{v_2} \right\rangle$

$\pi : \; P + \underbrace{\langle v_1 , \overrightarrow{PQ} \rangle}_{\dim 2}$ (non posso prendere v_1 e v_2 poiché $v_1 // v_2$)

dim 2 (non posso prendere v_1 e v_2 poiché $v_1 // v_2$)

$$\vec{PQ} = Q - P = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

Eq parametriche di π sono
$$\begin{cases} x = 1 + \lambda + 2\mu & (1) \\ y = 1 + \lambda & (2) \\ z = 0 - \lambda - \mu & (3) \end{cases}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = P + \lambda v_1 + \mu \vec{PQ}$$

Per trovare le eq. cartesiane si osserva che

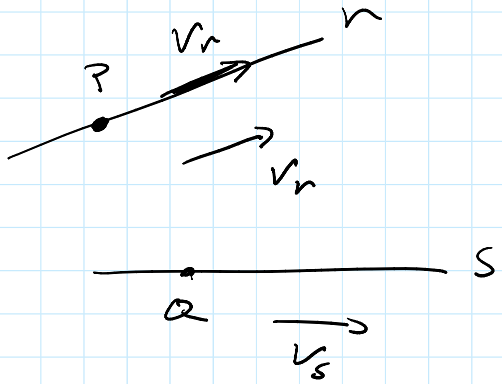
$$\vec{PX} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - P = \lambda v_1 + \mu \vec{PQ} \iff$$
$$\in \langle v_1, \vec{PQ} \rangle$$

$$\det \begin{pmatrix} x-1 & y-1 & z \\ 1 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix} = 0 \quad \begin{matrix} \vec{Px} \\ v_1 \\ \vec{PQ} \end{matrix} \iff \operatorname{rg} \underbrace{\begin{pmatrix} \vec{Px} \\ v_1 \\ \vec{PQ} \end{pmatrix}}_{= 2} \leq 2$$

Altro metodo: Pongo $\underset{(2)}{\lambda = y - 1}$ e $\underset{(3)}{\mu = -\lambda - z}$

$(1) \Rightarrow$
$$x = 1 + \lambda + 2\mu$$
$$x = \cancel{1} + y \cancel{-1} + 2(-\lambda - z)$$
$$x = y - 2\lambda - 2z$$
$$x \overset{(2)}{=} y - 2(y-1) - 2z$$

$$x = y - 2y + 2 - 2z$$
$$x = -y + 2 - 2z$$
$$x + y + 2z = 2$$ eq. cartesiana di π.

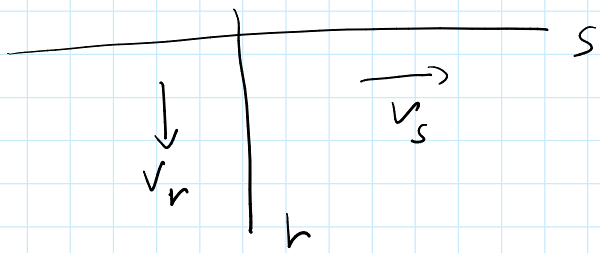


Def $r = P + \langle v_r \rangle$

$s = Q + \langle v_s \rangle$

s e r sono ortogonali $\iff$

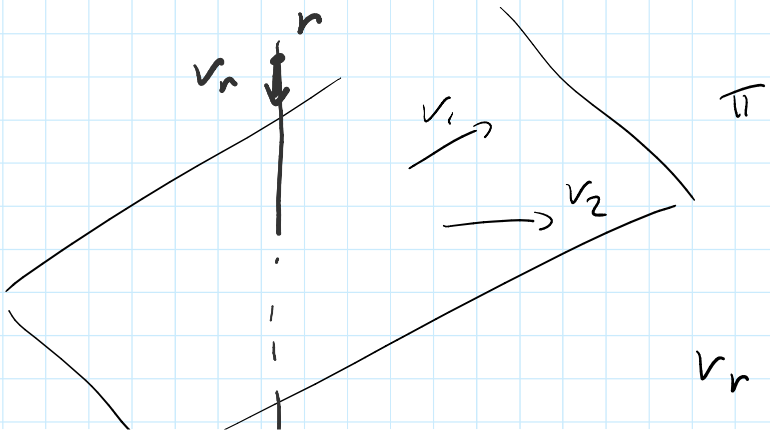
$s \perp r \iff v_r \cdot v_s = 0 \iff v_r \perp v_s$



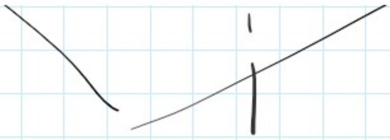
Def $r = P + \langle v_r \rangle \qquad \pi : Q + \langle v_1, v_2 \rangle$

r e π sono ortogonali $\iff r \perp \pi \iff$

$$\begin{cases} v_r \cdot v_1 = 0 \\ v_r \cdot v_2 = 0 \end{cases}$$



$v_r \in \langle v_1 \times v_2 \rangle$

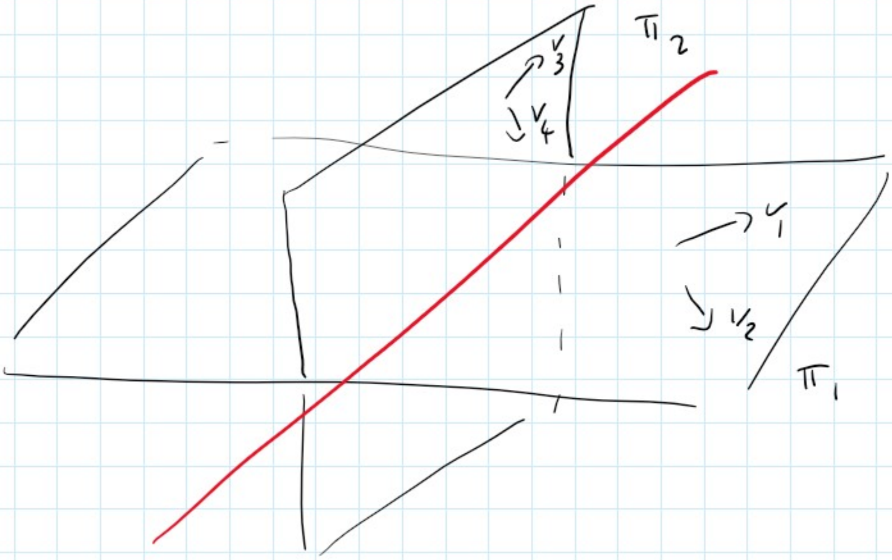


$v_r \in \langle v_1 \times v_2 \rangle$

Def $\pi_1 = P + V_{\pi_1}$ $\quad \dim V_{\pi_1} = 2$

$\pi_2 = Q + V_{\pi_2}$ $\quad \dim V_{\pi_2} = 2$

π_1 e π_2 sono ortogonali se $V_{\pi_1} \perp V_{\pi_2}$



$V_{\pi_1} = \langle v_1, v_2 \rangle$

$V_{\pi_2} = \langle v_3, v_4 \rangle$

Oss Sia $\pi : ax + by + cz = d$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \perp \pi$$

$$\pi : P + V_\pi \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V_\pi^\perp$$

Oss $\pi_1 \quad a_1 x + b_1 y + c_1 z = d_1$ $\qquad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \perp \pi_1$

$\pi_2 \qquad a_2 x + b_2 y + c_2 z = d_2 \qquad \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \perp \pi_2$

$$\pi_1 \perp \pi_2 \iff \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \perp \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$