$\underline{Es}$: $f: \mathbb{R}[x]^{\leq 2} \longrightarrow \mathbb{R}^3$

$$a + \underline{b}x + \underline{c}x^2 \longrightarrow \begin{pmatrix} b+c \\ b+c \\ b+c \end{pmatrix}$$

① Verificare che f è lineare

Devo verificare che se $p_1(x) = a + bx + cx^2$ e $p_2(x) = d + ex + gx^2 \in \mathbb{R}[x]^{\leq 2}$ e $\lambda_1, \lambda_2 \in \mathbb{R}$

$$f(\lambda_1 p_1(x) + \lambda_2 p_2(x)) = \lambda_1 f(p_1(x)) + \lambda_2 f(p_2(x))$$

Def $f: V \longrightarrow W$ è lineare $\Longleftrightarrow$

$f(v_1 + v_2) = f(v_1) + f(v_2) \quad \forall v_1, v_2 \in V$

e

$f(\lambda v) = \lambda f(v) \quad \forall v \in V$ e $\forall \lambda \in \mathbb{R}$

$\Longleftrightarrow \quad f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$

$\forall v_1, v_2 \in V$ e $\lambda_1, \lambda_2 \in \mathbb{R}$

$$f(\lambda_1 p_1(x) + \lambda_2 p_2(x)) =$$

$$f(\lambda_1 (a + bx + cx^2) + \lambda_2 (d + ex + gx^2))$$

$$f((\lambda_1 a + \lambda_2 d) + (\underline{\underline{\lambda_1 b + \lambda_2 e}})x + (\underline{\underline{\lambda_1 c + \lambda_2 g}})x^2)$$

$$= \begin{pmatrix} \lambda_1 b + \lambda_2 e + \lambda_1 c + \lambda_2 g \\ \lambda_1 b + \lambda_2 e + \lambda_1 c + \lambda_2 g \\ \lambda_1 b + \lambda_2 e + \lambda_1 c + \lambda_2 g \end{pmatrix} =$$

$$\overset{?}{=} \lambda_1 f(p_1(x)) + \lambda_2 f(p_2(x))$$

$$\lambda_1 f(a + \underline{b}x + \underline{c}x^2) + \lambda_2 f(d + \underline{e}x + \underline{g}x^2)$$

$$= \lambda_1 \begin{pmatrix} b+c \\ b+c \\ b+c \end{pmatrix} + \lambda_2 \begin{pmatrix} e+g \\ e+g \\ e+g \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1(b+c) + \lambda_2(e+g) \\ \lambda_1(b+c) + \lambda_2(e+g) \\ \lambda_1(b+c) + \lambda_2(e+g) \end{pmatrix} =$$

② **Calcolare la matrice associata ad f rispetto alle basi canoniche del dominio e del codominio.**

$$f : \underset{\mathcal{B}}{\mathbb{R}[x]^{\leq 2}} \longrightarrow \underset{\mathcal{C}}{\mathbb{R}^3} \qquad a + \underline{b}x + \underline{c}x^2 \longrightarrow \begin{pmatrix} b+c \\ b+c \\ b+c \end{pmatrix}$$

Base canonica di $\mathbb{R}[x]^{\leq 2}$ è $\{1, x, x^2\} = \mathcal{B}$

Base canonica di $\mathbb{R}^3$ è $\{e_1 = (100), e_2 = (010), e_3 = (001)\} = \mathcal{C}$

$$M_{\mathcal{B}}^{\mathcal{C}}(f) = \begin{pmatrix} 0 & \vdots & \vdots \\ 0 & \vdots & \vdots \\ 0 & \vdots & \vdots \end{pmatrix}$$

$$(f(1))_{\mathcal{C}} \quad (f(x))_{\mathcal{C}} \quad (f(x^2))_{\mathcal{C}}$$

$f(1) = f(1 + 0x + 0x^2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad f(x) = f(0 + 1x + 0x^2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$f(1) = f(1 + \underline{0}x + \underline{0}x^2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad f(x) = f(0 + \underline{1}x + \underline{0}x^2) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$f(x^2) = f(0 + \underline{0}X + \underline{1}x^2) = \begin{pmatrix} 0+1 \\ 0+1 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

③ Dire se f è iniettiva, suriettiva, biiettiva.

$f : V \longrightarrow W$

f iniettiva $\iff$ $\ker f = \{ v \in V \mid f(v) = \vec{0}_W \} = \vec{0}_V$

$\ker f = Sol(M^e_{\mathcal{B}}(f)) = \{ X \mid M^e_{\mathcal{B}}(f) X = \vec{0} \}$

f suriettiva $\iff$ $\operatorname{Im} f = W$

$\dim V = \dim \ker f + \underbrace{\dim \operatorname{Im} f}_{= rk(M^e_{\mathcal{B}}(f))}$

$\operatorname{Im} f = \langle f(1), f(x), f(x^2) \rangle$

$= \left\langle \cancel{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}, \cancel{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

$= \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ è base di $\operatorname{Im} f$ e $\dim \operatorname{Im} f = 1$

$\operatorname{Im} f \subseteq \mathbb{R}^3$, $\dim \mathbb{R}^3 = 3$, $\dim \operatorname{Im} f = 1$

$\Rightarrow \operatorname{Im} f \subsetneq \mathbb{R}^3 \Rightarrow f$ non è suriettiva.

$\Rightarrow \quad \mathrm{Im}\, f \subsetneq \mathbb{R}^3 \quad \Rightarrow f$ non è suriettiva.

$$\underbrace{\dim \mathbb{R}[x]^{\leq 2}}_{3} = \underbrace{\dim \mathrm{Im}\, f}_{1} + \underbrace{\dim \mathrm{Ker}\, f}_{\Downarrow \; 2}$$

$\dim \mathrm{Ker}\, f = 2 \Rightarrow f$ non è iniettiva

$\Big(f$ iniettiva $\Leftrightarrow \mathrm{Ker}\, f = \{\vec{0}_{\mathbb{R}[x]^{\leq 2}}\}$

$\Leftrightarrow \dim \mathrm{Ker}\, f = 0 \Big)$

$$\mathrm{Ker}\, f = \left\{ p(x) = a + bx + cx^2 \;\middle|\; f(p(x)) = \begin{pmatrix} b+c \\ b+c \\ b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$p(x) \in \mathrm{Ker}\, f \Leftrightarrow b + c = 0$

$\Leftrightarrow c = -b$

$$= \{ p(x) = a + bx + (-b)x^2 \}$$

$$= \{ p(x) = a + b(x - x^2) \} = \langle 1, x - x^2 \rangle$$

Una base di $\mathrm{Ker}\, f$ è $\{1, x - x^2\}$

$$\mathrm{Ker}\, f = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\mathcal{B}} \;\middle|\; \underline{\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$x_2 + x_3 = 0$$

$$x_3 = -x_2$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ -x_2 \end{pmatrix}_{\mathcal{B}} \right\} = \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}_{\mathcal{B}} \right\}$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}_{\mathcal{B}} \right\rangle$$

$$\mathcal{B} = \{1, x, x^2\}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} = 1 \cdot 1 + 0x + 0x^2 = 1$$

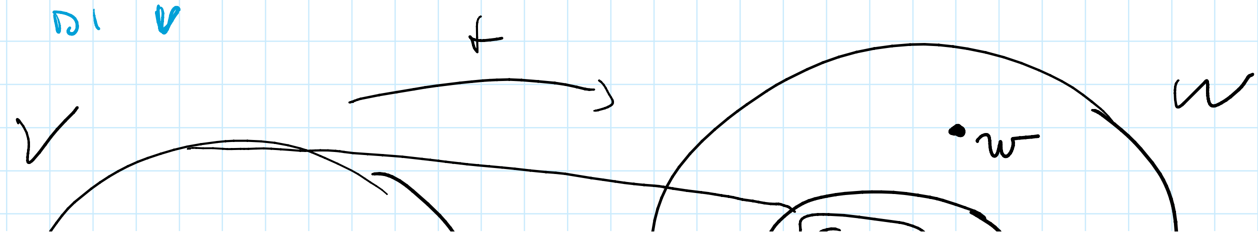
$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}_{\mathcal{B}} = 0 \cdot 1 + 1 \cdot x + (-1)x^2 = x - x^2$$

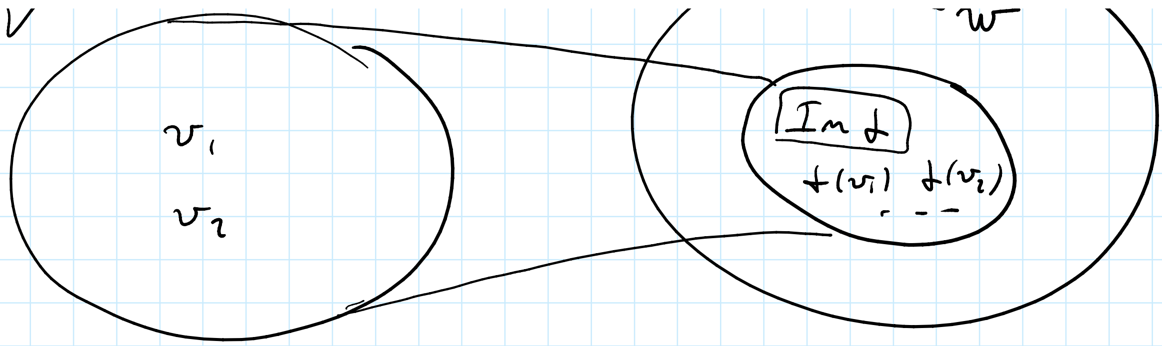
$$\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\mathcal{B}} = 1, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}_{\mathcal{B}} = x - x^2 \right\rangle$$

$f: V \longrightarrow W$, $w \in W$, $w \neq 0_W$

$$f^{-1}(w) = \overleftarrow{f}(w) = \begin{cases} \emptyset & \text{se } w \notin \mathrm{Im} f \\ v + \mathrm{Ker} f & \text{se } w \in \mathrm{Im} f \text{ con } f(v) = w \end{cases}$$

$f^{-1}(w)$ NON È UN SOTTOSPAZIO VETTORIALE DI V





④ Calcolare $f^{-1}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}\right)$

Mi domando se $\begin{pmatrix}1\\2\\0\end{pmatrix} \in \mathrm{Im}\, f = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$?

$$= \left\{ \begin{pmatrix}\lambda\\\lambda\\\lambda\end{pmatrix} \right\}$$

No e dunque $f^{-1}\begin{pmatrix}1\\2\\0\end{pmatrix} = \emptyset$.

$v_1, v_2 \in V \quad | \quad f(v_1) = f(v_2)$

$$f(v_1) - f(v_2) = 0_W$$

$$f(v_1 - v_2) = 0_W$$

Se $\mathrm{Ker}\, f = 0_V \implies v_1 - v_2 = 0_V$

$$\implies v_1 = v_2$$

---

⑤ Calcolare $f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \left\{ p(x) \;\middle|\; f(p(x)) = \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$

Mi domando se $\begin{pmatrix}1\\1\\1\end{pmatrix} \in \mathrm{Im}\, f = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$?

$$= \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}, \begin{pmatrix}3\\3\\3\end{pmatrix} \dots \right\}$$

Sì $\begin{pmatrix}1\\1\\1\end{pmatrix} \in \mathrm{Im}\, f$ Dunque $f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \bar{v} + \mathrm{Ker}\, f$

Sì, $\begin{pmatrix}1\\1\\1\end{pmatrix} \in \operatorname{Im} f$. Dunque $f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = v + \ker f$

con v t.c. $f(v) = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$$M^{e}_{\mathcal{B}}(f) = \begin{array}{ccc} \multicolumn{3}{c}{\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}} \\ f(1) & f(x) & f(x^2) \end{array}$$

Sia $f(x) = \begin{pmatrix}1\\1\\1\end{pmatrix}$ che $f(x^2) = \begin{pmatrix}1\\1\\1\end{pmatrix}$

Dunque $f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = x^2 + \ker f = x^2 + \langle 1, x - x^2 \rangle$

$= x + \ker f = x + \langle 1, x - x^2 \rangle$

METODO 2

$$f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \left\{ p(x) = a + bx + \underline{c}x^2 \;\middle|\; \begin{pmatrix} b+c \\ b+c \\ b+c \end{pmatrix} = f(p(x)) = \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

$p(x) = a + bx + cx^2 \in f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} \iff b + c = 1$

$\iff \underline{c} = 1 - b$

$= \{ a + bx + (1-b)x^2 \}$

$= \{ a + bx + x^2 - bx^2 \}$

$= \{ a + b(x - x^2) + x^2 \}$

$$= x^2 + \underbrace{\{ a + b(x - x^2) \}}_{\langle 1,\, x - x^2 \rangle = \ker f}$$

$= x^2 + \ker f.$

$= x^2 + \ker f$

METODO 3

$$f^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}_{\mathcal{B}} \;\middle|\; M^{e}_{\mathcal{B}}(f)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}_{\mathcal{B}} = \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}_{\mathcal{B}} \;\middle|\; \begin{pmatrix}0&1&1\\0&1&1\\0&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}_{\mathcal{B}} \;\middle|\; x_2 + x_3 = 1 \right\}$$

$$= \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}_{\mathcal{B}} \;\middle|\; x_3 = 1 - x_2 \right\}$$

$$= \left\{ \begin{pmatrix}x_1\\x_2\\1-x_2\end{pmatrix}_{\mathcal{B}} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}_{\mathcal{B}}}_{x^2} + x_1 \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{B}}}_{1} + x_2 \underbrace{\begin{pmatrix}0\\1\\-1\end{pmatrix}_{\mathcal{B}}}_{x-x^2} \right\}$$

$\mathcal{B} = \{1, x, x^2\}$

$$\begin{pmatrix}0\\0\\1\end{pmatrix}_{\mathcal{B}} = 0 \cdot 1 + 0x + 1x^2 = x^2$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}_{\mathcal{B}} = 1 \cdot 1 + 0x + 0x^2 = 1$$

$$\begin{pmatrix}0\\1\\-1\end{pmatrix}_{\mathcal{B}} = 0 \cdot 1 + 1x + (-1)x^2 = x - x^2$$

$$= \left\{ x^2 + \underbrace{x_1 1 + x_2 (x - x^2)} \right\}$$

$$= \{ x^2 + \underbrace{x_1 1 + x_2 (x - x^2)} \}$$

$$= x^2 + \langle 1, (x - x^2) \rangle = x^2 + \operatorname{Ker} f .$$

BASI CANONICHE

Base canonica di $\mathbb{R}$ $\quad \{ 1 \}$

" " di $\mathbb{R}^2$ $\quad \{ e_1 = (1 0), \; e_2 = (0 1) \}$

" " di $\mathbb{R}^3$ $\quad \{ e_1 = (1 0 0), e_2 = (0 1 0), e_3 = (0 0 1) \}$

" " di $\mathbb{R}^4$ $\quad \{ e_1 = (1 0 0 0), \; e_2 = (0 1 0 0), \; e_3 = (0 0 1 0), \; e_4 = (0 0 0 1) \}$

Base canonica di $\mathbb{R}[x]^{\leq 1}$ $\quad \{ 1, x \}$

" " di $\mathbb{R}[x]^{\leq 2}$ $\quad \{ 1, x, x^2 \}$

" " di $\mathbb{R}[x]^{\leq 3}$ $\quad \{ 1, x, x^2, x^3 \}$

. . . . . .

Base canonica di $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ $\quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Base canonica di $\mathrm{Mat}_{2\times 1}(\mathbb{R})$ $\quad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

" " di $\mathrm{Mat}_{2\times 3}(\mathbb{R})$ $\quad \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

$\mathbb{R}^2 \quad \{ e_1 = (1 0), e_2 = (0 1) \} = \mathcal{E}$

$\begin{pmatrix} 7 \\ 0 \end{pmatrix}_e = 7 e_1 + 0 e_2$

$\{ e_2 = (0\,1), \; e_1 = (1\,0) \} = \mathcal{B} \qquad 7 e_2 + 0 e_1 = \begin{pmatrix} 7 \\ 0 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$

$f : V \to W$ app. lineare .

$\dim V = \dim \operatorname{Ker} f + \underbrace{\dim \operatorname{Im} f}_{rg(M(f))}$

FORMULA DELLE DIMENSIONI

$\oplus, \cap, + \qquad V \supset W_1, W_2$

$\dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$

formula di GRASSMANN

Es

① Dimostrare che $\mathcal{U} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \;\middle|\; \operatorname{tr} A = 0 \right\}$

(Se $A \in Mat_{n \times n}(\mathbb{R}) \qquad A = (a_{ij})$

$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} \qquad \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$)

coincide con $\mathcal{U}' = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$

Cerchiamo una base di $\mathcal{U}$, cioè dei generatori LI

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U} \iff tr(A) = 0$$
$$\iff a + d = 0$$
$$\iff \underline{d = -a}$$

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\in \underbrace{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle}_{\text{generatori di } \mathcal{U}}$$

Devo verificare se $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ sono LI

$$\lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & -\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\iff \lambda_1 = 0 \quad \lambda_2 = 0 \quad e \quad \lambda_3 = 0$$

Dunque $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ sono LI

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ sono generatori di $\mathcal{U}$ e sono LI. Dunque sono una base di $\mathcal{U}$ e $\dim \mathcal{U} = 3$

$\mathcal{U}' = < \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} >$

A casa verifico che $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ sono LI

Dunque $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ sono una base di $\mathcal{U}'$

e $\dim \mathcal{U}' = 3$

Osserviamo che i vettori della base di $\mathcal{U}'$ hanno traccia nulla $\quad tr \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 = tr \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$0 = tr \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\boxed{\mathcal{U}' \subseteq \mathcal{U}}, \quad \dim \mathcal{U}' = 3, \quad \dim \mathcal{U} = 3$

$\Rightarrow \mathcal{U}' = \mathcal{U}$

② Determinare una base e la dimensione di un supplementare $\mathcal{V}$ di $\mathcal{U}$, cioè

cerco $\mathcal{V} \subseteq Mat_{2\times 2}(\mathbb{R})$ t.c. $\mathcal{V} \oplus \mathcal{U} = Mat_{2\times 2}(\mathbb{R})$

$\mathcal{V} \oplus \mathcal{U} \iff \mathcal{V} + \mathcal{U}$ e $\mathcal{V} \cap \mathcal{U} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

Voglio anche se $\mathcal{V} + \mathcal{U} = Mat_{2\times 2}(\mathbb{R})$

$$\underset{\substack{= \\ 4 \\ =}}{\dim \mathcal{V} + \mathcal{U}} = \underbrace{\dim \mathcal{V}} + \underbrace{\dim \mathcal{U}}_{3} - \underbrace{\dim \mathcal{V} \cap \mathcal{U}}_{0}$$

$$\begin{matrix} 4 & & 3 & 0 \\ = & \Downarrow & & \text{poiché } \mathcal{V}\oplus\mathcal{U} \\ \dim \mathrm{Mat}_{2\times 2}(\mathbb{R}) & 1 & & \end{matrix}$$

$$\mathcal{V} = \langle A \rangle, \quad A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A \notin \mathcal{U}$$

$$\Updownarrow$$

$$\operatorname{tr} A \neq 0$$

$$\mathcal{V} = \langle A \rangle \text{ con } A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \operatorname{tr}(A) = 1 \neq 0 .$$

oppure con $A = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$

oppure con $A = \begin{pmatrix} 324 & 8632 \\ 2 & 248 \end{pmatrix}$