Calculus Math 4U Lecture Ø3

Topics of todays lecture

↦ exercise on $\sup(E)$, $\max(E)$, E set of numbers

↦ complex numbers: inequalities & De Moivre

Exercise on $\sup(E)$ vs $\max(E)$ (+ important notes to solve <u>inequalities</u>)

Consider the set E defined as

$$E = \left\{ \frac{\sqrt{n+1}}{\sqrt{n}+1} \quad n \in \mathbb{N},\ n \geq 1 \right\}$$

Prove that 1) $\min(E) = \frac{\sqrt{2}}{2}$ (easy, let's do it by induction

2) $\sup(E) = 1$, $\sup(E) \neq \max(E)$, $\sup(E) \notin E$

Proof of statement 1, by induction

↦ step Ø: for $n = 1$ $\quad \frac{\sqrt{n+1}}{\sqrt{n}+1} = \frac{\sqrt{2}}{2}$, $\quad \underbrace{\frac{\sqrt{2}}{2}}_{\min(E)} \leq \underbrace{\frac{\sqrt{2}}{2}}_{E \text{ evaluated in } n=1}$ ✓

↦ inductive step: if $\frac{\sqrt{n+1}}{\sqrt{n}+1} \geq \frac{\sqrt{2}}{2}$

then $\frac{\sqrt{(n+1)+1}}{\sqrt{n+1}+1} \geq \frac{\sqrt{2}}{2}$

it is sufficient to prove that

$\forall x_1, x_2 \in \mathbb{R},\ 0 \leq x_1 < x_2$, $x_1^2 < x_2^2$

$$\frac{\sqrt{n+1}}{\sqrt{n}+1} < \frac{\sqrt{n+2}}{\sqrt{n+1}+1} \mapsto \frac{n+1}{n+1+2\sqrt{n}} < \frac{n+2}{n+2+2\sqrt{n+1}}$$

$$\frac{n+1+2\sqrt{n}}{n+1} > \frac{n+2+2\sqrt{n+1}}{n+2} \mapsto \cancel{1} + \frac{2\sqrt{n}}{n+1} > \cancel{1} + \frac{2\sqrt{n+1}}{n+2}$$

$\forall x_1, x_2 \in \mathbb{R},\ 0 \leq x_1 < x_2$, $1/x_1 > 1/x_2$

$$\frac{\cancel{4}n}{n^2+1+2n} > \frac{\cancel{4}(n+1)}{n^2+4+4n} \mapsto n(n^2+4+4n) > (n+1)(n^2+1+2n)$$

$$\cancel{n^3} + 4n^2 + 4n > \cancel{n^3} + 2n^2 + n + n^2 + 1 + 2n$$

$$4n^2 + 4n - 2n^2 - n^2 - 3n - 1 > 0$$

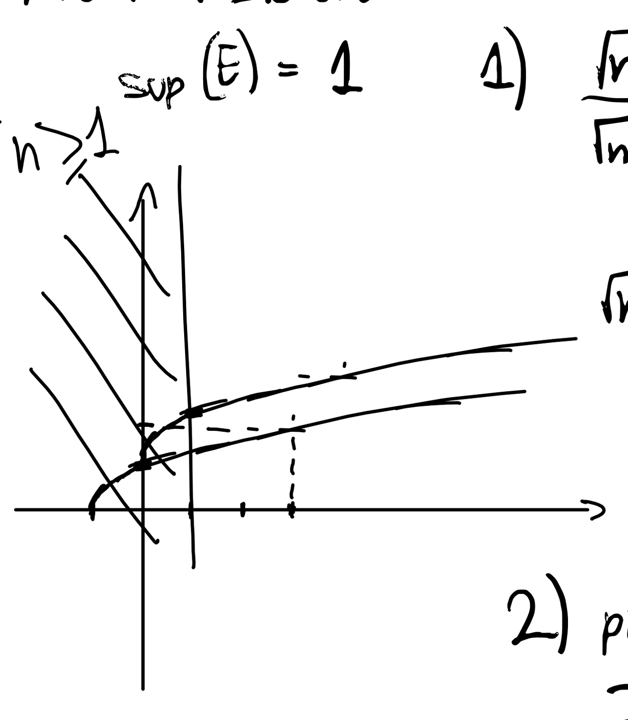
$$n^2 + n - 1 > \emptyset$$

>1, true (not true for $n = \emptyset$, but not included)

Proof of statement 2

$\sup(E) = 1$ $\quad$ 1) $\frac{\sqrt{n+1}}{\sqrt{n}+1} \leq 1 \quad \forall n \in \mathbb{N}^+$

$n \geq 1$



$$\sqrt{n+1} \leq (\sqrt{n}+1) \cdot 1 \quad \text{true!}$$

2) prove that $\forall \beta \in \mathbb{R},\ \beta < 1$

$\exists \alpha \in E : \beta \leq \alpha \leq 1$

(which means) $\exists n \in \mathbb{N}^+ : \beta \leq \overbrace{\frac{\sqrt{n+1}}{\sqrt{n}+1} \leq 1}^{\text{proved at step 1)}}$

$$\beta \leq \frac{\sqrt{n+1}}{\sqrt{n}+1} \mapsto (\sqrt{n}+1) \cdot \beta \leq \sqrt{n+1}$$

<u>Note that</u>: I implicitly assumed that $\sqrt{n}+1 > \emptyset$, which is always true;

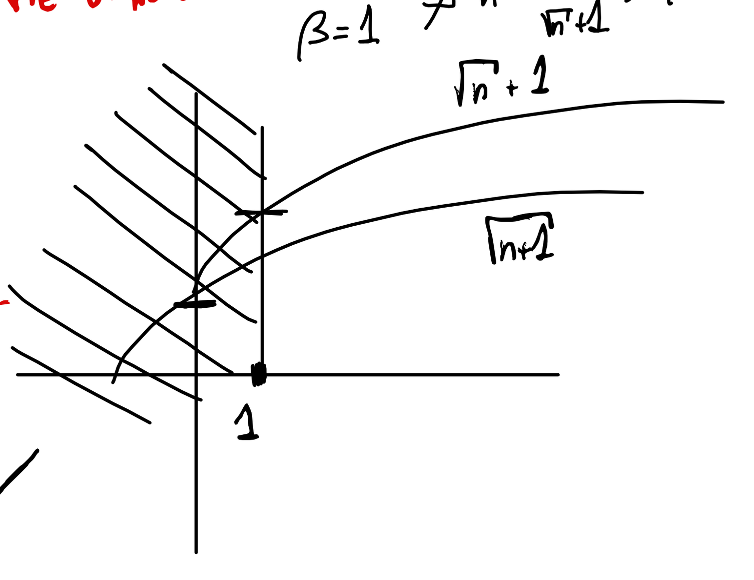
If in another exercise I have to multiply by an expression that may be $\leq \emptyset$ I need to distinguish all the different scenarios:

↦ multiplication by negative number

↦ flip the inequality sign

↦ multiplication by Ø, β can take any value

$\beta = 1 \quad \nexists n : \frac{\sqrt{n+1}}{\sqrt{n}+1} \geq \beta$



assumption that $\beta \geq \emptyset$!

$$\sqrt{n+1}^{\,2} \geq (\sqrt{n}+1)^2 \beta^2$$

$$n+1 \geq (n+1+2\sqrt{n})\beta^2$$

$$n(1-\beta^2) - 2\sqrt{n}\,\beta^2 + 1 - \beta^2 \geq \emptyset$$

consider the new variable $m = \sqrt{n}$

$$m^2(1-\beta^2) - 2m\beta^2 + 1 - \beta^2 \geq \emptyset$$

$$m_{1,2} = \frac{-\beta^2 \pm \sqrt{\beta^4 - (1-\beta^2)^2}}{1-\beta^2} = \frac{-\beta^2 \pm \sqrt{\cancel{\beta^4} - \cancel{\beta^4} + 2\beta^2 - 1}}{1-\beta^2}$$

$$= \frac{-\beta^2 \pm \sqrt{2\beta^2 - 1}}{1-\beta^2}$$

$\beta = 1$ issue! already adressed

$\beta^2 \geq \frac{1}{\sqrt{2}}$ ✓ $\quad$ (min(E))

there are no solutions i.e. all are ok

Complex numbers: inequalities

Exercise 1:

Solve the following equation: (not solved in class)

$$z^2 - \frac{1}{4}(z - \bar{z})^2 - 4 = 8i$$

$$z^2 - \frac{1}{4}(z - \bar{z})^2 = 4 + 8i$$

let $z = a + ib$, $\left[(a+ib) - (a-ib)\right]^2 = (2ib)^2 = -4b^2$

$$(a+ib)^2 + b^2 = 4 + 8i \mapsto a^2 - \cancel{b^2} + 2aib + \cancel{b^2} = 4 + 8i$$

$$\begin{cases} a^2 = 4 \\ \cancel{2}ab = \cancel{8}\,4 \end{cases}, \quad \begin{matrix} a = \pm 2 \\ b = \pm 2 \end{matrix} \text{ same sign!}$$

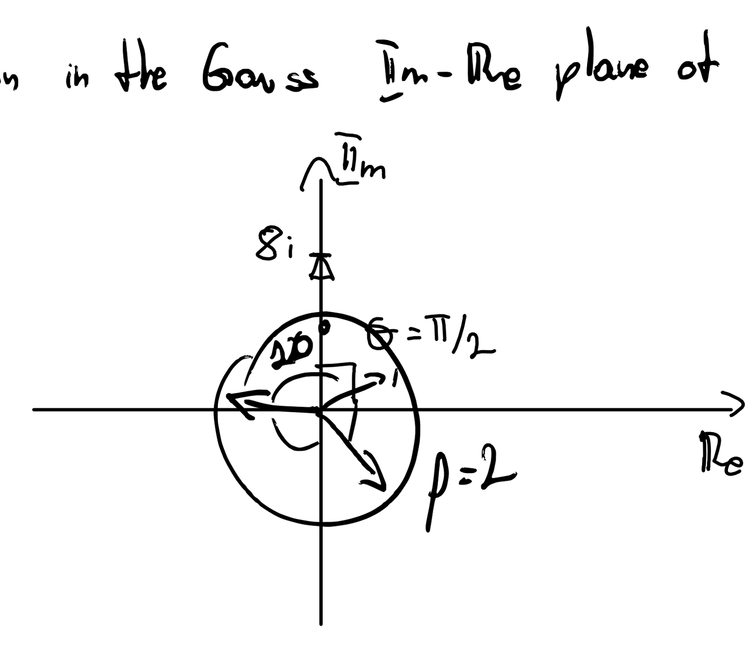
Exercise: $\quad$ from Exam of 17/01/2022

Solve the following equation:

$$\left(\frac{z}{i}\right)^3 = -8 \mapsto z^3 = -8i^3 = +8i$$

$$z^3 = 8i$$

↦ representation in the Gauss Im-Re plane of $z^* = 8i$



↦ switch to trigonometric form of $z^* = 8i$

$$z^3 = 8\left(\cos\left(\pi/2 + 2k\pi\right) + i\sin\left(\pi/2 + 2k\pi\right)\right),\ k \in \mathbb{Z}$$

↦ using de Moivre's formula

$$z = \cancel{\sqrt[3]{8}}\,2\left(\cos\left(\pi/6 + \frac{2k\pi}{3}\right) + i\sin\left(\pi/6 + \frac{2k}{3}\pi\right)\right),\ k \in \mathbb{Z}$$

Exercise 3

Solve the inequality

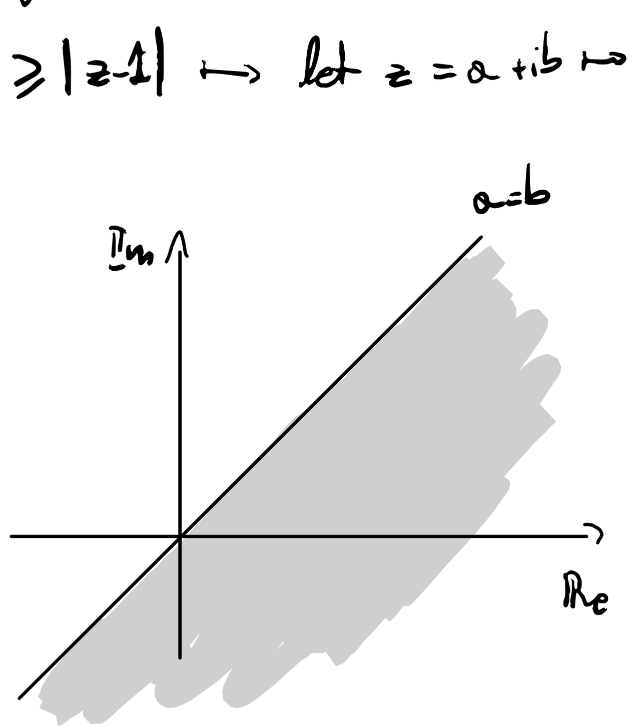
$$\left|\frac{(z-i)}{(z-1)}\right| \geq 1$$ (what happens for $|z-1| = \emptyset$??) $\quad$ from exam of 12/09/2022

$$|z-i| \geq |z-1| \mapsto \text{let } z = a+ib \mapsto |a + i(b-1)| \geq |a-1+ib|$$

$$\sqrt{a^2 + (b-1)^2} \geq \sqrt{(a-1)^2 + b^2}$$

$$\cancel{a^2} + \cancel{b^2} - 2b + \cancel{1} \geq \cancel{a^2} - 2a + \cancel{1} + \cancel{b^2}$$

$$-\cancel{2}b \geq -\cancel{2}a \mapsto b \leq a$$



Exercise

Solve the following equation $\quad$ from exam of 05/07/21

$$\mathrm{Im}(z^2) + |z|^2\,\mathrm{Re}\left(\frac{1}{z}\right) = \emptyset$$

let $z = a+ib$, $z^2 = (a^2 - b^2) + 2iab$, $\mathrm{Im}(z^2) = 2ab$

$$|z|^2 = a^2 + b^2$$

$$\frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{a - ib}{a^2+b^2}, \quad \mathrm{Re}\left(\frac{1}{z}\right) = \frac{a}{a^2+b^2}$$

$$2ab + \cancel{a^2+b^2}\,\frac{a}{\cancel{a^2+b^2}} = \emptyset \mapsto 2ab + a = \emptyset \mapsto a(2b+1) = \emptyset$$

↦ $a = \emptyset$, b any

↦ $b = -\frac{1}{2}$, a any