# Foglio 9

**Esercizio 1**
Calcolare il prodotto scalare $v \cdot w$ dei vettori $v, w \in \mathbb{R}^2$ quando assumono i valori indicati:

$$a) \quad v = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \quad [9]$$

$$b) \quad v = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix} \quad w = \begin{pmatrix} -2 \\ 8 \end{pmatrix} \quad [0]$$

$$c) \quad v = \begin{pmatrix} -\sqrt{2} \\ \frac{1}{3} \end{pmatrix} \quad w = \begin{pmatrix} \sqrt{8} \\ 9 \end{pmatrix} \quad [-1]$$

$$d) \quad v = \begin{pmatrix} -11 \\ 6 \end{pmatrix} \quad w = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad [3]$$

$$e) \quad v = \begin{pmatrix} 0 \\ 44 \end{pmatrix} \quad w = \begin{pmatrix} 68 \\ \frac{1}{12} \end{pmatrix} \quad \left[\frac{11}{3}\right]$$

$$f) \quad v = \begin{pmatrix} -10 \\ 7 \end{pmatrix} \quad w = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \quad [-4]$$

$$g) \quad v = \begin{pmatrix} 2 \\ \frac{1}{3} \end{pmatrix} \quad w = \begin{pmatrix} -\frac{1}{6} \\ \frac{5}{4} \end{pmatrix} \quad \left[\frac{1}{12}\right]$$

**Esercizio 2**
Calcolare il prodotto scalare $v \cdot w$ dei vettori $v, w \in \mathbb{R}^3$ quando $v$ e $w$ assumono i valori

indicati:

$$a) \quad v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 6 \\ 5 \\ -4 \end{pmatrix} \quad [-5]$$

$$b) \quad v = \begin{pmatrix} 2 \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix} \quad [1]$$

$$c) \quad v = \begin{pmatrix} 3 \\ \frac{1}{3} \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} \quad [8]$$

$$d) \quad v = \begin{pmatrix} 1 \\ 24 \\ -13 \end{pmatrix} \quad w = \begin{pmatrix} 25 \\ 6 \\ 13 \end{pmatrix} \quad [0]$$

$$e) \quad v = \begin{pmatrix} -1 \\ 0 \\ 44 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 68 \\ 0 \end{pmatrix} \quad [-1]$$

$$f) \quad v = \begin{pmatrix} \frac{1}{2} \\ -\frac{10}{3} \\ 7 \end{pmatrix} \quad w = \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} \quad [-10]$$

$$g) \quad v = \begin{pmatrix} \frac{2}{3} \\ 9 \\ \frac{1}{3} \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ -\frac{1}{6} \\ \frac{5}{2} \end{pmatrix} \quad [0]$$

**Esercizio 3**
Calcolare la norma $\|v\|$ del vettore $v \in \mathbb{R}^2$ definito come segue:

$$a) \; v = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad [\sqrt{5}] \quad b) \; v = \begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix} \quad [\sqrt{5}] \quad c) \; v = \begin{pmatrix} \sqrt{12} \\ 0 \end{pmatrix} \quad [\sqrt{12}]$$

$$d) \; v = \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{7}}{4} \end{pmatrix} \quad [1] \quad e) \; v = \begin{pmatrix} -\sqrt{17} \\ 2\sqrt{2} \end{pmatrix} \quad [5] \quad f) \; v = \begin{pmatrix} -\frac{3}{4} \\ -1 \end{pmatrix} \quad \left[\frac{5}{4}\right]$$

**Esercizio 4**
Calcolare la norma $\|v\|$ del vettore $v \in \mathbb{R}^3$ definito come segue:

$$a) \; v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad [\sqrt{14}] \quad b) \; v = \begin{pmatrix} \sqrt{2} \\ -1 \\ 3 \end{pmatrix} \quad [2\sqrt{3}] \quad c) \; v = \begin{pmatrix} \sqrt{12} \\ 0 \\ -2 \end{pmatrix} \quad [4]$$

$$d) \; v = \begin{pmatrix} \frac{1}{2} \\ \sqrt{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad [\sqrt{3}] \quad e) \; v = \begin{pmatrix} -\sqrt{7} \\ -1 \\ 2\sqrt{2} \end{pmatrix} \quad [4] \quad f) \; v = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{4} \\ -\frac{\sqrt{3}}{4} \end{pmatrix} \quad [1]$$

**Esercizio 5**
Determinare $k \in \mathbb{R}$ affinché $v, w \in \mathbb{R}^2$ siano ortogonali:

$$a) \quad v = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} k \\ 3 \end{pmatrix} \quad [k = 9]$$

$$b) \quad v = \begin{pmatrix} -k \\ \frac{1}{2} \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \left[k = \frac{1}{2}\right]$$

$$c) \quad v = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \quad w = \begin{pmatrix} \sqrt{2} \\ k \end{pmatrix} \quad [k = 0]$$

$$d) \quad v = \begin{pmatrix} k-2 \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ 1+k \end{pmatrix} \quad [k = 1]$$

$$e) \quad v = \begin{pmatrix} k \\ -2 \end{pmatrix} \quad w = \begin{pmatrix} k-7 \\ -6 \end{pmatrix} \quad [k = 3, 4]$$

$$f) \quad v = \begin{pmatrix} k \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} k \\ -7 \end{pmatrix} \quad [k = \pm\sqrt{7}]$$

**Esercizio 6**
Determinare $k \in \mathbb{R}$ affinché $v, w \in \mathbb{R}^3$ siano ortogonali:

$$a) \quad v = \begin{pmatrix} -k \\ 1 \\ k \end{pmatrix} \quad w = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} \quad [k = -1]$$

$$b) \quad v = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ -\frac{k}{2} \\ k+1 \end{pmatrix} \quad [k = 2]$$

$$c) \quad v = \begin{pmatrix} 1 \\ \sqrt{2} \\ 2-k \end{pmatrix} \quad w = \begin{pmatrix} -k \\ -\sqrt{2} \\ 1 \end{pmatrix} \quad [k = 0]$$

$$d) \quad v = \begin{pmatrix} k-2 \\ 1 \\ k \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 0 \\ k \end{pmatrix} \quad [k = -2, 1]$$

$$e) \quad v = \begin{pmatrix} 1 \\ -2 \\ k-6 \end{pmatrix} \quad w = \begin{pmatrix} k^2 \\ -2 \\ k \end{pmatrix} \quad [k = 1, 2]$$

$$f) \quad v = \begin{pmatrix} k \\ k+1 \\ -k \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ k-1 \\ 5 \end{pmatrix} \quad \left[k = \frac{3 \pm \sqrt{13}}{2}\right]$$

**Esercizio 7**
Determinare l'angolo compreso tra i vettori $v, w \in \mathbb{R}^2$ definiti come segue:

$$a) \quad v = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 9 \\ 3 \end{pmatrix} \quad \left[\frac{\pi}{2}\right]$$

$$b) \quad v = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix} \quad w = \begin{pmatrix} -8 \\ 2 \end{pmatrix} \quad [0]$$

$$c) \quad v = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \quad w = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \quad \left[\frac{2}{3}\pi\right]$$

$$d) \quad v = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 1-\sqrt{3} \\ 1+\sqrt{3} \end{pmatrix} \quad \left[\frac{\pi}{4}\right]$$

$$e) \quad v = \begin{pmatrix} \sqrt{7} \\ -2 \end{pmatrix} \quad w = \begin{pmatrix} -7 \\ 2\sqrt{7} \end{pmatrix} \quad [\pi]$$

$$f) \quad v = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \quad \left[\frac{\pi}{6}\right]$$

**Esercizio 8**
Determinare l'angolo compreso tra i vettori $v, w \in \mathbb{R}^3$ definiti come segue:

$$a) \quad v = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} \quad w = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix} \quad \left[\frac{\pi}{2}\right]$$

$$b) \quad v = \begin{pmatrix} -2 \\ 1 \\ \frac{1}{4} \end{pmatrix} \quad w = \begin{pmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{8} \end{pmatrix} \quad [0]$$

$$c) \quad v = \begin{pmatrix} -1 \\ \sqrt{2} \\ 2-\sqrt{3} \end{pmatrix} \quad w = \begin{pmatrix} -\sqrt{3} \\ -\sqrt{2} \\ 1 \end{pmatrix} \quad \left[\frac{\pi}{2}\right]$$

$$d) \quad v = \begin{pmatrix} -\sqrt{3} \\ 1 \\ 0 \end{pmatrix} \quad w = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix} \quad \left[\frac{5}{6}\pi\right]$$

$$e) \quad v = \begin{pmatrix} 1 \\ -2 \\ -6 \end{pmatrix} \quad w = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 3 \end{pmatrix} \quad [\pi]$$

$$f) \quad v = \begin{pmatrix} \sqrt{3} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ \sqrt{6} \\ \sqrt{6} \end{pmatrix} \quad \left[\frac{\pi}{6}\right]$$

**Esercizio 9**
Date le seguenti basi di $\mathbb{R}^3$, si determini una base ortonormale di $\mathbb{R}^3$ utilizzando il procedimento di Gram-Schmidt a partire da $\mathcal{B}$.

a) $\mathcal{B} = \{(-1,0,1),(0,1,0),(1,0,1)\}$.
$\left[\text{R. } \left\{ \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \ (0,1,0), \ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \right\}.\right]$

b) $\mathcal{B} = \{(0,0,1),(0,1,1),(1,1,1)\}$.
   [R. $\{(0,0,1),(0,1,0),(0,0,1)\}$].

c) $\mathcal{B} = \{(2,0,0),(1,2,0),(0,-1,-1)\}$.
   [R. $\{(1,0,0),(0,1,0),(0,0,-1)\}$].

**Esercizio 10**
Sia $U$ il sottospazio di $\mathbb{R}^3$ definito da $U = \{(x,y,z) \in \mathbb{R}^3 : \ 2x - y = 0\}$. Si determini una base ortonormale di $U$ rispetto al prodotto scalare ordinario di $\mathbb{R}^3$.
$\left[\text{R. } \left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right), \ (0,0,1)\right\}.\right]$

**Esercizio 11**
Siano dati i vettori $v_1 = (1,-1,0)$, $v_2 = (2,0,1)$, $v_3 = (0,-1,1)$ in $\mathbb{R}^3$.

a) Far vedere che formano una base di $\mathbb{R}^3$.

b) Ortonormalizzarla col metodo di Gram-Schmidt.
$\left[\text{R. } u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \ u_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \ u_3 = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right).\right]$

**Esercizio 12**
Sia $W$ il sottospazio di $\mathbb{R}^4$ generato da $\{(1,1,0,0),(1,2,1,0),(1,0,0,-1)\}$. Trovare una base ortonormale di $W$.
$\left[\text{R. } \left\{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0\right), \ \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, 0\right), \ \left(\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{2}\right)\right\}.\right]$

**Esercizio 13**
Sia $W$ il sottospazio di $\mathbb{R}^4$ (con il prodotto scalare canonico) generato dai vettori

$$v_1 = (1,1,0,1), \quad v_2 = (1,-2,0,0), \quad v_3 = (1,0,-1,2).$$

a) Trovare una base ortonormale di $W$.
$\left[\text{R. } \left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right), \ \left(\frac{4}{\sqrt{42}}, -\frac{5}{\sqrt{42}}, 0, \frac{1}{\sqrt{42}}\right), \ \left(\frac{4}{\sqrt{105}}, \frac{2}{\sqrt{105}}, \frac{7}{\sqrt{105}}, -\frac{6}{\sqrt{105}}\right)\right\}.\right]$

b) Trovare una base di $W^\perp$.
   [R. $W^\perp = \langle(-2,-1,4,3)\rangle$.]

**Esercizio 14**
Sia $W$ il sottospazio di $\mathbb{R}^4$ definito da $W = \langle(1,1,0,0),(1,2,-1,3)\rangle$. Si determini una base ortonormale di $W^\perp$.
$\left[\text{R. } \left\{\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right), \ \left(\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)\right\}.\right]$