Exercise 1 (punti 8) Consider the function

\[ f(x) = x e^{\frac{1}{x}} \]

(a) find its (maximal) domain, determine its sign and study the possibility that \( f \) is odd or even:

Solution:

**Domain:** the domain is clearly equal to \( D := \mathbb{R} \setminus \{0\} \)

The function is neither even nor odd.

**Sign:** since \( e^{\frac{1}{x}} > 0 \) for every \( x \neq 0 \) one has

\[ f(x) \geq 0 \iff x \geq 0 \]

(b) compute limits and possible asymptotes:

**Limits at 0:**

\[ \lim_{x \to 0^-} f(x) = 0 \cdot 0 = 0 \]
\[ \lim_{x \to 0^+} f(x) = 0 \cdot \infty \]

indeterminate form ... Let us try the change of variable \( y = \frac{1}{x} \)

\[ \lim_{x \to 0^+} f(x) = \lim_{y \to +\infty} \frac{e^y}{y} = +\infty \]

So \( x = 0 \) is a (right) vertical asymptote at \( x = 0 \).

**Limits at \( \pm \infty \)**

\[ \lim_{x \to +\infty} x e^{\frac{1}{x}} = +\infty \times 1 = +\infty \]
\[ \lim_{x \to -\infty} x e^{\frac{1}{x}} = -\infty \times 1 = -\infty \]

**Asymptotes:**

\[ \lim_{x \to +\infty} \frac{x e^{\frac{1}{x}}}{x} = \lim_{x \to +\infty} e^{\frac{1}{x}} = 1 \]

\[ \lim_{x \to +\infty} x \left( e^{\frac{1}{x}} - x \right) = \lim_{y \to 0^+} \frac{1}{y} (e^y - 1) = \lim_{y \to 0^+} \frac{1}{y} (y + o(y)) = 1 \implies \text{there is an asymptote at } +\infty : \ y = x + 1 \]

\[ \lim_{x \to -\infty} \frac{x e^{\frac{1}{x}}}{x} = \lim_{x \to -\infty} e^{\frac{1}{x}} = 1 \]

\[ \lim_{x \to -\infty} x \left( e^{\frac{1}{x}} - x \right) = \lim_{y \to 0^-} \frac{1}{y} (e^y - 1) = \lim_{y \to 0^-} \frac{1}{y} (y + o(y)) = 1 \implies \text{there is an asymptote at } -\infty : \ y = x + 1 \]
(c) study the differentiability of \( f \) and find the derivative where possible (if necessary study the limits of the derivative); discuss the monotonicity of \( f \) and determine its supremum and infimum; if existing determine relative (=local) and absolutely (=global) minima and maxima of \( f \):

\[
\forall x \neq 0, \quad f'(x) > 0 \iff e^x - xe^x \frac{1}{x^2} = e^x - e^{\frac{1}{x}} > 0 \iff 1 - \frac{1}{x} > 0 \iff x > 1 \text{ or } x < 0
\]

and

\[
f'(x) = 0 \iff x = 1
\]

\( f \) is increasing on \([1, +\infty[\) and on \(]-\infty, 0[\), and it is decreasing on \([0, 1[\). Hence it has a local minimum at \(x = 1\). The function is both upper unbounded and lower unbounded.

Finally

\[
\lim_{x \to 0^-} f'(x) = 0
\]

so \( y = 0 \) is a left tangent at \( x = 0 \)

(d) determine the second derivative and study the convexity of the function :

\[
f''(x) = \frac{d}{dx} \left( e^x \left( 1 - \frac{1}{x} \right) \right) = -e^x \frac{1}{x^2} \left( 1 - \frac{1}{x} \right) + e^x \frac{1}{x^2} = e^x \frac{1}{x^3}
\]

so that

\[
f''(x) > 0 \iff x > 0, \quad f''(x) < 0 \iff x < 0.
\]

Hence the function is convex on the interval \([0, +\infty[\) and concave on the interval \(]-\infty, 0[\)

(e) draw a qualitative graph of \( f \).
Exercise 2 (punti 8) Consider the equation on complex numbers

\[ z^6 + 2iz^3 - 1 = 0. \]

Find the solutions with their multiplicity, and draw the on the complex plane.

**Solution** Consider the substitution \( w = z^3 \), so that the equation becomes the second order equation

\[ w^2 + 2iw - 1 = 0. \]

whose left-hand side is easily seen to be the square of the first order polynomial \((w + 1)\), namely we obtain

\[ (w + i)^2 = 0. \]

Therefore we have the only solution \( w = -i \) with multiplicity equal to 2. If we write \( z^3 = w = -i = e^{3\pi i/2} \) we obtain the three solutions \( z_1 = e^{\pi i/2} \), \( z_2 = e^{7\pi i/6} \), \( z_3 = e^{11\pi i/6} \), each one with multiplicity 2.

![Equilateral triangle with vertices labeled]

Exercise 3 (punti 8) Study the behaviour of the following series for the values of the parameter \( \alpha > 0 \)

\[ \sum_{n=1}^{+\infty} n^\alpha \left(1 - \sqrt{\frac{n^2}{n^2 + 1}}\right)^{\alpha - 1}. \]

Let us investigate the order of the sequence (this is not the only method):

\[ n^\alpha \left(1 - \sqrt{\frac{n^2}{n^2 + 1}}\right)^{\alpha - 1} = n^\alpha \left(1 - \sqrt{\frac{n^2}{n^2 + 1}}\right)^{\alpha - 1} \frac{(1 + \sqrt{\frac{n^2}{n^2 + 1}})^{\alpha - 1}}{(1 + \sqrt{\frac{n^2}{n^2 + 1}})^{\alpha - 1}} = \]

\[ n^\alpha \left(1 - \frac{n^2}{n^2 + 1}\right)^{\alpha - 1} = n^\alpha \left(\frac{1}{n^2 + 1}\right)^{\alpha - 1} \sim n^{\alpha - 2\alpha + 2} = n^{-\alpha + 2} \]
Therefore, the series (which has positive terms) converges if and only if $-\alpha + 2 < -1$, i.e. if and only if $\alpha > 3$.

**Exercise 4 (punti 8)**

(a) Use De L’Hôpital Theorem to show that

$$\lim_{x \to \infty} \frac{\arctan(x + 1) - \arctan(x)}{\frac{1}{x^2}} = 1;$$

**Solution** The limit is a form $0/0$ so that we can apply De L’Hôpital Theorem provided the limit of the ratio of the derivatives does exist. Denoting with $f$ and $g$ the numerator and the denominator we have

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{(1 + (x + 1)^2)^{-1}}{(1 + (x)^2)^{-1}} = 1$$

(b) Use the result stated in the previous point to discuss the behaviour of the generalized integral

$$\int_{1}^{\infty} \frac{1}{x^\alpha [\arctan(x + 1) - \arctan(x)]} \, dx$$

for all values of the parameter $\alpha \in \mathbb{R}$.

From

$$\lim_{x \to \infty} \frac{\arctan(x + 1) - \arctan(x)}{\frac{1}{x^2}} = 1$$

we get that the integrand

$$\frac{1}{x^\alpha [\arctan(x + 1) - \arctan(x)]}$$

is of the same order as

$$\frac{1}{x^{\alpha - 2}}$$

so that the integral converges if and only if $\alpha - 2 > 1$, i.e. if and only if $\alpha > 3$. 