Program of Differential Equations, Prof. M. Bardi  
Master Degree in Mathematics, a.y. 2022/2023

1. Introduction (references: [E], sect. 1.1, 1.2, 1.3, 2.1)

Generalities on 1st order Partial Differential Equations, linear and nonlinear operators. Examples and motivations: the transport equation, conservation laws, Hamilton-Jacobi (HJ) equations.

2. The method of characteristics (ref.: [E], sect. 3.2, [L] chapter 1, sect. 1.2)

Derivation of the characteristic equations for the PDE \( F(x, u, Du) = 0 \) and inversion of the characteristic flow. Characteristics for conservation laws and examples of shocks. Characteristics for the HJ equation \( u_t + H(x, D_x u) = 0 \). Local existence theorem of solutions to the Cauchy for the equation \( u_t + H(D_x u) = 0 \). Examples of global existence of classical solutions, examples of crossing of characteristics and shock of the gradient; links with conservation laws.

3. Links between HJ equations and Calculus of Variations, introduction to Convex Analysis (ref. [E], sect. 3.3 and appendix B.)


5. The Hopf-Lax formula (ref. [E], chapter 3, sect. 3.3)

The Hopf-Lax formula for the H-J equation as value function of a problem in Calculus of Variations. Dynamic programming and Lipschitz continuity. The Hopf-Lax formula satisfies the H-J equation a.e. Counterexample to the uniqueness of a.e. solutions. Uniqueness and comparison principle for classical solutions. Semiconcave functions, properties and examples. Uniqueness of solutions to HJ semiconcave in \( x \). The Hopf-Lax formula is semiconcave in \( x \) if the initial data are.

6. Viscosity solutions of Hamilton-Jacobi equations (refs. [BCD], ch. II, sect. 1, 2, 3; [E], ch. 10)

Definition of viscosity solution for Hamilton-Jacobi equations and its motivation as the uniform limit of viscous approximations with vanishing viscosity. The Hopf-Lax formula is a viscosity solution. Definition for general equations and consistency of viscosity solutions with classical ones. Sub- and superdifferential of semicontinuous functions, equivalent definition of viscosity solution. Stability with respect to uniform convergence. The example \( |u'| = 1 = 0, \ u(-1) = 0 = u(1) \). Comparison Principle for classical and for viscosity solutions in bounded domains; uniqueness of solution to the Dirichlet problem. Change of dependent variable in H-J equations. The distance function from the boundary is the unique solution of the eikonal equation with null boundary condition. Comparison theorem for evolutive equations: behaviour of the solutions at the terminal time. The Hopf-Lax formula is the unique solution of the Cauchy problem.

7. Introduction to optimal control via Dynamic Programming (refs. [BCD], ch. III, sect. 1 and 3 and appendix 5; [E], ch. 10)

properties. Dynamic Programming Principle and Hamilton-Jacobi-Bellman equation. The value function is the unique solution of a terminal value problem for the HJB equation.

8. Introduction to Linear-Quadratic control (refs. [FR] cap. IV, sez. 4 and 5; [B])

Feedback controls. Verification theorem and synthesis of an optimal feedback for finite horizon optimal control problems. Linear-Quadratic regulator with finite horizon and Riccati differential equation. Solvability of the Riccati equation.

9. Introduction to game theory (refs.[Bar] ch. 1 and 3; [Bre], [B])


Two person, non-0-sum games: Nash equilibria and examples (the prisoner’s dilemma,...). Pareto optima. Existence of Nash equilibria under convexity assumptions by means of Brouwer fixed point theorem; equilibria in mixed strategies.

10. Introduction to differential games (refs. [Bre], [ES], [BCD] cap. VIII, [B])


11. A short introduction to deterministic Mean Field Games (ref. [C], [B])

The continuity equation in weak form for a measure transported by a flow. Motivation and derivation of the Mean Field Games system of 1st order PDEs (MFG). The Monge-Kantorovitch distance on probability measures; continuity of the cost with respect to the Kantorovich-Rubinstein metrics $d_1$, examples. Uniqueness of a (classical) solution to the MFG system under a monotonicity condition. An existence theorem for solutions of the MFG system (the outline of the proof is optional).

N.B.: Only the proofs presented in class are part of the program, see the pdf files of the lectures available on Moodle.

Bibliography

The items with asterisk * are available on Moodle.


