PROBLEM 1

The feedback system shown in the figure is based on an amplifier and a feedback network whose transfer functions are:

\[ A(s) = \frac{A_0}{1 + s/\omega_p} \quad \beta(s) = \beta \in [0, 1] \]

Derive the expression of the low-frequency gain and of the -3 dB bandwidth \( \omega_\beta \) of the feedback amplifier transfer function \( H(s) = Y(s)/X(s) \). Then, show that the expression of the step response of the feedback amplifier is:

\[ x(t) = \begin{cases} 0, & t < 0 \\ X_{\text{step}}, & t \geq 0 \end{cases} \quad \Rightarrow \quad y(t) = X_{\text{step}} \frac{1}{\beta} \frac{T_0}{1 + T_0} (1 - e^{-\omega_p t}) \]

where \( T_0 = \beta \cdot A_0 \). Note that \( \omega_\beta \approx \omega_c \) if \( T_0 \gg 1 \), where \( \omega_c \) is the gain crossover frequency of the loop transmission \( T(s) = \beta(s) \cdot A(s) \).

\[
\begin{aligned}
Y(s) &= A(s) X_e(s) \\
X_e(s) &= X(s) - X_f(s) \quad \Rightarrow \quad Y(s) = A(s) \left[ X(s) - \beta(s) Y(s) \right] \\
X_f(s) &= \beta(s) Y(s) 
\end{aligned}
\]

The closed-loop transfer function is:

\[
H(s) = \frac{Y(s)}{X(s)} = \frac{A(s)}{1 + \beta(s) A(s)} = \frac{A_0}{(1 + s/\omega_p) + \beta A_0}
\]

\[
= \frac{1}{\beta} \left( \frac{\beta A_0}{1 + \beta A_0} \right) \frac{1}{1 + s/\omega_c} \]

\[
= H_\infty \cdot \frac{T_0}{1 + T_0} \frac{1}{1 + s/\omega_c}
\]

where

\[ T_0 = \beta A_0 \quad \text{loop gain} \]

\[ H_\infty = \frac{1}{\beta} \quad \text{ideal closed-loop gain, i.e. gain of the feedback system when } T_0 \rightarrow \infty \]
\[ \omega_p^1 = \omega_p (1 + T_p) \text{ -3dB bandwidth of the feedback system} \]

Note that if \( T_p \gg 1 \), then \( \omega_c \approx \omega_p T_p \). This is approximately equal to the loop transmission crossover frequency \( \omega_c \):

\[
\omega_c = \left| T(\omega_c) \right| = 1 \quad \left| \frac{\beta A}{\omega_c^2 + \left( \frac{\omega_c}{\omega_p} \right)^2} \right| = 1 \\
= \frac{\omega_c}{\sqrt{1 + \left( \frac{\omega_c}{\omega_p} \right)^2}} = \sqrt{\frac{T_p^2}{1 + \left( \frac{\omega_c}{\omega_p} \right)^2}} = \omega_p \sqrt{T_p^2 - 1}
\]

If \( T_p \gg 1 \), then \( T_p^2 \) is even closer, so

\[ \omega_c \approx \omega_p T_p \approx \omega_p^1 \]

---

**Step response of the feedback system**

\[
x(t) = \begin{cases} 
0, & t < \alpha \\
X_{step}, & t \geq \alpha 
\end{cases} \Rightarrow \mathcal{L} \{ x(t) \} = X(s) = \frac{X_{step}}{s}
\]

\[
Y(s) = H(s) X(s) = H_0 \frac{T_p}{1 + T_p s} \frac{1}{1 + s/\omega_p^1} \cdot \frac{X_{step}}{s}
\]

\[
= H_0 \frac{T_p}{1 + T_p} \cdot \frac{X_{step}}{s} \cdot \frac{\omega_p^1}{(s + \omega_p^1)}
\]

Given that:

\[
\mathcal{L}^{-1} \left\{ \frac{\alpha}{s(s + \alpha)} \right\} = (1 - e^{-\alpha t}) \cdot \text{unit step}
\]

then we have
\[ y(t) = \mathcal{L}^{-1} \{ Y(s) \} = \begin{cases} \infty, & t < 0 \\ Y_{\text{step}}(t) \frac{T_\phi}{1 + T_\phi} \left( 1 - e^{-\omega_0 t} \right), & t \geq 0 \end{cases} \]
PROBLEM 2

Consider a feedback amplifier based on an OTA whose voltage transfer function \( A(s) \) features two poles and no zeroes in the frequency range of interest, with a capacitive feedback network. Assume that the low-frequency gain of the OTA \( A_0 = A(j0) \) is large enough so that \( \beta A_0 \gg 1 \) and that \( \rho = \frac{\omega_p2}{\omega_p1} \gg 1 \), where \( \omega_p1 \) and \( \omega_p2 \) are the frequencies of the dominant and of the non-dominant pole of the OTA, respectively.

1) Derive the expression of the closed-loop transfer function \( H(s) \) of the feedback amplifier and of the frequency \( \omega'_p1 \) and \( \omega'_p2 \) of its two poles.

2) Derive the expression of \( \rho \) corresponding to: (a) real coincident poles \( \omega'_p1 = \omega'_p2 \) and (b) complex conjugate poles of the feedback amplifier with \( \text{Re}\{p'_1\} = \text{Im}\{p'_1\} \) and \( \text{Re}\{p'_2\} = -\text{Im}\{p'_2\} \). [Note: the expression of \( \rho \) contains \( A_0, \beta, \omega_p1 \) and \( \omega_p2 \).]

3) Compute the phase margin \( PM \) corresponding to scenario (a) and (b) of the previous item.

SOLUTION

1) Closed-loop TF and frequency of its poles

\[
H(s) = \frac{V_o(s)}{V_i(s)} = \frac{A(s)}{1 + \beta A(s)}
\]

Given that the OTA voltage transfer function is:

\[
A(s) = \frac{A_0}{(1 + s/\omega_p1)(1 + s/\omega_p2)}
\]

then, the closed-loop voltage transfer function becomes:

\[
H(s) = \frac{A_0}{(1 + s/\omega_p1)(1 + s/\omega_p2) + \beta A_0}
\]

The poles of \( H(s) \) are the roots of the polynomial:

\[
s^2 + \left( \frac{1}{\omega_p1} + \frac{1}{\omega_p2} \right) s + \frac{1}{\omega_p1 \omega_p2} = 0
\]

Given the assumptions that \( T_p = \beta A_0 \gg 1 \) and \( \rho = \frac{\omega_p2}{\omega_p1} \gg 1 \)

the polynomial can be approximated as
\[ s^2 + \omega_p^2 s + \omega_p \omega_{p2} T \phi = 0 \]

\[ p_1, p_2 = -\frac{\omega_p^2}{2} \pm \sqrt{\left( \frac{\omega_p^2}{4} - \omega_p \omega_{p2} T \phi \right)} \]

\[ = -\frac{\omega_p}{2} \left( 1 \pm \sqrt{1 - 4 \frac{\omega_p \omega_{p2} T \phi}{\omega_p^2}} \right) \]

**Real negative poles** if \( \omega_p^2 \geq 4 \omega_p \omega_{p2} T \phi \)

\[ \omega_{p1} = \frac{\omega_p^2}{2} \left( 1 - \sqrt{1 - 4 \frac{\omega_p \omega_{p2} T \phi}{\omega_p^2}} \right) = \omega_p T \phi \text{ if } \omega_p \gg 4 \omega_p \omega_{p2} T \phi \]

\[ \omega_{p2} = \frac{\omega_p^2}{2} \left( 1 + \sqrt{1 - 4 \frac{\omega_p \omega_{p2} T \phi}{\omega_p^2}} \right) = \omega_p - \omega_p T \phi \text{ if } \omega_p \gg 4 \omega_p \omega_{p2} T \phi \]

**Complex conjugate poles** if \( \omega_p^2 < 4 \omega_p \omega_{p2} T \phi \)

\[ p_1, p_2 = -\frac{\omega_p}{2} \left( 1 \pm j \sqrt{\frac{4 \omega_p \omega_{p2} T \phi}{\omega_p^2} - 1} \right) \]

2.a) **Real coincident poles** when \( \omega_p = 4 \omega_p T \phi \)

\[ B = \frac{\omega_p^2}{\omega_{p1}} = 4 T \phi \Rightarrow \omega_{p1} = \omega_{p2} = \frac{\omega_p}{2} \]

2.b) **Complex conjugate poles** with \( Re(p_1') = Im(p_1') \quad Re(p_2') = -Im(p_2') \)

\[ \Rightarrow \sqrt{\frac{4 \omega_p T \phi}{\omega_{p2}} - 1} = 1 \quad \frac{4 \omega_p}{\omega_{p2}} T \phi = 2 \]

\[ \phi = \frac{\omega_{p2}}{\omega_p} = 2 T \phi \]
\[ S = \frac{\omega_p^2}{\omega_{p_1}} = 2 T_\phi \]

3.a) \text{PM when } \ p_1 = p_2^1

\[ \text{PM} = 180^\circ - \arctan \left( \frac{\omega_c}{\omega_{p_1}} \right) - \arctan \left( \frac{\omega_c}{\omega_{p_2}} \right) \]

\[ \omega_c : \quad | T_{\omega_c} | = 1 \quad \frac{B A_\phi}{\sqrt{[1 + \left( \frac{\omega_c}{\omega_{p_1}} \right)^2] [1 + \left( \frac{\omega_c}{\omega_{p_2}} \right)^2]}} = 1 \]

\[ T_\phi^2 = \left( 1 + x \right) \cdot \left( 1 + \left( \frac{\omega_{p_1}}{\omega_{p_2}} \right)^2 \right) \quad x = \left( \frac{\omega_c}{\omega_{p_1}} \right)^2 \]

Given that if \( p_1 = p_2^1 \) then \( \frac{\omega_{p_1}}{\omega_{p_2}} = \frac{1}{\Delta T_\phi} \) (see 2.a above)

we have

\[ x^2 + \left[ 1 + \left( \Delta T_\phi \right)^2 \right] x + \left( \Delta T_\phi \right)^2 \left( 1 - T_\phi^2 \right) = 0 \]

Since \( T_\phi \gg 1 \):

\[ x^2 + 16 T_\phi^2 x - 16 T_\phi^4 = 0 \]

\[ x = -8 T_\phi^2 \pm \sqrt{64 T_\phi^4 + 16 T_\phi^4} \]

Since \( x = \left( \frac{\omega_c}{\omega_{p_1}} \right)^2 \) must be positive:

\[ x = \left( \frac{\omega_c}{\omega_{p_1}} \right)^2 = -8 T_\phi^2 + \sqrt{64 T_\phi^4 + 16 T_\phi^4} = T_\phi^2 \left( \frac{\sqrt{80}}{\omega_{p_1}} - 5 \right) \approx T_\phi^2 \]

\[ \omega_c \equiv \omega_{p_1}, \quad T_{\omega} \]
\[ \Phi = 180^\circ - \arctan \left( \frac{\omega_p T_\theta}{\omega_p T_\phi} \right) - \arctan \left( \frac{\omega_p T_\phi}{4 \omega_p T_\theta} \right) \]

\[ = 180^\circ - 80^\circ - 14^\circ = 76^\circ \]

3.6) \( \Phi \) when \( p_1^* \), \( p_2^* \) are complex conjugate with \( |Re| = |Im| \)

\[ p_1^* , p_2^* = -\frac{\omega_p}{2} \left( 1 \pm i 1 \right) \quad \text{with} \quad \omega_p = 2 \omega_p \cdot T_\phi \quad \text{(see 2.6 above)} \]

\[ \omega_c \quad \left| T(\omega_c) \right| = 1 \quad T_\phi^2 = \left[ 1 + \left( \frac{\omega_c}{\omega_p} \right)^2 \right] \left[ 1 + \left( \frac{\omega_c}{\omega_p^2} \right)^2 \right] \]

\[ T_\theta^2 = \left( 1 + x \right) \left[ 1 + \frac{4}{(2 T_\theta)^2} x \right] \quad x = \left( \frac{\omega_c}{\omega_p} \right)^2 \]

\[ x^2 + \left[ 1 + (2 T_\theta)^2 \right] x + (2 T_\theta)^2 \left( 1 - T_\theta^2 \right) = 0 \]

\[ x^2 + 4 T_\theta^2 x - 4 T_\theta^4 = 0 \]

\[ x = -2 T_\theta^2 \pm \sqrt{4 T_\theta^4 + 4 T_\theta^4} \quad \text{again, the negative root must be discarded since} \ x > 0 \]

\[ x = T_\theta^2 \left( \sqrt{2} - 2 \right) = 0.828 T_\theta^2 \]

\[ \omega_c = \sqrt{0.828} \omega_p T_\theta \approx 0.81 \omega_p \cdot T_\phi \quad \approx \omega_p T_\phi \]

\[ \Phi = 180^\circ - \arctan \left( \frac{0.81 \omega_p T_\phi}{\omega_p} \right) - \arctan \left( \frac{0.81 \omega_p T_\phi}{2 \omega_p T_\theta} \right) \]

\[ = 180^\circ - 90^\circ - 24.5^\circ = 65.5^\circ \]
PROBLEM 3
An amplifier is realized using a single-stage OTA with capacitive feedback. The OTA transconductance is $G_m = 4 \text{ mS}$, the output resistance $R_o = 100 \text{ k\Omega}$, the parasitic capacitance of each input and output node $C_i = C_o = 100 \text{ fF}$, and a non-dominant pole at frequency $\omega_p = 2\pi (1 \text{ GHz})$. Compute:

1) the value of capacitance $C_f$ so that the ideal (i.e. assuming $T_0 = \beta A_{\text{add}} \rightarrow \infty$) low-frequency closed-loop gain of the amplifier is $A_{\text{cl0}} = -4$ assuming $C_s = 400 \text{ fF}$;
2) the value of the feedback factor $\beta$ and of the capacitive load due to the feedback network $C_{\text{fb}}$;
3) the actual value of low-frequency closed-loop gain of the amplifier $A_{\text{cl}}$, taking into account the finite value of $T_0 = \beta A_{\text{add}}$;
4) the bandwidth of the amplifier, that you may estimate as the gain crossover frequency $\omega_c$ of the loop transmission $T(j\omega)$, and the phase margin $\text{PM}$ (assume $C_L = 1 \text{ pF}$).

Finally, discuss the effect of an increase of the parasitic capacitance $C$ on the feedback amplifier gain and bandwidth.

SOLUTION

1) Leveraging the symmetry of the circuit topology, the feedback amplifier can be studied using the half circuit equivalent model:

where $\text{Add}(s)$ is the OTA voltage transfer function in differential mode.

$$\text{Add}(s) = \frac{\text{Addx}}{1 + s/\omega_p}$$

$$\text{Addx} = C_o R_o$$

$$\omega_p = \frac{1}{R_o C_i}$$

$$C_i = C_o + C_l + C_f$$

$$A_{cl}(s) = -\frac{\alpha}{\beta} \frac{T(s)}{1 + T(s)}$$

$$\alpha = \frac{C_s}{C_i + C_s + C_f}$$

$$\beta = \frac{C_f}{C_i + C_s + C_f}$$

$$T(s) = \beta \text{Add}(s)$$

$$A_{cl}(s) = -\frac{C_s}{C_f} \frac{\beta \text{Addx}}{1 + s/\omega_p + \beta \text{Addx}} = -\frac{C_s}{C_f} \frac{T_f}{1 + T_f} \frac{1}{1 + \frac{s}{\omega_p(1 + T_f)}}$$

$$\omega_p = \omega_c (1 + T_f)$$

$$\omega_c = \frac{1}{T_f + T_0}$$
If \( T \to \infty \) then \( \text{Ae}_\text{c} \to \text{Ae}_\text{c} \text{d} \) (or \( \text{A}_\infty \)) = \(- \frac{C_s}{C_t} \)

So, for \( \text{Ae}_\text{c} \text{d} = -4 \)
\[
C_t = \frac{C_s}{\text{Ae}_\text{c} } = 100 \text{ fF}
\]

2) \( \beta \) and \( C_{fb} \)
\[
\beta = \frac{C_t}{C_i + C_s + C_t} = 0.167 \quad C_{fb} = C_t(1-\beta) = 83.3 \text{ fF}
\]

3) \( \text{Ae}_\text{c} \text{d} \)
\[
T_\infty = \beta \text{A} \text{d} \text{d} = \beta \text{C} \text{r} \text{R}_0 = 66.8
\]
\[
\text{Ae}_\text{c} = - \frac{C_s}{C_t} \frac{T_\infty}{1 + T_\phi} = \frac{\text{Ae}_\text{c}}{C_t} \frac{T_\phi}{1 + T_\phi} = - 3.89
\]

4) \( \omega_p' \) and \( \text{PM} \)
\[
\omega_p' = \omega_p (1 + T_\infty) = \frac{1}{R_0 C_s} (1 + T_\phi) = 2\pi (831.2 \text{ MHz})
\]
\[
\omega_c = \omega_p \sqrt{T_\infty^2 - 1} = \omega_p T_\infty = 2\pi (935.8 \text{ MHz}) \equiv \omega_p'
\]
\[
\text{PM} = 180^\circ - \arctan \left( \frac{\omega_c}{\omega_p} \right) = 180^\circ - \arctan \left( \frac{\text{A}R T_\phi}{\omega_p} \right)
\]
\[
= 180^\circ - 88.1^\circ = 91.9^\circ
\]

5) If \( C_i < 4 \) then \( \beta \downarrow \), \( T_\phi = \beta \text{A} \text{d} \text{d} \downarrow \) and
\[
|\text{Ae}_\text{c} \text{d}| = \frac{C_s}{C_t} \frac{T_\phi}{1 + T_\phi} \downarrow
\[ \omega_p' \equiv \omega_p (1 + \tau p) \downarrow \]