Logic for knowledge representation, learning, and inference

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Contents

CHAPTER 1

Resolution and Unification

1. Propositional resolution

to be done

2. Unification

to be done

3. Deciding (un)satisfiability in FOL

to be done

4. Exercises

Exercise 1:
Let $\theta = [x/f(y)]$,$\lambda = [y/z]$ and $\mu = [z/a]$. compute:

(1) $\theta \circ \lambda \circ \mu$
(2) $\theta \circ \mu \circ \lambda$
(3) $\lambda \circ \theta \circ \mu$
(4) $\lambda \circ \mu \circ \theta$
(5) $\mu \circ \theta \circ \lambda$
(6) $\mu \circ \lambda \circ \theta$

Exercise 2:
Find two substitutions α and β such that $\alpha \circ \beta \neq \beta \circ \alpha$.

Exercise 3:
Prove that the composition of substitutions is associative. I.e., that for every substitutions α, β, and γ:

$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$$

Solution Let α, β and γ the following substitutions

$$\alpha = [x_1/t_1, \ldots, x_n/t_n]$$
$$\beta = [y_1/u_1, \ldots, y_n/u_n]$$
$$\gamma = [z_1/v_1, \ldots, z_n/n_n]$$

To prove the associativity property, we use the fact that for every substitution $\sigma = [x_1/t_1, \ldots, x_n/t_n]$, then, for every substitution θ, then

$$\sigma \circ \theta = [x_1/t_1\theta, \ldots, x_n/t_n\theta]$$

i.e., $\sigma \circ \theta$ is the substitution obtained by applying the substitution θ to all the terms t_i of the substitution σ. We therefore have that

$$\alpha \circ (\beta \circ \gamma) = [x_1/t_i(\beta \circ \gamma), \ldots, x_n/t_n\beta \circ \gamma)]$$

with

$$\beta \circ \gamma = [y_1/u_1\gamma, \ldots, y_n/y_n\gamma]$$

And therefore we have that

$$\alpha \circ (\beta \circ \gamma) = [x_1/t_i[y_1/u_1\gamma, \ldots, y_n/y_n\gamma] \ldots, x_n/t_n[y_1/u_1\gamma, \ldots, y_n/y_n\gamma]]$$

We also thave that

$$(\alpha \circ \beta) = [x_1/t_1[y_1/u_1 \ldots, y_n/u_n], \ldots x_n/t_n[y_1/u_1, \ldots, y_n/u_n]]$$

and therefore

$$\begin{aligned}(\alpha \circ \beta) \circ \gamma &= [x_1/t_1[y_1/u_1 \ldots, y_n/u_n]\gamma, \ldots x_n/t_n[y_1/u_1, \ldots, y_n/u_n]\gamma] \\ &= [x_1/t_1[y_1/u_1\gamma \ldots, y_n/u_n\gamma], \ldots x_n/t_n[y_1/u_1\gamma, \ldots, y_n/u_n\gamma]]\end{aligned}$$

□

Exercise 4:
Find the most general unifier (MGU) of the set of atoms $\{P(a, y), P(xf(b))\}$
Exercise 5:

Find a most general unifier for the set $\{P(a, x, f(g(y))).P(z, f(z), f(u))\}$ **Solution** $\theta = [z/a.x/f(a), u/g(y)]$ □

Exercise 6:
Determine whether or not the set $W = \{Q(f(a), g(x)), Q(y, y)\}$ is unifiable. **Solution** W is not unifiable. □

Exercise 7:
Determine whether each of the following sets of expressions are unifiable. If yes give a MGU:

(1) $\{Q(a, x, f(x)), Q(a, y, y)\}$
(2) $\{Q(x, y, z), Q(u, h(v, v), u)\}$

Exercise 8:
Transform the following formula in prenex Skolemized conjunctive normal form:

$\forall x \exists y \exists z (father(x, y) \wedge mother(x, z)) \wedge$

$\forall xyzw(father(x, z) \wedge mother(x, w) \wedge father(y, z) \wedge mother(y, w) \rightarrow sibling(x, y)$

Solution

$father(x, f(x))$

$mother(x, m(x))$

$\neg father(x, z) \lor \neg mother(x, w) \lor \neg father(y, z) \lor \neg mother(y, w) \lor sibling(x, y)$

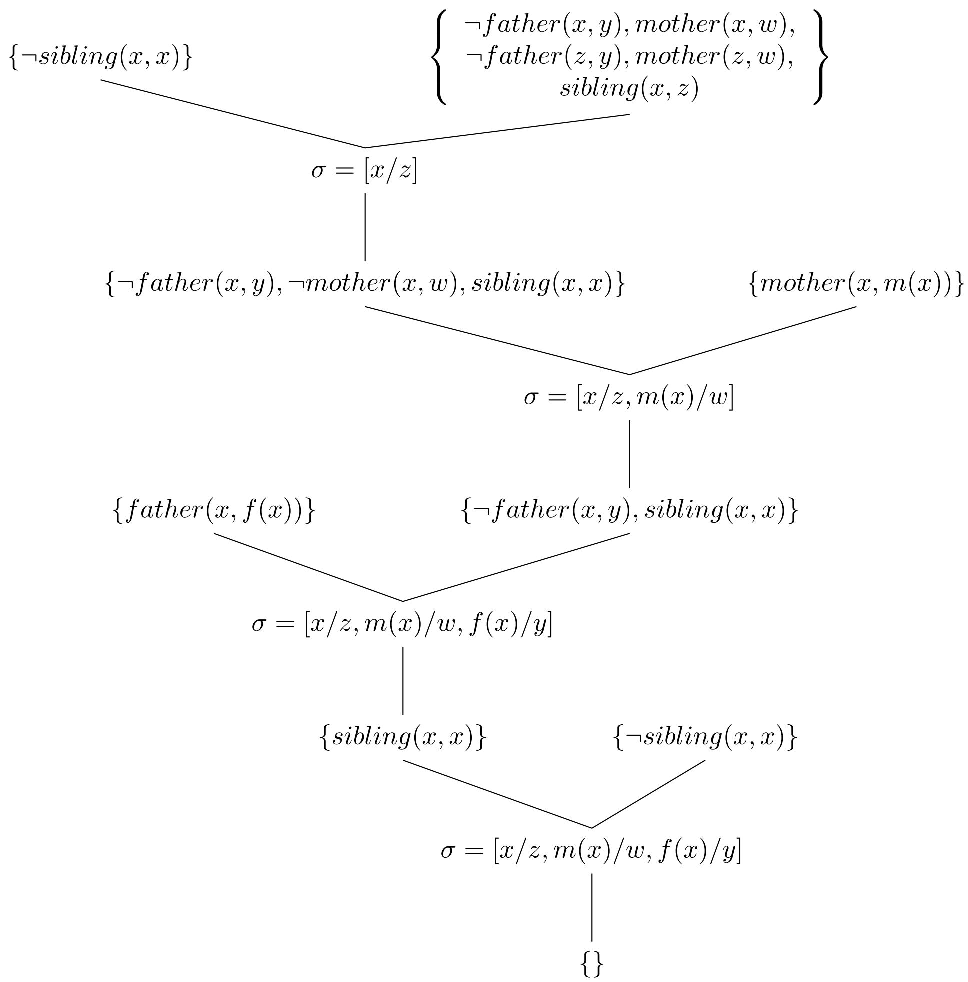
□

Exercise 9:

From the clauses of the previous exercise prove that

$$sibling(x, x)$$

Solution



□ **Exercise 10:**

Find a most general unifier for the set

$$\{P(a, x, f(g(y))).P(z, f(z), f(u))\}$$

Solution

$$\sigma = [z/a, x/f(a), u/g(y)]$$

□ **Exercise 11:**

Apply the resolution and unification rule to the following clauses

$$\neg P(x,y) \vee \neg Q(x,b,y)$$
$$Q(a,z,f(z,w)) \vee m(w,b)$$

x, y, z, w are variables, and a, b are constants **Solution** The two clauses contains two opposite literals on the predicate Q that unify which are:

$$Q(x,b,y), Q(a,z,f(z,w))$$

Their most general unifier is

$$\sigma = [x/a, y/f(b,w), z/b]$$

By applying the resolution rule we obtain the clause

$$\neg P(a, f(b,w)) \vee m(w,b)$$

□

Exercise 12:

Find all resolvents (i.e., all the clauses that can be derived from the application of first order resolution) of the following two clauses:

$$\phi_1 = \{\neg P(x,y), \neg P(f(a), g(u,b)), Q(x,u)\}$$
$$\phi_2 = \{P(f(x), g(a,b)), \neg Q(f(a),b), \neg Q(a,b)$$

where x, y, and u are variables and a and b are constants. **Solution** Let us first rename the variables in order to be sure that there is no clashing. We rename the variable x of the second clause with z obtaining

$$\phi_2 = \{P(f(z), g(a,b)), \neg Q(f(a),b), \neg Q(a,b)$$

The two clauses contains four pairs of opposite literals that can bee unified. In the following table we report each pair of literal the most general unifier, and the corresponding resolvent

Lit. in ϕ_1	Lit. in ϕ_2	Unifier	resolvent
$\neg P(x,y)$	$P(f(z), g(a,b))$	$x/f(z), y/b$	$\neg P(f(a), g(u,b)), Q(f(z),u), \neg Q(f(a),b), \neg Q(a,b)$
$\neg P(f(a), g(u,b))$	$P(f(z), g(a,b))$	$z/a, u/a$	$\neg P(x,y), Q(x,a), \neg Q(f(a),b), \neg Q(a,b)$
$Q(x,y)$	$\neg Q(f(a),b)$	$x/f(a), y/b$	$\neg P(f(a),b), \neg P(f(a), g(u,b), P(z, g(a,b)), \neg Q(a,b)$
$Q(x,y)$	$\neg Q(a,b)$	$x/a, y/b$	$\neg P(a,b), \neg P(f(a), g(u,b)), P(f(z), g(a,b)), \neg Q(f(a),b)$

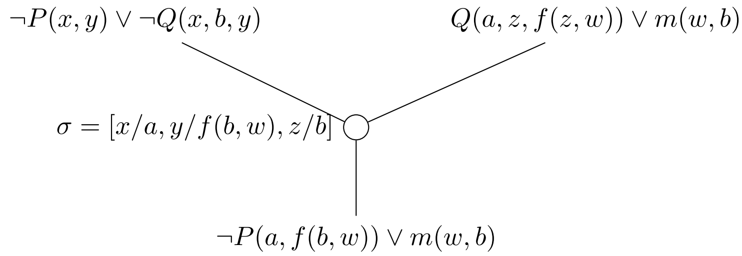
□

Exercise 13:

Apply the resolution and unification rule to the following clauses

$$\neg P(x,y) \vee \neg Q(x,b,y)$$
$$Q(a,z,f(z,w)) \vee m(w,b)$$

x, y, z, w are variables, and a, b are constants **Solution**



□

Exercise 14:
Conisder the following facts:

(1) Married people are humans;
(2) Every human has a mother;
(3) A parson is the mother in low of sombody, if she is the mother of his/her wife/husband;

Formalize them in a formula ϕ by using the predicates

- Human(x): x is a Human;
- Mother(x, y): x is the mother of y;
- MotherInLow(x, y): x is the mother in low of y;
- Married(x, y): x is merried with y.

and show by resolution that from ϕ is follows that every married person has a mother-in-low. **Solution**

(1) Married people are humans:

$$\forall xy(\mathit{Married}(x, y) \rightarrow \mathit{Human}(x) \wedge \mathit{Human}(y))$$

(2) Every human has a mother:

$$\forall x(\mathit{Human}(x) \rightarrow \exists y \mathit{Mother}(y, x))$$

(3) A parson is the mother in low of sombody, if she is the mother of his/her wife/husband:

$$\forall xyz.(\mathit{Mother}(x, y) \wedge \mathit{Married}(y, z) \rightarrow \mathit{MotInLow}(x, z))$$

We can transform in prenex CNF obtaining

$$\forall xy \neg \mathit{Married}(x, y) \vee \mathit{Human}(x)$$
$$\forall xy \neg \mathit{Married}(x, y) \vee \mathit{Human}(y)$$
$$\forall x \exists y \neg \mathit{Human}(x) \vee \mathit{Mother}(y, x)$$
$$\forall xyz \neg \mathit{Mother}(x, y) \vee \neg \mathit{Married}(y, z) \vee \mathit{MotInLow}(x, z)$$

The third clause need to be scolemized by introducing a new function say f obtaining, the set of clauses

(1) $\neg \mathit{Married}(x, y) \vee \mathit{Human}(x)$
(2) $\neg \mathit{Married}(x, y) \vee \mathit{Human}(y)$
(3) $\neg \mathit{Human}(x) \vee \mathit{Mother}(f(x), x)$
(4) $\neg \mathit{Mother}(x, y) \vee \neg \mathit{Married}(y, z) \vee \mathit{MotInLow}(x, z)$

From this set of clauses we want to derive the fact that every married person has a mother-in-low, which can be translated into

$$\forall xz(\mathit{Married}(x,z) \rightarrow \exists y \mathit{MotInLow}(y,z))$$

We first need to negate and transform it in prenex CNF. obtaining

$$\exists x,z \forall y(\mathit{Married}(x,z) \wedge \neg \mathit{MotInLow}(y,z))$$

By applying skolemization we introduce two new constants a and b, and we obtain the clauses

$$\mathit{Married}(a,b) \tag{5}$$

$$\neg \mathit{MotInLow}(y,b) \tag{6}$$

We can now apply the following resolution and unification chain:

(7)	$\mathit{Human}(a)$	(5), (1), x/a, y/b
(8)	$\mathit{Mother}(f(a),a)$	(7), (3), x/a
(9)	$\neg \mathit{Married}(a,z) \vee \mathit{MotInLow}(f(a),z)$	(8), (4) x/a, $y/f(a)$
(10)	$\neg \mathit{Married}(a,b)$	(9),(6), z/b, $y/f(a)$
(11)	$\bot$	(10),(5)

Since we can derive the empty clause $\bot$ form the set of clauses and the negation of the conclusion, it means that the conclusion logicall follows from the initial clauses. □

Exercise 15:

Use resolution to decide if

$$\forall x Q(x) \rightarrow \forall x P(x) \leftrightarrow \exists z \forall y(Q(y) \vee P(z))$$

is valid, satisfiable, non valid, or unsatisfiable, and explain your answer.

Solution We have to negate the formula and then transform it into Skolemized negatd normal form: For the CNF transformation, We use the fact that $\neg(A \leftrightarrow B)$ is equivalent to $(A \vee B) \wedge (\neg A \vee \neg B)$.

$$\neg((\forall x Q(x) \rightarrow \forall x P(x)) \leftrightarrow (\exists z \forall y(Q(y) \vee P(z)))) \leftrightarrow$$
$$(\forall x Q(x) \rightarrow \forall x P(x)) \vee (\exists z \forall y(Q(y) \vee P(z))) \wedge$$
$$(\neg(\forall x Q(x) \rightarrow \forall x P(x)) \vee \neg(\exists z \forall y(Q(y) \vee P(z))))$$

We treat the two conjuct separately. Let us start with the first one.

$$\forall x Q(x) \rightarrow \forall x P(x)) \vee (\exists z \forall y(Q(y) \vee P(z))$$

We rename variables.

$$\forall x Q(x) \rightarrow \forall w P(w)) \vee (\exists z \forall y(Q(y) \vee P(z))$$

We rewrite the $\rightarrow$ in terms of $\vee$.

$$\neg \forall x Q(x) \vee \forall w P(w)) \vee (\exists z \forall y(Q(y) \vee P(z))$$

and push the negation close to the atoms

$$(\exists x \neg Q(x) \vee \forall w P(w)) \vee (\exists z \forall y(Q(y) \vee P(z))$$

We apply Skolemization

$$(\neg Q(a) \vee \forall w P(w)) \vee (\forall y(Q(b) \vee P(z))$$

We mode the universal quantifiers into the front

$$\forall yw(\neg Q(a) \vee P(w) \vee Q(y) \vee P(b))$$

We therefore obtain the clause

$$\{\neg Q(a), P(w), Q(y), P(b)\}$$

Let us now conider the second conjunct

$$(\neg(\forall x Q(x) \to \forall x P(x)) \vee \neg(\exists z \forall y (Q(y) \vee P(z))))$$

We push the negation close to the atoms

$$(\forall x Q(x) \wedge \exists x \neg P(x)) \vee \forall z \exists y (\neg Q(y) \wedge \neg P(z))$$

We apply skolemization, introducting a new constant c (notice that you cannot use the constants used in the previous clauses) and the skolem function f.

$$(\forall x Q(x) \wedge P(c)) \vee \forall z (\neg Q(f(z)) \wedge \neg P(z))$$

We move the quantifiers in the front

$$\forall xz((Q(x) \wedge \neg P(c)) \vee (\neg Q(f(z)) \wedge \neg P(z)))$$

and transform the matrix in CNF obtaining the clauses

$$\{Q(x), \neg Q(f(z))\}.\{Q(x), \neg P(z)\}, \{\neg P(c), \neg Q(f(z))\}, \{\neg P(c), \neg P(z)\}$$

Summing up the set of clauses we have obtained are the following:

$$\{\neg Q(a), P(w), Q(y), P(b)\} \tag{12}$$

$$\{Q(x), \neg Q(f(z))\} \tag{13}$$

$$\{Q(x), \neg P(z)\} \tag{14}$$

$$\{\neg P(c), \neg Q(f(z))\} \tag{15}$$

$$\{\neg P(c), \neg P(z)\} \tag{16}$$

Notice that the set of clauses can be satisfied by the following interpretation:

$$\begin{aligned} \Delta^{\mathcal{I}} &= \{0\} \\ \mathcal{I}(a) = \mathcal{I}(b) = \mathcal{I}(c) &= 0 \\ \mathcal{I}(f) &= f(0) \mapsto 0 \\ \mathcal{I}(P) = \mathcal{I}(Q) &= \emptyset \end{aligned}$$

Since the negated of the formula is satisfiable, the formula is not valid. To check that the formula is satisfiable we have to build the an interpretation that satisfies it. To this purpopse it is convenient to transform it in CNF and then try to find an interpretation that satisfies each clause; This will show that the formula is satisfiable, Alternatively, we can try to derive the empty clause; In this case we would have proven that the formula is not satisfiable. Let us first transform the formula in CNF, We have that $A \leftrightarrow B$ is equivalent to the conjunction of $A \to B$ and $B \to A$; therefore let us transform each of the two part of the equivalence in CNF We start from the first equivalente, and we push the negation inside, and rewrite the implication

$$\begin{aligned}(\forall x Q(x) \to \forall x P(x)) \to \exists z \forall y (Q(y) \vee P(z)) \leftrightarrow \\ \neg(\forall x Q(x) \to \forall x P(x)) \vee \exists z \forall y (Q(y) \vee P(z)) \leftrightarrow \\ (\forall x Q(x) \wedge \neg \forall x P(x)) \vee \exists z \forall y (Q(y) \vee P(z)) \leftrightarrow \\ (\forall x Q(x) \wedge \exists x \neg P(x)) \vee \exists z \forall y (Q(y) \vee P(z))\end{aligned}$$

Now let us rename the variables

$$(\forall x Q(x) \wedge \exists w \neg P(w)) \vee \exists z \forall y (Q(y) \vee P(z))$$

Distribute the and w.r.t., or

$$(\forall x Q(x) \vee \exists z \forall y (Q(y) \vee P(z))) \wedge (\exists w \neg P(w) \vee \exists z \forall y (Q(y) \vee P(z)))$$

Skolemization

$$(\forall x Q(x) \vee \forall y (Q(y) \vee P(a))) \wedge (\neg P(b) \vee \forall y (Q(y) \vee P(c)))$$

Prenex normal form

$$\forall xy ((Q(x) \vee Q(y) \vee P(a)) \wedge (\neg P(b) \vee (Q(y) \vee P(c)))$$

and clausal form

$$\begin{gathered}\{Q(x), Q(y), P(a)\} \\ \{\neg P(b), Q(y), P(c)\}\end{gathered}$$

Let us now consider the opposite implication

$$\begin{aligned}\exists z \forall y (Q(y) \vee P(z)) \to (\forall x Q(x) \to \forall x P(x)) \leftrightarrow \\ \neg(\exists z \forall y (Q(y) \vee P(z))) \vee (\neg \forall x Q(x) \vee \forall x P(x)) \leftrightarrow \\ \forall z \exists y (\neg Q(y) \wedge \neg P(z)) \vee (\exists x \neg Q(x) \vee \forall x P(x)) \leftrightarrow\end{aligned}$$

Now let us rename the variables

$$\forall z \exists y (\neg Q(y) \wedge \neg P(z)) \vee (\exists x \neg Q(x) \vee \forall w P(w))$$

Skolemization and prenex normal form

$$\begin{aligned}\forall z (\neg Q(f(z) \wedge \neg P(z)) \vee (\neg Q(c) \vee \forall w P(w)) \leftrightarrow \\ \forall zw (\neg Q(f(z) \wedge \neg P(z)) \vee (\neg Q(c) \vee P(w)) \leftrightarrow \\ \forall zw (\neg Q(f(z)) \vee \neg Q(c) \vee P(w)) \wedge (\neg P(z) \vee \neg Q(c) \vee P(w))\end{aligned}$$

and rewrite in CNF

$$\begin{aligned}\{\neg Q(f(z)), \neg Q(c), P(w)\} \\ \{\neg P(z), \neg Q(c), P(w)\}\end{aligned}$$

Therefore the set of clauses are:

$$\begin{aligned}\{Q(x), Q(y), P(a)\} \\ \{\neg P(b), Q(y), P(c)\} \\ \{\neg Q(f(z)), \neg Q(c), P(w)\} \\ \{\neg P(z), \neg Q(c), P(w)\}\end{aligned}$$

Notice that the interpretation that interpret $\mathcal{I}(P)$ in the entire domain, will satisfy all the four clauses. Therefore the initial formula is also satisfiable. □

Exercise 16:

Consider the following facts:

(1) John likes all kind of food.
(2) Anything anyone eats and not killed is food.
(3) Anil eats peanuts and she is still alive

Prove by resolution that John likes peanuts. (Suggestion: you have to perform the following steps: 1. Formalize the statements in first order logic, 2. trasform in CNF 3. negate the goal, 4 derive the empty clause by resolution and unification).

Solution Step-1: Conversion of Facts into FOL

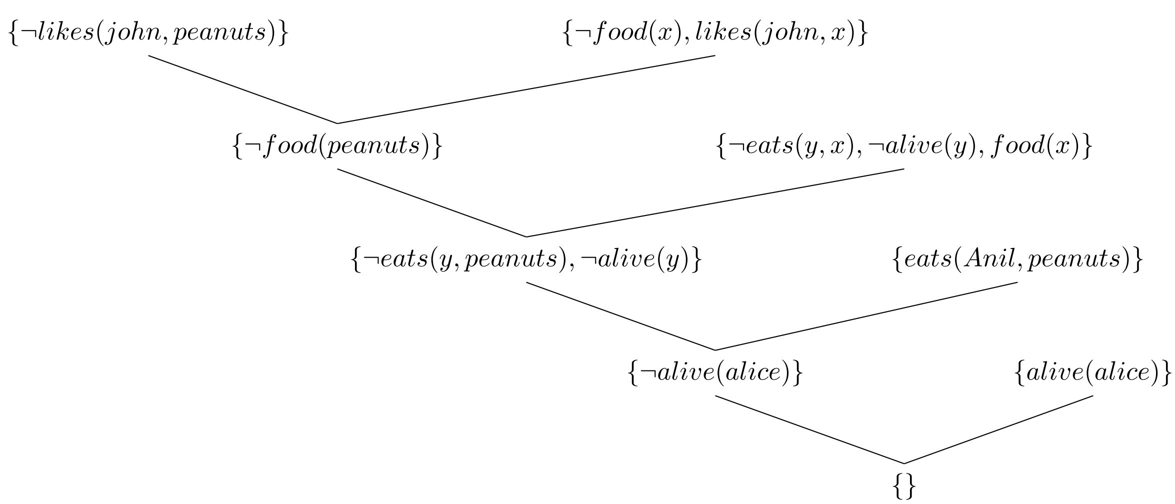
(1) $\forall x(food(x) \rightarrow likes(john, x))$
(2) $\forall x \forall y(eats(y, x) \wedge alive(y) \rightarrow food(x))$
(3) $eats(Anil, peanuts) \wedge alive(alice)$

Step-3: Conversion in CNF

(1) $\{\neg food(x), likes(john, x)\}$
(2) $\{\neg eats(y, x), \neg alive(y), food(x)\}$
(3) $\{eats(Anil, peanuts)\}$
(4) $alive(alice)$

Step-1: add negation of the goal

(1) $\{\neg food(x), likes(john, x)\}$
(2) $\{\neg eats(y, x), \neg alive(y), food(x)\}$
(3) $\{eats(Anil, peanuts)\}$
(4) $alive(alice)$
(5) $\neg likes(john, peanuts)$



Since we derived the empty clasue it means that the goal logically follows from the premises. □