LECTURE 20, May 18, 2023

GENERAL THEORY of PDEs for DIFFERENTIAL GAMES?

- NOT for NON-ZERO SUM GAMES!
  because systems of 1st. order nonlinear
  are NASTY.
- YES for 0-sum games & the Isaccs equal

PLAN
- def. a value function.
- DYNAMIC PROGRAM, PRINCIPLE
- value is a vise. sol. of Isaacs eq. ... the unique one.

Q: How to define the value function?

N.B. open loop vs. open loop is not good!

History:
- R. Isaacs 50-60! method of characteristics
  piecewise C' solutions. $n \leq 2$
- W. Fleming (1961)
  Euler scheme: solve static $\Delta t$
  games at discrete times, let $\Delta t \to 0$.
- A. Friedman, time localization $\Delta t \to 0$...
"Positional differential games."
also involves a limit procedure.

- **Non-anticipating (causal or markovian) strategies**

Vainberg, Roxin, Elliott-Kalton (1967-72)

Notations: \( \Theta_t : = \{ \alpha : [t, T] \to A \text{ measurable} \} \)

\( B_t : = \{ B : [t, T] \to B \text{ measurable} \} \)

**Def.** A strategy of 1st player is \( \alpha : B_t \to \Theta_t \) i.e. nonanticipating if \( \forall t \leq s \leq T, \forall B, \hat{B} \in B_t, b(\alpha) = \hat{b}(\alpha) \forall t < \tau \leq s \Rightarrow \alpha[\hat{b}](\tau) = \alpha[b](\tau) \forall \tau \leq 2t \leq T \)

\[
\begin{array}{c}
0 \\
\hline
\dots \quad t \quad \dots \quad T \\
\hline
0 \quad 1 \quad 2 \quad \ldots \quad T
\end{array}
\]

\( \alpha[\hat{b}](\tau) = \alpha[b](\tau) \)

\( T_t = \{ \text{nonantic. strat. 1st player} \} \)

\( \Delta_t = \{ B \subset A | b \in B \} \)

\( \beta : \Theta_t \to B_t, \forall t \in [t, T], \forall \alpha, \hat{\alpha} \in \Theta_t, \alpha(\tau) = \hat{\alpha}(\tau) \forall t \leq \tau \leq T \Rightarrow \beta[\alpha](\tau) = \beta[\hat{\alpha}](\tau) \forall \tau \in [t, T] \)

\( \text{N.B.} \quad \forall B \in \Theta_t \forall \alpha \in \Theta_t \exists \text{unique trajectory of } \)

\[
\begin{cases}
\dot{y}(\tau) = f(y(\tau), \alpha(t), b(\tau)) & \tau \geq t \\
y(t) = x
\end{cases}
\]
Define \( y(x; t, a, \beta [b]) \).

Similarly, \( \dot{y} = \mathcal{H}(y, \alpha, \beta [e]) \).

Payoff (for 1st) - cost (for 2nd) functional

\[
J(x; t, a(\cdot), b(\cdot)) = \int_t^T \mathcal{L}(y_x(x), a(x), b(x)) \, dx + g(y_x(T))
\]

**DEF.** The lower value of the D.C. is

\[
V(t, x) := \inf_{x \in \Xi} \sup_{a \in A_t} \inf_{\beta \in B_t} J(x, t, \alpha, \beta [e])
\]

\[
V^*(t, x) := \sup_{a \in A_t} \inf_{\beta \in B_t} J(x, t, \alpha, \beta [e], b)
\]

If \( V(t, x) = V^*(t, x) \) the D.C. has a value.

Q: \( V \leq V^* ? \)

**Example** (Bekovitz)

\[
\begin{align*}
g(t) &= (a - b)^2 \geq 0 \\
g(x) &= x
\end{align*}
\]

\( A = B = \{0, 1\} \), \( e = 0 \), \( g' > 0 \)

\( b \) minimizes the worst \( \hat{y} = 0 \)

\( \beta^* \equiv \hat{y} = 0 \)

\( V(x; t, a, \beta) \equiv y_x(x) \forall a \in A_t \)

\[
\Rightarrow \quad V(t, x) \leq \sup_{a \in A_t} g(y_x(T)) = g(x)
\]
Example of STRATEGIES.

Ex.1. CONSTANT STRAT. Fix $x \in \mathbb{B}_t$

$\beta \in \mathcal{A}_g \implies \beta \in \Delta_t \implies A_t \subseteq \mathbb{B}_t$

$\Rightarrow \beta \in \mathcal{A}_g \implies \beta \in \mathcal{A}_g$ is measurable, $\forall \in \mathcal{A}_g$

Ex.2. Feedback: $\Phi : R^n \times [0,T] \rightarrow B$ s.t. $\forall \in \mathcal{A}_g$

$\Phi (q(x), 0) = q(x)$ measurable. Then $\beta \in \Delta_t$

$\Rightarrow \beta \in \mathcal{A}_g \implies \beta \in \mathcal{A}_g$ is measurable.
Remark. Information pattern is not very realistic, but all more realistic values $e \in \mathcal{V}$, so if $V = \tilde{V}$ they all coincide.

Es. If $V = \tilde{V}$ and saddle $\mathcal{V}$ among admissible feedbacks $\Rightarrow \tilde{V} = V = V$.

Assumptions. \( T > 0 \), \( A \), \( B \) metric compact.

1. \( f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \) cont., \( |f| \leq C_1 \)
   \[ |f(x, a, b) - f(x', a, b)| \leq C_1 \| x - x' \| \quad \forall x, x' \in \mathbb{R}^n, a \in \mathbb{A}, b \in \mathbb{B} \]

2. \( e: \mathbb{R}^n \times \mathbb{A} \times \mathbb{B} \rightarrow \mathbb{R} \) cont., \( |e| \leq C_2 \)
   \[ |e(x, a, b) - e(x', a', b')| \leq C_2 \| x - x' \| \quad \forall x, x' \in \mathbb{R}^n, a, b \in \mathbb{A} \]

3. \( g: \mathbb{R}^n \rightarrow \mathbb{R} \) \( |g| \leq C_3 \), \( |g(x) - g(x')| \leq C_3 \| x - x' \| \)

Remark. I can add \( g_{n+1} = e(g, a, b) \)

\[ g((x, g_{n+1})) = g(x) + g_{n+1} \] set an equivalent game with \( \tilde{e} \equiv e \tilde{g} \).

Dynamic Programming Principle (TENET OF TRANSITION).

Then. \( 0 < t < t + \sigma \leq T \)

\[ V(t, x) = \inf_{A} \sup_{y(t, x)} \int_{t}^{t+\sigma} \left\{ e(y(t), x, \sigma), \tilde{g}(x(t), x, \sigma) \right\} dt + V(t + \sigma, g) \]
where \( y(x) = y_x(\cdot; t, a, b \in \mathcal{B}) \).

\[ W(t, x) = \sup_{\gamma \in \Gamma} \left\{ \int_{0}^{t+a} p(\gamma(s), a, b) \, ds + O(t+\alpha, y(t+\alpha)) \right\} \]

\[ y(\cdot) = y_x(\cdot; t, a, b \in \mathcal{B}) \]

If \( \alpha = 0 \), then \( \leq " \) ("\geq " in the latter optional)

1. If \( \alpha > 0 \), then \( \exists \Delta_1 \in \mathbb{R} \) (def. of inf).

\[ W(t, x) \geq \sup_{\gamma \in \Gamma} W(t+\alpha, y_x(t+\alpha; t, a, \delta[a])) - \varepsilon \]

\[ \forall 2 \in \mathbb{R}^+ \quad \text{def. of inf.} \Rightarrow \exists \Delta_2 \in \mathbb{R} \]

\[ W(t+\alpha, x) \geq \sup_{\gamma \in \Gamma} g(y_x(T; t+\alpha, a, \delta[a])) - \varepsilon \]

Def. \( \hat{\beta}[a](0) = \left\{ \begin{array}{ll} \delta[a](0) & \text{if } t \leq s \leq t + \alpha \\
 \delta_2[a](0) & \text{if } t + \alpha < s \leq t \end{array} \right. \)

with \( z = y_x(t+\alpha; t, a, \delta[a]) \)

\[ \hat{\beta} \in \Delta \Rightarrow \gamma_s \in \mathbb{R}^+ \]

\[ W(t, x) \geq \sup_{\gamma \in \Gamma} g(y_x(T; t+\alpha, a, \delta[a])) - 2\varepsilon \]

\[ \Rightarrow \sup_{\gamma \in \Gamma} g(y_x(T; t, a, \hat{\beta}[a])) - 2\varepsilon \]
\[ W(t, x) \geq \inf_{\beta \in B_t} \sup_{a \in \mathcal{A}_t} g(y_x(t, \alpha, \beta, [\alpha])) \geq V(t, x). \]

Let \( \varepsilon \to 0 \) and set \( W \geq V \). \( \forall t, x \) \( \forall \varepsilon \)

Estimates \( \forall \varepsilon \in V \)

Then, \( \exists C_4 \) s.t. \( T, C_1, C_2, C_3 \) s.t.

1. \( |V(t, x)|, |V(t, x)| \leq C_4 \)

2. \( |V(t, x) - V(t, x)| \leq C_4 (|t - T| + |x - x_1|) \)

Proof:

Part of proof: \( 1. \) \( \left| \int_T^t L(t, x) \, dt + g(t) \right| \leq (T-t) C_2 + C_3 \leq T C_2 + C_3 =: C_4 \)

2. Optional, see Notes.

Isaacs' Hamiltonians

\[ H^+(x, p) := \min_{a \in \mathcal{A}} \max_{b \in B} p \cdot f(x, a, b) + l(t, q, b) \]

\[ H^{-}(x, p) := \max_{a \in \mathcal{A}} \min_{b \in B} \text{some} \]

\( \forall \alpha \in \mathcal{A}, \beta \in B \) \( \leq H^+(x, p) \).
\[ \text{Prop. } H^+, H^- : \mathbb{R}^n \to \mathbb{R} \text{ are continuous and for } F \in H^+, F^- \]

\[ \exists K > 0 : \]

\[ (L_x) \quad |F(x, P) - F(x', P)| \leq K|x - x'|(1 + |P|) \]

\[ (L_P) \quad |F(x, P) - F(x, \tilde{P})| \leq c_1 |P - \tilde{P}| \]

1. \( (L_x) \) for \( F = H^+ (H^-, H^-) \)

\[ H^+(\tilde{x}, P) - H^+(x, P) \leq - \text{ close } b' \in B ; \]

\[ H^+(x, P) = \min_{a \in A} \left\{ P \cdot f(x, a, b') + Ec(x, a, b') \right\} \]

\[ \leq \min_{a \in A} \left\{ P \cdot f(x, a, b') + Ec(x, a, b') \right\} \]

\[ H^+(\tilde{x}, P) - H^+(x, P) \leq \phi + P \cdot f(x, a, b') + Ec(x, a, b') \]

\[ \leq |P| \leq 1 \quad \text{since } \leq 1 \]

\[ \text{if } \quad x = c_1, v < c_2 \text{. } \text{Exchange } x \text{ for } \tilde{x} \quad \Rightarrow \]

\[ (L_x) \] for \( H^+ \), \( H^- \)

\[ \geq (L_P) \text{ Hw} \]

\[ 3. \quad (L_x) + (L_P) \Rightarrow \text{conv. fr } \text{ of } H^+ \# H^- \]

\[ \]