LEC T URE 18, May 11, 2023

LAST LECTURES OF THE COURSE:

May 16-19 at 12:30 pm (200-60)

23 No 25

30 June 1st.

NON-ZERO SUM GAMES IN MIXED STRATEGIES

\( \mu \in \mathcal{P}(A), \nu \in \mathcal{P}(B) \) prob measures

\[
\tilde{\Phi}^A (\mu, \nu) = \int_{A \times B} \Phi^A (a, b) \, d\mu(a) \, d\nu(b)
\]

\[
\tilde{\Phi}^B (\mu, \nu) = \int_{A \times B} \Phi^B (a, b) \, d\mu(a) \, d\nu(b)
\]

Con (Nash) All bi-matrix games (i.e. A & B finite sets) have at least one Nash Equi. in mixed strategies.

\( \Phi : A \times B \mapsto \mathbb{R} \)

\( A = \{ 1, \ldots, m \}, \quad B = \{ 1, \ldots, n \} \)

\( \mathcal{P}(A) \leftrightarrow \Delta^m, \quad \mathcal{P}(B) \leftrightarrow \Delta^n \)

\( \mu \leftrightarrow \mathsf{\pi} \) discrete density \( \nu \leftrightarrow \mathsf{\gamma} \)

\( \Phi^A (i, j) = (\mu^A)_{i,j} \), \( \Phi^B (i, j) = (\nu^B)_{i,j} \)

\[ \tilde{\Phi}^A (x, y) = x^T \mu^A y, \quad \tilde{\Phi}^B (x, y) = x^T \nu^B y \]

\( x^T \mathsf{\pi} \mathsf{\gamma} \) bilinear
Ass. of Nash then are verified (cont. + concavity)
\[ \exists \text{ Nash Eq. } f^* \left( \Delta u, \Delta v, \hat{\Phi}^A, \hat{\Phi}^B \right) \]

Thus (Nash games) \( A, B \) connect \( \hat{\Phi}^A, \hat{\Phi}^B \in C(A + B) \)
\[ \Rightarrow \text{ the game in mixed strategies has an equil.} \]
\[ \text{as for Von Neumann... see } [\text{Bressan}] \]

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**Dynamical Games.** i.e. involving time:

- repeated (static) games
- discrete -time games
- continuous-time : differential games

2-person diff. games : dynamical system with controls of 2 players:

\[
\begin{align*}
\dot{y}(t) &= f(y(t), a(t), b(t)) \quad t > 0 \\
y(0) &= x \quad \text{controls}
\end{align*}
\]

\( f : \mathbb{R}^n \times A \times B \to \mathbb{R}^m \) cont. & Lip in \( y, u, \text{uniform in } a, b \)

\( T > 0 \) fixed.
\( A = \{ a(\cdot) : [0, T] \to A \}, \text{meas. leb} \)
\( B = \{ b(\cdot) : [0, T] \to B \}, \text{meas. leb} \)

Know: \exists \text{unique trajectory } y(t) \ & \text{fixed } a(\cdot), b(\cdot)

\[ y(t) := y_x(s; t, a(\cdot), b(\cdot)) \]
Payoffs :  \( J_A(t, x, a(t), b(t)) := \int_t^t e^A(y(s), a(s), b(s)) ds + g_A(y(t)) \)
\( J_B(\cdots) := \int_t^t e^B(y(s)) ds + g_B(y(t)) \)

1st player wants to maximize \( J_A \)
2nd \( - - - - - - - - - - J_B \)

Game is 0-sum if \( J_A = - J_B \), i.e. if

\[ e_A = - e_B, \quad s_A = - s_B. \]

Other possible payoffs :

* INFINITE HORIZON \( J^i \) \( i \geq 0 \)

\[ J^i := \int_0^\infty e^i(y(s), a(s), b(s)) e^{-\lambda s} ds \quad i = A, B \]

* games with a TARGET : given \( \mathcal{T} \subseteq \mathbb{R}_+ \)

\[ t^i \cdot (a(\cdot), b(\cdot)) := \begin{cases} \min \{ t^i \colon y^i(s; 0, 0, b(s)) \in \mathcal{T} \} & \text{if } \mathcal{T} \neq \emptyset \\ +\infty & \text{if } \mathcal{T} = \emptyset \end{cases} \]

Pursuit – Evasion \( y = (y_A, y_B) \in \mathbb{R}_+^k \times \mathbb{R}_+^k \)

\( x = 0 \)

\[ \mathbf{\dot{y}}_A = f_A(y_A, a) \quad y_0 = \frac{1}{2} y_A = y_B \]
\[ \mathbf{\dot{y}}_B = f_B(y_B, b) \quad J^B = e^B(x(0), c(0)) \]

A = pursuer
B = evader

\( J_A = - J_B = - t^i(a, b) \)  Rufus Isaacs ~ 1954–6
(Pioneering work URSS) Rand Corp.

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**FUNDAMENTAL QUESTION : where do I choose the control functions?**
1st try

\[ a \in \mathcal{A}, \ b \in \mathcal{B} \rightarrow \text{"STANDARD CHASE \ (A, B, J^A, J^B)"} \]. Is this an interesting model?

**Example**: CHASING GAME (Acchi appendix)

\[ \dot{y}_A = a \in \mathcal{B}_1(0) \]
\[ \dot{y}_B = b \in \mathcal{B}_1(0) \]
\[ J^A = -t_x \quad J^B = t_x \]

\[ V^+ = \sup_{a \in \mathcal{A}} \inf_{t \in (0, \infty)} \int_{t}^{t+x} f(a(\cdot), b(\cdot)) \leq \infty \]
\[ \in \mathbb{R} \]

V_- = \sup_{b \in \mathcal{B}} \inf_{a \in \mathcal{A}} \int_{0}^{t+x} f(a(\cdot), b(\cdot)) < \infty

Conclusion: the information pattern is NOT realistic, an interesting! We want a better model!

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Recall Feedback controls for 2-player (Markovian strategies)

\[ \Phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \]

**Admissible if** 3 unique solutions

\[ \begin{cases} \dot{y}(s) = f(y(s), \Phi(s, y(s))) \quad s > t \\ y(t) = x \end{cases} \]

**Remark**: 
\[ \begin{array}{l}
\Phi \text{ admissible if unique solution.}
\end{array} \]

\[ V(x, t) = \inf_{a \in \mathcal{A}} \int_{t}^{\infty} f(x, t, a(\cdot)) \leq \infty \]

**Ans.**: YES. Pf HW.
Notations: \( u_A : [0,T] \times \mathbb{R}^k \to A \) feeds for 1st player,
\( u_B : [0,T] \times \mathbb{R}^k \to B \) feeds for 2nd player.

Define a pair \((u_A, u_B)\) of measurable feedbacks is \texttt{ADMISSIBLE}
if for all \( t \),
\[
\begin{cases}
  y(t) = f(y(t), u_A(t, y(t)), u_B(t, y(t))) \\
  y(0) = x
\end{cases}
\]

Define the set \( [E,T] \) of
\[
\begin{align*}
  u_A(t, y(t)) &= u_A(t, y(t)) \\
  u_B(t, y(t)) &= u_B(t, y(t))
\end{align*}
\]

Define \((u^*_A, u^*_B)\) \texttt{admissible}. \texttt{Nash} is \texttt{equilib.} for \( x_0, t \)
among adm. feedb. \texttt{Nash} if:

- \( u^*_A \) is opt. for 1st \( u_B \) if 2nd plays \( u^*_B \)
  i.e., \( u^*_A \) maximizes \( u_A \to J^A(x_0, t; u_A, u^*_B) \)
  among all \( u_A \); \((u^*_A, u^*_B)\) is \texttt{admissible}.

- \( u^*_B \) is optimal for 2nd \( u_A \) if 1st uses \( u^*_A \)
  \( u^*_B \) maximizes \( u_B \to J^B(x_0, t; u^*_A, u_B) \) and \( u^*_B \)
  \((u^*_A, u_B)\) is \texttt{admissible}.

Plan: build such Nash equilib via a Verif. The
1 player
Pre-Hamiltonian: \( p = f(x, a) + e(x, a) \) param. \( x, p \)

For 2 players 2 pre-Ham. with 3 parameters: \( x, p_1, p_2 \)
\[
\begin{align*}
  \Phi^A(q, x; p_1) &= p_1 f(x, a) + e_A(x, q, b) \\
  \Phi^B(q, x; p_2) &= p_2 f(x, q, b) + e_B(x, q, b)
\end{align*}
\]
Hypothesis #: I continuous \( f_n: (u_1^i, u_2^i) \) \( \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^5 \) s.t. \( A \times B \) s.t. \( \forall x, p_1, p_2 \in \mathbb{R}^5 \):

\[
(u_1^i, u_2^i)(x, p_1, p_2) \text{ is a Nash eq. of the same } (A, B, \Phi^A, \Phi^B)
\]

\[
\begin{align*}
&u_1^i(x, p_1, p_2) \in \arg \max_{a} \Phi^A(a, u_2^i(x, p_1, p_2), x, p_1) \\
&u_2^i(x, p_1, p_2) \in \arg \max_{b} \Phi^B(u_1^i(x, p_1, p_2), b, x, p_2)
\end{align*}
\]

Suff. conditions for \( \alpha \) from Nash thm.,... see Lemma in the Notes...

Verification Thm.: Supp. \( \exists w_1, w_2 \in C^1([t_0, T], \mathbb{R}^n) \)

Cont. of \( t = T \)

\[
\begin{cases}
\frac{dw_1}{dt} + D_x w_1 \cdot f(x, u_1^i, u_2^i) + e_A(x, a_1, u_2^i) = 0 & \text{in } (t_0, T) \times \mathbb{R}^5 \\
\frac{dw_2}{dt} + D_x w_2 \cdot f(x, u_1^i, u_2^i) + e_B(x, a_1, u_2^i) = 0
\end{cases}
\]

\[
\begin{align*}
u_i^# &= u_i^i(x, D_x w_1, D_x w_2) & i = 1, 2 \\
w_1(x, T) &= \delta_A(x) & w_2(x, T) = \delta_B(x)
\end{align*}
\]

\( \alpha \) \( \triangleright \) \( (u_1^i(x, D_x w_1(t), D_x w_2(t)), u_2^i(\text{same})) \) is a pair of admissible feedbacks, \( \Rightarrow \) such pair is Nash equil. for diff. game (Simon, edw. feeds.)

\text{Rmk:} 1st. eq. is

\[
\frac{dw_1}{dt} + \max_{a} \frac{1}{2} D_x w_1 \cdot f(x, a_1, u_2^i) + e_A(x, a_1, u_2^i) = 0
\]

\( 2^{\text{nd}} \) eq. is
\[ \frac{\partial w_2}{\partial t} + \text{no} + \int D_x w_2 \cdot f(x, u_1^*, b) + e_B(x, u_1^*, b) = 0 \]

2 H-J-B equations. coupled via \( u_2^* \) and \( u_1^* \).

Proof. Def. \( \tilde{f}(x, t, a) := f(x, a, u_2^*(x, D_x w_1, 0, D_x w_2(x, t))) \)

\[ \tilde{e}(x, t, a) = e_A(x, 0, u_2^*(x, D_x w_1, 0, D_x w_2(x, t))) \rightarrow \text{1st eq. is} \]

\[ \begin{align*}
\frac{2w_1}{\partial t} + \text{no} + \int D_x w_1 \cdot \tilde{f}(x, t, a) + \tilde{e}(x, t, a) &= 0 \\
\tilde{w}_1(x, T) &= \delta_A(x) \rightarrow \text{standard H-J-B eq. for 1st player}.
\end{align*} \]

For 1st player, can apply a Vequil, then, with dependence on time in \( f \) and \( e \).

\[ \text{(see L. Notes, pf... same as without } f \ldots \text{H-W).} \]

\[ \Rightarrow u_1^*(x, D_x w_1, D_x w_2) \text{ is optimal feedback for a} \]

if 2nd uses \( u_2^*(x, D_x w_1, D_x w_2) \), \( u_1^*, u_2^* \) admits \( (u_1^*, u_2^*) \) Nash equl.

\[ \Rightarrow \text{1st part of being a Nash equl.} \]

In 2nd player: use 2nd eq. \& 2nd terminal condition use Vequil, then, with fixed \( \tilde{w}_2(x, T) = \delta_B(x) \)

\[ u_2^* \text{ is opt} \ldots \Rightarrow \text{2nd part of Nash equl.} \]
APPLICATION: LINEAR - QUADRATIC DIFF. CASES.

\[
\begin{aligned}
\begin{cases}
    \dot{y} = Ay + B_1a + B_2b \quad A \in \mathbb{R}^{n \times n}, \quad a \in \mathbb{R}^n, \\
    y(t) = x \\
    B_1 \in \mathbb{R}^{n \times m}, \quad B_2 \in \mathbb{R}^{n \times m_2}
\end{cases}
\end{aligned}
\]

\( a(t) \in L^2(C_0, T; \mathbb{R}^n), \quad b \in L^2(C_0, T; \mathbb{R}^{m_2}) \)

\[
J^A(x, t, a, b) := -\int_0^T \left[ y(t)^T M_1 y(t) + \frac{1}{2} (a(t))^2 \right] dt + y(T)^T \Phi_1 y(T)
\]

\[
J^B(x, t, a, b) := -\int_0^T \left[ y(t)^T M_2 y(t) + \frac{1}{2} (b(t))^2 \right] dt + y(T)^T \Phi_2 y(T)
\]

\( M_1, \Phi_1, M_2, \Phi_2 \in \text{Sym} (n) \)

Q.1. Hyp. $\# \text{ holder} ? \quad n \times X \quad \Phi^A \# n \times X \quad \Phi^B ?$

\[
\Phi^A(a, b; x, p_1) = p_1^T (A x + B_1 a + B_2 b) - x^T B_1^T x - \frac{1}{2} a^2 =
\]

\[
= p_1^T B_1 a - \frac{1}{2} a^2 + \text{ terms indep. of } A,
\]

\[
\Rightarrow -\infty \quad \Rightarrow a \to -\infty \Rightarrow \exists \text{ unique } \Phi^A
\]

\[ \Phi^B \]

\[
\begin{array}{l}
\end{array}
\]