Chapter 1
An introduction to Mean Field Game theory

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Abstract These notes are an introduction to Mean Field Game (MFG) theory, which models differential games involving infinitely many interacting players. We focus here on the Partial Differential Equations (PDE) approach of MFGs. The two main parts of the text correspond to the two emblematic equations in MFG theory: the first part is dedicated to the MFG system, while the second part is devoted to the master equation.
The MFG system describes Nash equilibrium configurations in the mean field approach to differential games with infinitely many players. It consists in the coupling between a backward Hamilton-Jacobi equation (for the value function of a single player) and a forward Fokker-Planck equation (for the distribution law of the individual states). We discuss the existence and the uniqueness of the solution to the MFG system in several frameworks, depending on the presence or not of a diffusion term and on the nature of the interactions between the players (local or nonlocal coupling). We also explain how these different frameworks are related to each other. As an application, we show how to use the MFG system to find approximate Nash equilibria in games with a finite number of players and we discuss the asymptotic behavior of the MFG system.
The master equation is a PDE in infinite space dimension: more precisely it is a kind of transport equation in the space of measures. The interest of this equation is that it allows to handle more complex MFG problems as, for instance, MFG problems involving a randomness affecting all the players. To analyse this equation, we first discuss the notion of derivative of maps defined on the space of measures; then we present the master equation in several frameworks (classical form, case of finite state space and case with common noise); finally we explain how to use the master equation to prove the convergence of Nash equilibria of games with finitely many players as the number of players tends to infinity.
As the works on MFGs are largely inspired by P.L. Lions' courses held at the Collége de France in the years 2007-2012, we complete the text with an appendix describing the organization of these courses.

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1.1 Introduction

Mean field game (MFG) theory is devoted to the analysis of optimal control problems with a large number of small controllers in interaction. As an example, they can model crowd motions, in which the evolution of a pedestrian depends on the crowd which is around. Similar models are also used in economics: there, macroeconomic quantities are derived from the microeconomic behavior of the agents who interact through aggregate quantities, such as the prices or the interest rates. In the Mean Field Game formalism, the controllers are assumed to be “rational” (in the sense that they optimize their behavior by taking into account the behavior of the other controllers), therefore the central concept of solution is the notion of Nash equilibrium, in which no controller has interest to deviate unilaterally from the planned control. In general, playing a Nash equilibrium requires for a player to anticipate the other players’s responses to his/her action. For large population dynamic games, it is unrealistic for a player to collect detailed information about the state and the strategies of the other players. Fortunately this impossible task is useless: mean field game theory explains that one just needs to implement strategies based on the distribution of the other players. Such a strong simplification is well documented in the (static) game community since the seminal works of Aumann [20]. However, for differential games, this idea has been considered only very recently: the starting point is a series of papers by Lasry and Lions [143, 144, 145, 150], who introduced the terminology in around 2005. The term *mean field* comes for an analogy with the mean field models in mathematical physics, which analyse the behavior of many identical particles in interaction (see for instance [111, 176, 177]). Here the particles are replaced by agents or players, whence the name of mean field games. Related ideas have been developed independently, at about the same time, by Caines, Huang and Malhamé [132, 133, 134, 135], under the name of Nash certainty equivalence principle. In the economic literature, similar models (often in discrete time) were introduced in the 1990s as “heterogeneous agent models” (see, for instance, the pioneering works of Aiyagari [13] and Krussell and Smith [138]).

Since these seminal works, the study of mean field games has known a quick growth. There are by now several textbooks on this topic: the most impressive one is the beautiful monograph by Carmona and Delarue [68], which exhaustively covers the probability approach of the subject. One can also quote the Paris-Princeton Lectures by Gueant, Lasry and Lions [128] where the authors introduce the theory with sample of applications, the monograph by Bensoussan, Frehse and Yam [31], devoted to both mean field games and mean field control with a special emphasis on the linear-quadratic problems, and the monograph by Gomes, Pimentel and Voskanyan [119], on the regularity of the MFG system. Finally, [56] by the first author with Delarue, Lasry and Lions studies the master equation (with common noise) and the convergence of Nash equilibria as the number of player tends to infinity.

This text is a basic introduction to mean field games, with a special emphasis on the PDE aspects. The central ideas were largely developed in Pierre-Louis Lions’ series of lectures at the CollÃ¨ge de France [149] during the period 2007-2012. As these courses contain much more material than what is developed here, we added in the appendix some notes on the organization of these courses in order to help the interested reader.

The main mathematical object of the text is the so-called mean field game system, which takes the form

$$\left\{\begin{array}{ll} (i) \quad -\partial_t u - \nu \Delta u + H(x, Du, m) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) \;\; \partial_t m - \nu\Delta m - \operatorname{div}\left(D_pH(x,Du,m)m\right) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) \; m(0) = m_0\;,\; u(x,T) = G(x, m(T)) & \text{in } \mathbb{R}^d \end{array}\right. \tag{1.1}$$

In the above system, the unknown u and m are scalar and depend on time $t \in [0,T]$ and space $x \in \mathbb{R}^d$. The two equations are of (possibly degenerate) parabolic type (i.e., $\nu \geq 0$); the first equation is backward in time while the second one is forward in time. There are two other crucial structure conditions for this system: the first one is the convexity of $H = H(x,p,m)$ with respect to the second variable. This condition means that the first equation (a Hamilton-Jacobi equation) is associated with an optimal control problem and is interpreted as the value function associated with a typical small player. The second structure condition is that m_0 (and therefore $m(t,\cdot)$) is (the density of) a probability measure on $\mathbb{R}^d$. The Hamiltonian $H = H(x,p,m)$, which couples the two equations, depends on space, on the variable $p \in \mathbb{R}^d$ and on the probability measure m.

Let us briefly explain the interpretation of this system as a Nash equilibrium problem in a game with infinitely many small players. An agent (=a player) controls through his/her control α the stochastic differential equation (SDE)

$$dX_s = b(X_s, \alpha_s, m(s))ds + \sqrt{2\nu}dB_s \tag{1.2}$$

where (B_t) is a standard Brownian motion. He/She aims at minimizing the quantity

$$\mathbb{E}\left[\int_0^T L(X_s, \alpha_s, m(s))ds + G(X_T, m(T))\right] ,$$

where the running cost $L = L(x, \alpha, m)$ and the terminal cost $G = G(x, m)$ depend on the position x of the player, the control α and the distribution m of the other players. Note that in this cost the evolution of the measure $m(s)$ enters as a parameter. To solve this problem one introduces the value function:

$$u(t, x) = \inf_{\alpha} \mathbb{E}\left[\int_t^T L(X_s, m(s), \alpha_s)ds + G(X_T, m(T))\right] ,$$

where the infimum is taken over admissible controls α and where X solves the SDE (1.2) with initial condition $X_t = x$. The value function u then satisfies the PDE (1.1-(i)) where

$$H(x, p, m) = \sup_{\alpha} \left[-b(x, \alpha, m) \cdot p - L(x, m, \alpha)\right].$$

Given the value function u, it is known that the agent plays in the optimal way by using the feedback control $\alpha^* = \alpha^*(t, x)$ such that the drift is of the form $b(x, \alpha^*(t, x), m(t)) = -D_pH(x, Du(t, x), m(t))$. Now, if all agents argue in this way and if their associated noises are independent, the law of large numbers implies that their distribution evolves with a velocity which is due, on the one hand, to the diffusion, and, on the other hand, on the drift term $-D_pH(x, Du(t, x), m(t))$. This leads to the Kolmogorov-Fokker-Planck equation (1.1-(ii)). The fact that system (1.1) describes a Nash equilibrium can be seen as follows. As the single player is "small" (compared to the collection of the other agents), his/her deviation does not change the population dynamics. Hence the behavior of the other agents, and therefore their time dependent distribution $m(t)$, can be taken as given in the individual optimization. This corresponds to the concept of Nash equilibrium where all players play an optimal strategy while freezing the others' choices.

The main part of these notes (Section 1.3) is devoted to the analysis of the mean field game system (1.1): we discuss the existence and uniqueness of the solution in various settings and the interpretation of the system. This analysis takes some time since the PDE system behaves in a quite different way according to whether the system is parabolic or not (i.e., ν is positive or zero) and according to the regularity of H with respect to the measure. These various regimes correspond to different models: for instance, in many application in finance, the diffusion is nondegenerate (i.e., $\nu > 0$), while ν often vanishes in macroeconomic models. In most applications in economy the dependence of the Hamiltonian $H = H(x, p, m)$ with respect to the probability measure m is through some integral form of m (moments, variance), but in models of crowd motion it is very often through the value at position x of the density $m(x)$ of m. We will discuss these different features of the mean field game system, with, hopefully, a few novelties in the treatment of the equations. To keep these notes as simple as possible, the analysis is done for systems with periodic in space coefficients: the analysis for other boundary problems follows the same lines, with additional technicalities. We will also mention other relevant aspects of the MFG systems: their application to differential games with finitely many players, the long time ergodic behavior, the vanishing viscosity limits,...

The second focus of these notes is a (short and mostly formal) introduction to the "master equation" (Section 1.4). Indeed, it turns out that, in many applications, the MFG system (1.1) is not enough to describe the MFG equilibria. On the one hand, the MFG system does not explain how the agents take their decision in function of their current position and of the current distribution of the players (in "feedback form"). Secondly, it does not explain why one can expect the system to appear as the limit of games with finitely many players. Lastly, the PDE system does not allow to take into account problems with common noise, in which the dynamic of the agents is subject to a common source of randomness. All these issues can be overcome by the introduction of the master equation. This equation (introduced by Lions in his courses at CollÃ¨ge de France) takes the form of a partial differential equation in the space of measures which reads as follows (in the simplest setting):

$$
\begin{cases}
-\partial_t U(t,x,m) - \nu \Delta_x U(t,x,m) + H(x, D_x U(t,x,m), m) - \nu \int_{\mathbb{R}^d} \mathrm{div}_y D_m U(t,x,m,y)\, m(dy) \\
\qquad + \int_{\mathbb{R}^d} D_m U(t,x,m,y) \cdot D_p H(y, D_x U(t,y,m), m) m(dy) = 0 \\
\qquad\qquad\qquad \text{in } (0,T) \times \mathbb{R}^d \times \mathcal{P}_2 \\
U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}_2
\end{cases}
$$

where $\mathcal{P}_2$ is the space of probability measures on $\mathbb{R}^d$ (with finite second order moment). Here the unknown is the scalar quantity $U = U(t,x,m)$ depending on time and space and on the measure m (representing the distribution of the other players). This equation involves the derivative $D_m U$ of the unknown with respect to the measure variable (see Subsection 1.4.2). We will briefly explain how to prove the existence and the uniqueness of a solution to the master equation and its link with the MFG system. We will also discuss how to extend the equation to problems with a common noise (a noise which affects all the players). Finally, we will show how to use this master equation to prove that Nash equilibria in games with finitely many players converge to MFG equilibria.

These notes are organized as follows: in a preliminary part (Section 1.2), we introduce fundamental tools for the understanding and the analysis of MFG problems: a brief recap of the dynamic programming approach in optimal control theory, the description of the space of probability measures and some basic aspects of mean field theory. Then we concentrate on the MFG system (1.1) (Section 1.3). Finally, the analysis on the space of measures and the master equation are discussed in the last part (Section 1.4). We complete the text by an appendix on the organization of P.L. Lions' courses on MFGs at the CollÃ¨ge de France (Section 1.5).

1.2 Preliminaries

In this Section we recall some basic notion on optimal control and dynamic programming, on the space of probability measures and on mean field limits. As mean field games consist in a combination of these three topics, it is important to collect some preliminary knowledge of them.

1.2.1 Optimal control

We briefly describe, in a very formal way, the optimal control problems we will meet in these notes. We refer to the monographs by Fleming and Rischel [106], Fleming and Soner [107], Yong and Zhou [180] for a rigorous treatment of the subject.

Let us consider a stochastic control problem where the state (X_s) of the system is governed by the stochastic differential equation (SDE) with values in $\mathbb{R}^d$:

$$
X_s^\alpha = x + \int_t^s b(r, X_r^\alpha, \alpha_r) dr + \int_t^s \sigma(r, X_r^\alpha, \alpha_r) dB_r. \tag{1.3}
$$

In the above equation, $B = (B_s)_{s\geq 0}$ is a $N-$dimensional Brownian motion (starting at 0) adapted to a fixed filtration $(\mathcal{F}_t)_{t\geq 0}$, $b : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times N}$ satisfy some regularity conditions given below and the process $\alpha = (\alpha_s)$ is progressively measurable with values in some set A. We denote by $\mathcal{A}$ the set of such processes. The elements of $\mathcal{A}$ are called the control processes.

A generic agent controls the process X through the control α in order to reach some goal: here we consider optimal control problems, in which the controller aims at minimizing some cost J. We will mostly focus on the finite horizon problem, where J takes the form:

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T L(s, X_s^\alpha, \alpha_s)ds + g(X_T)\right].$$

Here $T > 0$ is the finite horizon of the problem, $L : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are given continuous maps (again we are more precise in the next section on the assumptions on L and g). The controller minimizes J by using controls in $\mathcal{A}$. We introduce the value function as the map $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$u(t,x) = \inf_{\alpha \in \mathcal{A}} J(t,x,\alpha)\ .$$

Dynamic programming and the verification Theorem.

The main interest of the value function is that it indicates how the controller should choose his/her control in order to play in an optimal way. We explain the key ideas in a very informal way. A rigorous treatment of the question is described in the references mentioned above.

Let us start with the **dynamic programming principle**, which states the following identity: for any $t_1 \leq t_2$,

$$u(t_1,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_{t_1}^{t_2} L(s, X_s^\alpha, \alpha_s)ds + u(t_2, X_{t_2}^\alpha)\right]. \tag{1.4}$$

The interpretation is that, to play optimally at time t_1, the controller does not need to predict in one shot the whole future strategy provided he/she knows what would be the best reward at some future time t_2, in which case it is enough to focus on the optimization between t_1 and t_2. So far, the optimization process can be built step by step like in semigroup theory. This relation has a fundamental consequence: to play in an optimal way the agent only needs to know the current state and play accordingly (and not the whole filtration at time t).

Fix now $t \in [0,T)$. Choosing $t_1 = t$, $t_2 = t+h$ (for $h > 0$ small) and assuming that u is smooth enough, we obtain by ItÃ´'s formula and (1.4) that

$$\begin{aligned} u(t,x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\int_t^{t+h} L(s, X_s^\alpha, \alpha_s)ds + u(t,x) + \int_t^{t+h} (\partial_t u(s, X_s^\alpha) + Du(s, X_s^\alpha) \cdot b(s, X_s^\alpha, \alpha_s) \\ + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(s, X_s^\alpha, \alpha_s) D^2 u(s, X_s^\alpha)))ds\Big]. \end{aligned}$$

Simplifying by $u(t,x)$, dividing by h and letting $h \to 0^+$ gives (informally) the Hamilton-Jacobi equation

$$0 = \inf_{a \in A}\left[L(t,x,a) + \partial_t u(t,x) + Du(t,x) \cdot b(t,x,a) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,a)D^2u(t,x))\right].$$

Let us introduce the Hamiltonian H of our problem: for $p \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d\times d}$,

$$H(t,x,p,M) := \sup_{a \in A}\left[-L(t,x,a) - p \cdot b(t,x,a) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,a)M)\right].$$

Then the **Hamilton-Jacobi equation** can be rewritten as a terminal value problem:

$$\begin{cases} -\partial_t u(t,x) + H(t,x,Du(t,x), D^2u(t,x)) = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ u(T,x) = g(x) & \text{in } \mathbb{R}^d. \end{cases}$$

The first equation is backward in time (the map H being nonincreasing with respect to D^2u). The terminal condition comes just from the definition of u for $t = T$.

Let us now introduce $\alpha^*(t,x) \in A$ as a maximum point in the definition of H when $p = Du(t,x)$ and $M = D^2u(t,x)$. Namely

$$H(t,x,Du(t,x),D^2u(t,x)) = -L(t,x,\alpha^*(t,x)) - Du(t,x)\cdot b(t,x,\alpha^*(t,x)) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,\alpha^*(t,x))D^2u(t,x)). \tag{1.5}$$

We assume that α^* is sufficiently smooth to justify the computation below. We are going to show that α^* is the **optimal feedback**, namely the optimal strategy to play at time t in the state x. Indeed, one has the following "**Verification Theorem**": Let $(X_s^{\alpha^*})$ be the solution of the stochastic differential equation

$$X_s^{\alpha^*} = x + \int_t^s b(r, X_r^{\alpha^*}, \alpha^*(r, X_r^{\alpha^*}))dr + \int_t^s \sigma(r, X_r^{\alpha^*}, \alpha^*(r, X_r^{\alpha^*}))dB_r$$

and set $\alpha_s^* = \alpha^*(s, X_s^{\alpha^*})$. Then

$$u(t,x) = J(t,x,\alpha^*).$$

Note that, with a slight abuse of notation, here $\alpha^* = (\alpha_s^*)$ is a control, namely it belongs to $\mathcal{A}$. Strictly speaking, (α_t^*) is the optimal control, $\alpha^*(t,x)$ being the optimal feedback.
Heuristic argument. By ItÃ´'s formula, we have

$$g(X_T^{\alpha^*}) = u(T, X_T^{\alpha^*}) = u(t,x) + \int_t^T (\partial_t u(s, X_s^{\alpha^*}) + Du(s, X_s^{\alpha^*})\cdot b(s, X_s^{\alpha^*}, \alpha_s^*) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(s, X_s^{\alpha^*}, \alpha_s^*)D^2u(s, X_s^{\alpha^*})))ds + \int_t^T \sigma^*(s, X_s^{\alpha^*}, \alpha_s^*)Du(s, X_s^{\alpha^*})\cdot dB_s.$$

Taking expectation, using first the optimality of α^* in (1.5) and then the Hamilton-Jacobi equation satisfied by u, we obtain

$$\begin{aligned}\mathbb{E}\left[g(X_T^{\alpha^*})\right] &= u(t,x) + \mathbb{E}\left[\int_t^T (\partial_t u(s, X_s^{\alpha^*}) - H(s, X_s^{\alpha^*}, Du(s, X_s^{\alpha^*}), D^2u(s, X_s^{\alpha^*})) - L(s, X_s^{\alpha^*}, \alpha_s^*))ds\right] \\ &= u(t,x) - \mathbb{E}\left[\int_t^T L(s, X_s^{\alpha^*}, \alpha_s^*)ds\right].\end{aligned}$$

Rearranging terms, we find

$$u(t,x) = \mathbb{E}\left[\int_t^T L(s, X_s^{\alpha^*}, \alpha_s^{\alpha^*})ds + g(X_T^{\alpha^*})\right],$$

which shows the optimality of α^*. □

The above arguments, although largely heuristic, can be partially justified. Surprisingly, the dynamic programming principle is the hardest step to prove, and only holds under strong restrictions on the probability space. In general, the value function is smooth only under very strong assumptions on the system. However, under milder conditions, it is at least continuous and then it satisfies the Hamilton-Jacobi equation in the viscosity sense. Besides, the Hamilton-Jacobi has a unique (viscosity) solution so that it characterizes the value function. If the diffusion is strongly non degenerate (e.g. if $N = d$ and σ is invertible with a smooth and bounded inverse) and if the Hamiltonian is smooth, then the value function is smooth as well. In this setting the above heuristic argument can be justified and the verification Theorem can be proved to hold.

We finally recall that, whenever α^* is uniquely defined from (1.5), then the Hamiltonian H is differentiable at (Du, D^2u) and

$$\begin{cases} H_p(t,x,Du(t,x),D^2u(t,x)) = -b(t,x,\alpha^*(t,x)), \\ H_M(t,x,Du(t,x),D^2u(t,x)) = -\frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(t,x,a^*(t,x))D^2(\cdot)) \end{cases} \tag{1.6}$$

This is a consequence of the so-called Envelope Theorem:

Lemma 1 *Let A be a compact metric space, $\mathcal{O}$ be an open subset of $\mathbb{R}^d$ and $f : A \times \mathcal{O} \to \mathbb{R}$ be continuous and such that $D_x f$ is continuous on $A \times \mathcal{O}$. Then the marginal map*

$$V(x) = \inf_{a \in A} f(a, x)$$

is differentiable at each point $x \in \mathcal{O}$ such that the infimum in $V(x)$ is a unique point $a_x \in A$, and we have

$$DV(x) = D_x f(a_x, x).$$

Proof. Let $x \in \mathcal{O}$ be such that the infimum in $V(x)$ is a unique point $a_x \in A$. Then an easy compactness argument shows that, if a_y is a minimum point of $V(y)$ for $y \in \mathcal{O}$ and $y \to x$, then $a_y \to a_x$.

Fix $y \in \mathcal{O}$. Note first that, as $a_x \in A$,

$$V(y) \leq f(a_x, y) = f(a_x, x) + D_x f(a_x, z_y) \cdot (y - x) = V(x) + D_x f(a_x, z_y) \cdot (y - x),$$

for some $z_y \in [x, y]$.

Conversely,

$$V(x) \leq f(a_y, x) = f(a_y, y) + D_x f(a_y, z'_y) \cdot (x - y) = V(y) + D_x f(a_y, z'_y) \cdot (x - y),$$

for some $z'_y \in [x, y]$.

By continuity of $D_x f$ and convergence of a_y, we infer that

$$\begin{aligned} &\lim_{y \to x} \frac{|V(y) - V(x) - D_x f(a_x, x) \cdot (y - x)|}{|y - x|} \\ &\qquad \leq \liminf_{y \to x} |D_x f(a_x, z_y) - D_x f(a_x, x)| + \left| D_x f(a_y, z'_y) - D_x f(a_x, x) \right| = 0. \end{aligned}$$

□

Estimates on the SDE.

In the previous introduction, we were very fuzzy about the assumptions and the results. A complete rigorous treatment of the problem is beyond the aim of these notes. However, we need at least to clarify a bit the setting of our problem. For this, we assume the maps b and σ to be continuous on : $[0, T] \times \mathbb{R}^d \times A$ and Lipschitz continuous in x independently of t and a: There is a constant $K > 0$ such that, for any $x, y \in \mathbb{R}^d$, $t \in [0, +\infty)$, $a \in A$,

$$|b(t, x, a) - b(t, y, a)| + |\sigma(t, x, a) - \sigma(t, y, a)| \leq K|x - y|.$$

Under these assumptions, for any bounded control $\alpha \in \mathcal{A}$, there exists a unique solution to (1.3). By a solution we mean a progressively measurable process X such that, for any $T > 0$,

$$\mathbb{E}\left[\int_t^T |X_s^\alpha|^2 ds \right] < +\infty$$

and (1.3) holds $\mathbb{P}-$a.s. More precisely, we have:

Lemma 2 *Let α be a bounded control in $\mathcal{A}$. Then there exists a unique solution X^α to* (1.3) *and this solution satisfies, for any $T > 0$ and $p \in [2, +\infty)$,*

$$\mathbb{E}\left[\sup_{t\in[t,T]}|X_t^\alpha|^p\right] \le C(1+|x|^p) + \|b(\cdot,0,\alpha.)\|_\infty^p + \|\sigma(\cdot,0,\alpha.)\|_\infty^p),$$

where $C = C(T,p,d,K)$.

Remark 1 In view of the above result, the cost J is well-defined provided, for instance, that the maps $L : [0,T]\times\mathbb{R}^d\times A\to\mathbb{R}$ and $g:\mathbb{R}^d\to\mathbb{R}$ are continuous with at most a polynomial growth.

Proof. The existence can be proved by a fixed point argument, exactly as in the more complicated setting of the McKean-Vlasov equation (see the proof of Theorem 2 below). Let us show the bound. We set $M := \|b(\cdot,0,\alpha.)\|_\infty + \|\sigma(\cdot,0,\alpha.)\|_\infty$. We have, by Hölder's inequality

$$|X_s^\alpha|^p \le C(p,T,d)\left(|x|^p + \int_t^s |b(r,X_r^\alpha,\alpha_r)|^p dr + \left|\int_t^s \sigma(r,X_r^\alpha,\alpha_r)dB_r\right|^p\right)$$

where the constant $C(p,T,d)$ depends only on p, T and d. Thus

$$\mathbb{E}\left[\sup_{t\le r\le s}|X_r^\alpha|^p\right] \le C(p,T,d)\left(|x|^p + \int_t^s \mathbb{E}\left[|b(r,X_r^\alpha,\alpha_r)|^p\right]dr + \mathbb{E}\left[\sup_{t\le r\le s}\left|\int_t^r \sigma(u,X_u^\alpha,\alpha_u)dB_u\right|^p\right]\right).$$

Note that

$$|b(s,X_s^\alpha,\alpha_s)| \le |b(s,0,\alpha_s)| + L|X_s^\alpha| \le M + L|X_s^\alpha|$$

and, in the same way,

$$|\sigma(s,X_s^\alpha,\alpha_s)| \le M + L|X_s^\alpha|. \tag{1.7}$$

So we have

$$\int_t^s \mathbb{E}\left[|b(r,X_r^\alpha,\alpha_r)|^p\right]dr \le 2^{p-1}(M^p(s-t) + L^p\int_t^s \mathbb{E}\left[|X_r^\alpha|^p\right]dr).$$

By the Burkholder-Davis-Gundy inequality (see Theorem IV.4.1 in [172]), we have

$$\mathbb{E}\left[\sup_{t\le r\le s}\left|\int_t^r \sigma(u,X_u^\alpha,\alpha_u)dB_u\right|^p\right] \le C_p\mathbb{E}\left[\left(\int_t^s \mathrm{Tr}(\sigma\sigma^*(r,X_r^\alpha,\alpha_r))dr\right)^{p/2}\right],$$

where the constant C_p depends on p only. Combining Hölder's inequality (since $p/2\ge 1$) with (1.7), we then obtain

$$\mathbb{E}\left[\sup_{t\le r\le s}\left|\int_t^r \sigma(u,X_u^\alpha,\alpha_u)dB_u\right|^p\right] \le C_p(s-t)^{p/2-1}2^{p-1}\left(M^p(s-t) + L^p\int_t^s \mathbb{E}\left[|X_r^\alpha|^p\right]dr\right).$$

Putting together the different estimates we get therefore, for $s\in[t,T]$,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le r\le s}|X_r^\alpha|^p\right] &\le C(p,T,d)\left(1+|x|^p+M^p+\int_t^s \mathbb{E}\left[|X_r^\alpha|^p\right]dr\right)\\ &\le C(p,T,d)\left(1+|x|^p+M^p+\int_t^s \mathbb{E}\left[\sup_{t\le u\le r}|X_u^\alpha|^p\right]dr\right).\end{aligned}$$

We can then conclude by Gronwall's Lemma. □

1.2.2 The space of probability measures

In this section we describe the space of probability measures and a notion of distance on this space. Classical references on the distances over the space of probability measures are the monographs by Ambrosio, Gigli and SavarÃ© [19], by Rachev and RÃ¼schendorf [171], Santambrogio [174], and Villani [178], [179].

The Monge-Kantorovitch distance.

Let (X, d) be a Polish space (= complete metric space). We have mostly in mind $X = \mathbb{R}^d$ endowed with the usual distance. We denote by $\mathcal{P}(X)$ the set of Borel probability measures on X. Let us recall that a sequence (m_n) of $\mathcal{P}(X)$ narrowly converges to a measure $m \in \mathcal{P}(X)$ if, for any test function $\phi \in C_b^0(X)$ (= the set of continuous and bounded maps on X), we have

$$\lim_n \int_X \phi(x) m_n(dx) = \int_X \phi(x) m(dx).$$

Let us recall that the topology associated with the narrow convergence corresponds to the weak-* topology of the dual of $C_b^0(X)$: for this reason we will also call it weak-* convergence. According to *Prokhorov compactness criterium*, a subset $\mathcal{K}$ of $\mathcal{P}(X)$ is (sequentially) relatively compact for the narrow convergence if and only if it is tight: for any $\varepsilon > 0$ there exists a compact subset K of X such that

$$\sup_{\mu \in \mathcal{K}} m(X \backslash K) \leq \varepsilon.$$

In particular, for any $\mu \in \mathcal{P}(X)$ and any $\varepsilon > 0$, there is some X_ε compact subset of X with $\mu(X \backslash X_\varepsilon) \leq \varepsilon$ (Ulam's Lemma).

We fix from now on a point $x_0 \in X$ and we denote by $\mathcal{P}_1(X)$ the set of measures $m \in \mathcal{P}(X)$ such that

$$\int_X d(x, x_0) m(dx) < +\infty.$$

By the triangle inequality, it is easy to check that the set $\mathcal{P}_1(X)$ does not depend on the choice of x_0. We endow $\mathcal{P}_1(X)$ with the Monge-Kantorovitch distance:

$$\mathbf{d}_1(m_1, m_2) = \sup_\phi \int_X \phi(x)(m_1 - m_2)(dx) \qquad \forall m_1, m_2 \in \mathcal{P}_1(X),$$

where the supremum is taken over the set of maps $\phi : X \to \mathbb{R}$ such that ϕ is $1-$Lipschitz continuous. Note that such a map ϕ is integrable against any $m \in \mathcal{P}_1(X)$ because it has at most a linear growth.

We note for later use that, if $\phi : X \to \mathbb{R}$ is $Lip(\phi)-$Lipschitz continuous, then

$$\int_X \phi(x)(m_1 - m_2)(dx) \leq Lip(\phi) \mathbf{d}_1(m_1, m_2).$$

Moreover, if X_1 and X_2 are random variables on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$ such that the law of X_i is m_i, then

$$\mathbf{d}_1(m_1, m_2) \leq \mathbb{E}\left[|X_1 - X_2|\right], \tag{1.8}$$

because, for any $1-$Lipschitz map $\phi : X \to \mathbb{R}$,

$$\int_X \phi(x)(m_1 - m_2)(dx) = \mathbb{E}\left[\phi(X_1) - \phi(X_2)\right] \leq \mathbb{E}\left[|X_1 - X_2|\right].$$

Taking the supremum in ϕ gives the result. Actually one can show that, if the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is "rich enough" (namely it is a "standard probability space"), then

$$\mathbf{d}_1(m_1, m_2) = \inf_{X_1, X_2} \mathbb{E}\left[|X_1 - X_2|\right],$$

where the infimum is taken over random variables X_1 and X_2 such that the law of X_i is m_i.

Lemma 3 *$\mathbf{d}_1$ is a distance over $\mathcal{P}_1(X)$.*

Proof. The reader can easily check the triangle inequality. We now note that $\mathbf{d}_1(m_1, m_2) = \mathbf{d}_1(m_2, m_1) \geq 0$ since one can always replace ϕ by $-\phi$ in the definition. Let us show that $\mathbf{d}_1(m_1, m_2) = 0$ implies that $m_1 = m_2$. Indeed, if $\mathbf{d}_1(m_1, m_2) = 0$, then, for any $1-$Lipschitz continuous map ϕ, one has $\int_X \phi(x)(m_1 - m_2)(dx) \leq 0$. Replacing ϕ by $-\phi$, one has therefore $\int_X \phi(x)(m_1 - m_2)(dx) = 0$. It remains to show that this equality holds for any continuous, bounded map $\phi : X \to \mathbb{R}$. Let $\phi \in C_b^0(X)$. We show in Lemma 4 below that there exists a sequence of maps (ϕ_k) such that ϕ_k is $k-$Lipschitz continuous, with $\|\phi_k\|_\infty \leq \|\phi\|_\infty$, and the sequence (ϕ_k) converges locally uniformly to ϕ. By Lipschitz continuity of ϕ_k, we have $\int_X \phi_k d(m_1 - m_2) = 0$. Since we can apply Lebesgue convergence theorem (because the ϕ_k are uniformly bounded and m_1 and m_2 are probability measures), we obtain that $\int_X \phi(x)(m_1 - m_2)(dx) = 0$. This proves that $m_1 = m_2$. □

Lemma 4 *Let $\phi \in C_b^0(X)$ and let us define the sequence of maps (ϕ_k) by*

$$\phi_k(x) = \inf_{y \in X} \phi(y) + kd(y, x) \qquad \forall x \in X$$

Then $\phi_k \leq \phi$, ϕ_k is $k-$Lipschitz continuous with $\|\phi_k\|_\infty \leq \|\phi\|_\infty$, and the sequence (ϕ_k) converges locally uniformly to ϕ.

Proof. We have

$$\phi_k(x) = \inf_{y \in X} \phi(y) + kd(y, x) \leq \phi(x) + kd(x, x) = \phi(x),$$

so that $\phi_k \leq \phi$. Let us now check that ϕ_k is $k-$Lipschitz continuous. Indeed, let $x_1, x_2 \in X$, $\varepsilon > 0$ and y_1 be $\varepsilon-$optimal in the definition of $\phi_k(x_1)$. Then

$$\phi_k(x_2) \leq \phi(y_1) + kd(y_1, x_2) \leq \phi(y_1) + kd(y_1, x_1) + kd(x_1, x_2) \leq \phi_k(x_1) + \varepsilon + kd(x_1, x_2).$$

As ε is arbitrary, this shows that ϕ_k is $k-$Lipschitz continuous. Note that $\phi_k(x) \geq -\|\phi\|_\infty$. As $\phi_k \leq \phi$, this shows that $\|\phi_k\|_\infty \leq \|\phi\|_\infty$.

Finally, let $x_k \to x$ and y_k be $(1/k)-$optimal in the definition of $\phi_k(x_k)$. Our aim is to show that $(\phi_k(x_k))$ converges to $\phi(x)$, which will show the local uniform convergence of (ϕ_k) to ϕ. Let us first remark that, by the definition of y_k, we have

$$kd(y_k, x_k) \leq \phi_k(x_k) - \phi(y_k) + 1/k \leq 2\|\phi\|_\infty + 1.$$

Therefore

$$d(y_k, x) \leq d(y_k, x_k) + d(x_k, x) \to 0 \qquad \text{as } k \to +\infty.$$

This shows that $(\phi(y_k))$ converges to $\phi(x)$ and thus

$$\liminf_k \phi_k(x_k) \geq \liminf_k \phi(y_k) + kd(y_k, x_k) - 1/k \geq \liminf_k \phi(y_k) - 1/k = \phi(x).$$

On the other hand, since $\phi_k \leq \phi$, we immediately have $\limsup_k \phi_k(x_k) \leq \phi(x)$, from which we conclude the convergence of $(\phi_k(x_k))$ to $\phi(x)$. □

Proposition 1 *Let (m_n) be a sequence in $\mathcal{P}_1(X)$ and $m \in \mathcal{P}_1(X)$. There is an equivalence between:*
i) $\mathbf{d}_1(m_n, m) \to 0$,

ii) (m_n) *narrowly converges to* m *and* $\int_X d(x,x_0)m_n(dx) \to \int_X d(x,x_0)m(dx)$.

iii) (m_n) *narrowly converges to* m *and* $\lim_{R\to+\infty} \sup_n \int_{B_R(x_0)^c} d(x,x_0)m_n(dx) = 0$.

Sketch of proof. $(i) \Rightarrow (ii)$. Let us assume that $\mathbf{d}_1(m_n, m) \to 0$. Then, for any Lipschitz continuous map ϕ, we have $\int \phi m_n(dx) \to \int \phi m(dx)$ by definition of $\mathbf{d}_1$. In particular, if we chose $\phi(x) = d(x,x_0)$, we have $\int_X d(x,x_0)m_n(dx) \to \int_X d(x,x_0)m(dx)$. We now prove the weak-* convergence of (m_n). Let $\phi : X \to \mathbb{R}$ be continuous and bounded and let (ϕ_k) be the sequence defined in Lemma 4. Then

$$\int \phi(m_n - m)(dx) = \int \phi_k(m_n - m)(dx) + \int (\phi - \phi_k)(m_n - m)(dx).$$

Fix $\varepsilon > 0$. As $(\int_X d(x,x_0)m_n(dx))$ converges and $m \in \mathcal{P}_1(X)$, we can find $R > 0$ large such that

$$\sup_n m_n(X\backslash B_R(x_0)) + m(X\backslash B_R(x_0)) \leq \varepsilon.$$

On the other hand, we can find k large enough such that $\|\phi_k - \phi\|_{L^\infty(B_R(x_0))} \leq \varepsilon$, by local uniform convergence of (ϕ_k). Finally, if n is large enough, we have $|\int \phi_k(m_n - m)(dx)| \leq \varepsilon$, by the convergence of (m_n) to m in $\mathbf{d}_1$. So

$$\begin{aligned}
\left|\int \phi(m_n - m)(dx)\right| &\leq \left|\int \phi_k(m_n - m)(dx)\right| + \left|\int_{X\backslash B_R(x_0)} (\phi - \phi_k)d(m_n - m)\right| + \left|\int_{B_R(x_0)} (\phi - \phi_k)(m_n - m)(dx)\right| \\
&\leq \left|\int \phi_k(m_n - m)(dx))\right| + (\|\phi_k\|_\infty + \|\phi\|_\infty)(m_n(X\backslash B_R(x_0)) + m(X\backslash B_R(x_0))) \\
&\qquad + 2\|\phi_k - \phi\|_{L^\infty(B_R(x_0))} \\
&\leq \varepsilon + 2\|\phi\|_\infty \varepsilon + 2\varepsilon.
\end{aligned}$$

This shows the weak-* convergence of (m_n) to m.

$(ii) \Rightarrow (iii)$. Let us assume that (m_n) narrowly converges to m and $\int_X d(x,x_0)m_n(dx) \to \int_X d(x,x_0)m(dx)$. We have to check that $\lim_{R\to+\infty} \sup_n \int_{B_R(x_0)^c} d(x,x_0)m_n(dx) = 0$. For this we argue by contradiction, assuming that there is $\varepsilon > 0$ and a subsequence still denoted (m_n) and $R_n \to +\infty$ such that

$$\int_{B_{R_n}(x_0)^c} d(x,x_0)m_n(dx) \geq \varepsilon.$$

Then, for any $M > 0$ and any n large enough so that $R_n \geq M$,

$$\begin{aligned}
\int_X d(x,x_0)m_n(dx) &= \int_X (d(x,x_0) \wedge M)m_n(dx) + \int_{B_M(x_0)^c} d(x,x_0)m_n(dx) - M\int_{B_M(x_0)^c} m_n(dx) \\
&\geq \int_X (d(x,x_0) \wedge M)m_n(dx) + \varepsilon - Mm_n(B_M(x_0)^c).
\end{aligned}$$

We let $n \to +\infty$ in the above inequality to get, as $(\int_X d(x,x_0)m_n(dx))$ converges to $\int_X d(x,x_0)m(dx)$ and (m_n) converges to m narrowly,

$$\int_X d(x,x_0)m(dx) \geq \int_X (d(x,x_0) \wedge M)m(dx) + \varepsilon - Mm(\overline{B_M(x_0)^c}).$$

As $\int_X d(x, x_0)m(dx)$ is finite, the last term in the right-hand side tends to 0 as M tends to infinity while the first one tends to $\int_X d(x, x_0)m(dx)$ by monotone convergence: this leads to a contradiction.

$(ii) \Rightarrow (iii)$. Let us assume that (m_n) weakly-* converges to m and that $\lim_{R\to+\infty} \sup_n \int_{B_R(x_0)^c} d(x, x_0)m_n(dx) = 0$. Fix $\varepsilon > 0$. In view of the last condition, we can find $R > 0$ large enough such that

$$\sup_n \int_{B_R(x_0)^c} d(x, x_0)m_n(dx) \leq \varepsilon \qquad \text{and} \qquad \int_{B_R(x_0)^c} d(x, x_0)m(dx) \leq \varepsilon.$$

As the sequence (m_n) converges, it is tight by Prokhorov theorem, and we can find a compact subset K of X such that

$$\sup_n \int_{K^c} m_n(dx) \leq R^{-1}\varepsilon \qquad \text{and} \qquad \int_{K^c} m(dx) \leq R^{-1}\varepsilon.$$

Let $\mathcal{K}_0$ be the set of $1-$Lipschitz continuous maps on X which vanish at x_0. Note that, for any $\phi \in \mathcal{K}_0$, we have

$$|\phi(x)| = |\phi(x) - \phi(x_0)| \leq d(x, x_0).$$

Therefore

$$\begin{aligned}
\mathbf{d}_1(m_n, m) &= \sup_{\phi\in\mathcal{K}_0} \int_X \phi(x)(m_n - m)(dx) \\
&\leq \sup_{\phi\in\mathcal{K}_0} \left[\int_K \phi(x)(m_n - m)(dx) + \int_{B_R(x_0)\setminus K} d(x, x_0)(m_n + m)(dx) + \int_{B_R(x_0)^c} d(x, x_0)(m_n + m)(dx)\right] \\
&\leq \sup_{\phi\in\mathcal{K}_0} \left[\int_K \phi(x)(m_n - m)(dx)\right] + R(m_n + m_n)(K^c) + 2\varepsilon \\
&\leq \sup_{\phi\in\mathcal{K}_0} \left[\int_K \phi(x)(m_n - m)(dx)\right] + 4\varepsilon.
\end{aligned}$$

By Ascoli-ArzelÃ , there exists $\phi_n \in \mathcal{K}_0$ optimal in the right-hand side. In addition, we can assume that (ϕ_n) converges uniformly, up to a subsequence, to some $1-$Lipschitz continuous map $\phi : K \to \mathbb{R}$. We can extend ϕ_n and ϕ to X by setting

$$\tilde{\phi}_n(x) = \sup_{y\in K}[\phi_n(y) - d(y, x_0)], \qquad \tilde{\phi}(x) = \sup_{y\in K}[\phi(y) - d(y, x_0)].$$

Then one easily checks that $(\tilde{\phi}_n)$ converges uniformly to $\tilde{\phi}$ in X, so that, by weak-* convergence of (m_n) to m we have:

$$\lim_n \int_X \tilde{\phi}_n(x)(m_n - m)(dx) = 0.$$

As the $(\tilde{\phi}_n)$ are 1-Lipschitz continuous and coincide with ϕ_n on K, we have, arguing as above,

$$\mathbf{d}_1(m_n, m) \leq \int_K \phi_n(x)(m_n - m)(dx) + 4\varepsilon \leq \int_X \tilde{\phi}_n(x)(m_n - m)(dx) + 6\varepsilon.$$

Letting $n \to +\infty$ in this inequality implies $\mathbf{d}_1(m_n, m) \to 0$. More precisely, we have proved that this holds for at least a subsequence of m_n. But since this argument applies to m_n as well as to any of its subsequences, a standard argument allows us to conclude the desired result. □

In the case where $X = \mathbb{R}^d$, we repeatedly use the following compactness criterium:

Lemma 5 *Let $r > 1$ and $\mathcal{K} \subset \mathcal{P}_1(\mathbb{R}^d)$ be such that*

$$\sup_{\mu\in\mathcal{K}} \int_{\mathbb{R}^d} |x|^r \mu(dx) < +\infty \ .$$

Then $\mathcal{K}$ is relatively compact for the $\mathbf{d}_1$ distance.

Note that bounded subsets of $\mathcal{P}_1(\mathbb{R}^d)$ are not relatively compact for the $\mathbf{d}_1$ distance. For instance, in dimension $d = 1$, the sequence of measures $\mu_n = \frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_n$ satisfies $\mathbf{d}_2(\mu_n, \delta_0) = 1$ for any $n \geq 1$ but μ_n narrowly converges to δ_0.

Proof of Lemma 5. Let $\varepsilon > 0$ and $R > 0$ sufficiently large. We have for any $\mu \in \mathcal{K}$:

$$\mu(\mathbb{R}^d \backslash B_R(0)) \leq \int_{\mathbb{R}^d \backslash B_R(0)} \frac{|x|^r}{R^r} \mu(dx) \leq \frac{C}{R^r} < \varepsilon \ , \tag{1.9}$$

where $C = \sup_{\mu\in\mathcal{K}} \int_{\mathbb{R}^d} |x|^r \mu(dx) < +\infty$. So $\mathcal{K}$ is tight.

Let now (μ_n) be a sequence in $\mathcal{K}$. From the previous step we know that (μ_n) is tight and therefore there is a subsequence, again denoted (μ_n), which narrowly converges to some μ. By (1.9) and (iii) in Proposition 1 the convergence also holds for the distance $\mathbf{d}_1$. □

The $\mathbf{d}_2$ distance.

Here we assume for simplicity that $X = \mathbb{R}^d$. Another useful distance on the space of measures is the Wasserstein distance $\mathbf{d}_2$. It is defined on the space $\mathcal{P}_2(\mathbb{R}^d)$ of Borel probability measures m with a finite second order moment (i.e., $\int_{\mathbb{R}^d} |x|^2 m(dx) < +\infty$) by

$$\mathbf{d}_2(m_1, m_2) := \inf_{\pi} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \pi(x,y) \right)^{1/2} ,$$

where the infimum is taken over the Borel probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal given by m_1 and second marginal by m_2:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(x)\pi(dx,dy) = \int_{\mathbb{R}^d} \phi(x) m_1(dx), \ \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(y)\pi(dx,dy) = \int_{\mathbb{R}^d} \phi(y) m_2(dy) \qquad \forall \phi \in C_b^0(\mathbb{R}^d).$$

Given a “sufficiently rich" probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the distance can be defined equivalently by

$$\mathbf{d}_2(m_1, m_2) = \inf_{X,Y} \left(\mathbb{E}\left[|X-Y|^2 \right] \right)^{1/2} ,$$

where the infimum is taken over random variables X, Y over Ω with law m_1 and m_2 respectively.

1.2.3 Mean field limits

We complete this preliminary part by the analysis of large particle systems. Classical references on this topic are the monographs or texbooks by Sznitman [177], Spohn [176] and Golse [111].

We consider system of $N-$particles (where $N \in \mathbb{N}^*$ is a large number) and we want to understand the behavior of the system as the number N tends to infinity. We work with the following system: for $i = 1, \ldots, N$,

$$\begin{cases} dX_t^i = b(X_t^i, m_{X_t}^N)dt + dB_t^i, \qquad m_{X_t}^N := \dfrac{1}{N} \displaystyle\sum_{j=1}^N \delta_{X_t^j} \\ X_0^i = Z^i \end{cases} \tag{1.10}$$

where the (B^i) are independent Brownian motions, the Z^i are i.i.d. random variables in $\mathbb{R}^d$ which are also independent of the (B^i). The map $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is assumed to be globally Lipschitz continuous. Note that, under these assumptions, the solution (X^i) to (1.10) exists and is unique, since this is an ordinary system of SDEs with Lipschitz continuous drift. A key point is that, because the (Z^i) have the same law and the equations satisfied by the X^i are symmetric, the X^i have the same law (they are actually "exchangeable").

We want to understand the limit of the (X^i) as $N \to +\infty$. The heuristic idea is that, as N is large, the (X^i) become more and more independent, so that they become almost i.i.d. The law of large numbers then implies that

$$\frac{1}{N}\sum_{j=1}^{N} b(X_t^i, X_t^j) \approx \tilde{\mathbb{E}}\left[b(X_t^i, \tilde{X}_t^i)\right] = \int_{\mathbb{R}^d} b(X_t^i, y)\mathbb{P}_{X_t^i}(dy),$$

where $\tilde{X}_t^i$ is an independent copy of X_t^i and $\tilde{\mathbb{E}}$ is the expectation with respect to this independent copy. Therefore we expect the X^i to be close to the solution $\bar{X}^i$ to the McKean-Vlasov equation

$$\begin{cases} d\bar{X}_t^i = b(\bar{X}_t^i, \mathcal{L}(\bar{X}_t^i))dt + dB_t^i, \\ \bar{X}_0^i = Z^i \end{cases} \tag{1.11}$$

This is exactly what we are going to show. For doing so, we proceed in 3 steps: firstly, we generalize the law of large numbers by considering the convergence of empirical measures (the Glivenko-Cantelli law of large numbers), secondly we prove the existence and the uniqueness of a solution to the McKean-Vlasov equation (1.11) and, thirdly, we establish the convergence.

The Glivenko-Cantelli law of large numbers.

Here we consider (X_n) a sequence of i.i.d. random variables on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}[|X_1|] < +\infty$. We denote by m the law of X_1. The law of large numbers states that, a.s. and in L^1,

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} X_n = \mathbb{E}[X_1].$$

Our aim is to show that a slightly stronger convergence holds: let

$$m_X^N := \frac{1}{N}\sum_{n=1}^{N} \delta_{X_n}$$

Note that m_X^N is a random measure, in the sense that m_X^N is a.s. a measure and that, for any Borel set $A \subset X$, $m_X^N(A)$ is a random variable. The following result is (sometimes) known as the Glivenko-Cantelli Theorem.

Theorem 1 *If* $\mathbb{E}[|X_1|] < +\infty$*, then, a.s. and in* L^1*,*

$$\lim_{N\to+\infty} \mathbf{d}_1(m_X^N, m) = 0.$$

Remark 2 It is often useful to quantify the convergence speed in the law of large numbers. Such results can be found in the text books [171] or, in a sharper form, in [68, Theorem 5.8], see also the references therein.

Sketch of proof. Let $\phi \in C_b^0(X)$. Then, by the law of large numbers,

$$\int_{\mathbb{R}^d} \phi(x) m_X^N(dx) = \frac{1}{N}\sum_{n=1}^{N} \phi(X_n) = \mathbb{E}[\phi(X_1)] \qquad a.s.$$

By a separability argument, it is not difficult to check that the set of zero probability in the above convergence can be chosen independent of ϕ. So (m_X^N) converge weakly-* to m a.s. Note also that

$$\int_{\mathbb{R}^d} d(x,x_0) m_X^N(dx) = \frac{1}{N}\sum_{n=1}^N d(X_n, x_0)$$

where the random variables $(d(X_n,x_0))$ are i.i.d. and in L^1. By the law of large numbers we have

$$\int_{\mathbb{R}^d} d(x,x_0) m_X^N(dx) \to \int_{\mathbb{R}^d} d(x,x_0) m(dx) \qquad \text{a.s.}$$

By Proposition 1, (m_X^N) converges a.s. in $\mathbf{d}_1$ to m. It remains to show that this convergence also holds in expectation. For this we note that

$$\mathbf{d}_1(m_X^N, m) = \sup_\phi \int_{\mathbb{R}^d} \phi(m_X^N - m)(dx) \leq \sup_\phi \frac{1}{N}\sum_{i=1}^N \phi(X_i) - \int_{\mathbb{R}^d} \phi(x) m(dx),$$

where the supremum is taken over the $1-$Lipschitz continuous maps ϕ with $\phi(0)=0$. So

$$\mathbf{d}_1(m_X^N, m) \leq \frac{1}{N}\sum_{i=1}^N |X_i| + \int_{\mathbb{R}^d} |x| m(dx).$$

As the right-hand side converges in L^1, $\mathbf{d}_1(m_X^N, m)$ is uniformly integrable which implies its convergence in expectation to 0. □

The well-posedness of the McKean-Vlasov equation.

Theorem 2 *Let us assume that $b : \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is globally Lipschitz continuous and let $Z \in L^2(\Omega)$. Then the McKean-Vlasov equation*

$$\begin{cases} dX_t = b(X_t, \mathcal{L}(X_t))dt + dB_t \\ X_0 = Z \end{cases}$$

has a unique solution, i.e., a progressively measurable process such that $\mathbb{E}\left[\int_0^T |X_s|^2 ds\right] < +\infty$ for any $T > 0$.

Remark 3 By ItÃ´'s formula, the law m_t of a solution X_t solves in the sense of distributions the McKean-Vlasov equation

$$\begin{cases} \partial_t m_t - \frac{1}{2}\Delta m_t + \operatorname{div}(m_t b(x, m_t)) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d \\ m_0 = \mathcal{L}(Z) \qquad \text{in } \mathbb{R}^d. \end{cases}$$

One can show (and we will admit) that this equation has a unique solution, which proves the uniqueness in law of the process X.

Proof. Let $\alpha > 0$ to be chosen later and E be the set of progressively measurable processes (X_t) such that

$$\|X\|_E := \mathbb{E}\left[\int_0^\infty e^{-\alpha t}|X_t| dt\right] < +\infty.$$

Then $(E, \|\cdot\|_E)$ is a Banach space. On E we define the map Φ by

$$\Phi(X)_t = Z + \int_0^t b(X_s, \mathcal{L}(X_s))ds + B_t, \qquad t \geq 0.$$

Let us check that the map Φ is well defined from E to E. Note first that $\Phi(X)$ is indeed progressively measurable. By the $L-$Lipschitz continuity of b (for some $L > 0$),

$$\begin{aligned}|\Phi(X)_t| &\leq |Z| + \int_0^t |b(X_s, \mathcal{L}(X_s))|ds + |B_t| \\ &\leq |Z| + t|b(0,\delta_0)| + L\int_0^t (|X_s| + \mathbf{d}_1(\mathcal{L}(X_s),\delta_0))ds + |B_t|,\end{aligned}$$

where one can easily check that $\mathbf{d}_1(\mathcal{L}(X_s),\delta_0) = \mathbb{E}\left[\int_0^t |X_s|ds\right]$. So

$$\begin{aligned}\mathbb{E}\left[\int_0^{+\infty} e^{-\alpha t}|\Phi(X)_t|dt\right] &\leq \alpha^{-1}\mathbb{E}[|Z|] + \alpha^{-2}|b(0,\delta_0)| + 2L\mathbb{E}\left[\int_0^{+\infty} e^{-\alpha t}\int_0^t |X_s|dsdt\right] + \int_0^{+\infty} e^{-\alpha t}\mathbb{E}\left[|B_t|\right]dt \\ &= \alpha^{-1}\mathbb{E}[|Z|] + \alpha^{-2}|b(0,\delta_0)| + \frac{2L}{\alpha}\mathbb{E}\left[\int_0^{+\infty} e^{-\alpha s}|X_s|ds\right] + C_d\int_0^{+\infty} t^{1/2}e^{-\alpha t}dt,\end{aligned}$$

where C_d depends only on dimension. This proves that $\Phi(X)$ belongs to E.

Let us finally check that Φ is a contraction. We have, if $X, Y \in E$,

$$\begin{aligned}|\Phi(X)_t - \Phi(Y)_t| &\leq \int_0^t |b(X_s,\mathcal{L}(X_s)) - b(Y_s,\mathcal{L}(Y_s))|\,dt \\ &\leq Lip(b)\left(\int_0^t \mathbf{d}_1(\mathbb{P}_{X_s},\mathbb{P}_{Y_s})dt + \int_0^t |X_s - Y_s|dt\right).\end{aligned}$$

Recall that $\mathbf{d}_1(\mathbb{P}_{X_s},\mathbb{P}_{Y_s}) \leq \mathbb{E}\left[|X_s - Y_s|\right]$. So multiplying by $e^{-\alpha t}$ and taking expectation, we obtain:

$$\begin{aligned}\|\Phi(X) - \Phi(Y)\|_E &= \mathbb{E}\left[\int_0^{+\infty} e^{-\alpha t}\,|\Phi(X)_s - \Phi(Y)_s|\,dt\right] \\ &\leq 2Lip(b)\int_0^{+\infty} e^{-\alpha t}\int_0^t \mathbb{E}\left[|X_s - Y_s|\right]dsdt \\ &\leq \frac{2Lip(b)}{\alpha}\|X - Y\|_E.\end{aligned}$$

If we choose $\alpha > 2Lip(b)$, then Φ is a contraction in the Banach space E and therefore has a unique fixed point. It is easy to check that this fixed point is the unique solution to our problem. □

The mean field limit.

Let (X^i) be the solution to the particle system (1.10) and $(\bar{X}^i)$ be the solution to (1.11). Let us note that, as the (B^i) and the (Z^i) are independent with the same law, the $(\bar{X}^i_t)$ are i.i.d. for any $t \geq 0$.

Theorem 3 *We have, for any $T > 0$,*

$$\lim_{N\to+\infty}\sup_{i=1,\dots,N}\mathbb{E}\left[\sup_{t\in[0,T]}|X^i_t - \bar{X}^i_t|\right] = 0.$$

Remark: a similar result holds when there is a non constant volatility term σ in front of the Brownian motion. The proof is then slightly more intricate.

Proof. We consider

$$X_t^i - \bar{X}_t^i = \int_0^t \left(b(X_t^i, m_{X_t}^N) - b(\bar{X}_t^i, \mathcal{L}(\bar{X}_t^i)) \right) dt.$$

By the uniqueness in law of the solution to the McKean-Vlasov equation we can denote by $m(t) := \mathcal{L}(\bar{X}_t^i)$ (it is independent of i). Then, setting $m_{\bar{X}_t}^N := \frac{1}{N} \sum_{j=1}^N \delta_{\bar{X}_t^j}$ and using the triangle inequality, we have

$$\begin{aligned}
|X_t^i - \bar{X}_t^i| &\leq \int_0^t \left| b(X_t^i, m_{X_t}^N) - b(\bar{X}_s^i, m_{\bar{X}_s}^N) \right| ds + \int_0^t \left| b(\bar{X}_s^i, m_{\bar{X}_s}^N) - b(\bar{X}_s^i, m(s)) \right| ds \\
&\leq Lip(b) \int_0^t (|X_s^i - \bar{X}_s^i| + \mathbf{d}_1(m_{X_t}^N, m_{\bar{X}_s}^N)) ds + Lip(b) \int_0^t \mathbf{d}_1(m_{\bar{X}_s}^N, m(s)) ds \\
&\leq Lip(b) \int_0^t (|X_s^i - \bar{X}_s^i| + \frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j|) ds + Lip(b) \int_0^t \mathbf{d}_1(m_{\bar{X}_s}^N, m(s)) ds, \qquad (1.12)
\end{aligned}$$

since

$$\mathbf{d}_1(m_{X_t}^N, m_{\bar{X}_s}^N) \leq \frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j|.$$

Summing over $i = 1, \ldots, N$, we get

$$\frac{1}{N} \sum_{i=1}^N |X_t^i - \bar{X}_t^i| \leq 2Lip(b) \int_0^t \frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j| ds + Lip(b) \int_0^t \mathbf{d}_1(m_{\bar{X}_s}^N, m(s)) ds.$$

Using Gronwall Lemma, we find, for any $T > 0$, and for some constant C_T depending on $Lip(b)$,

$$\sup_{t \in [0,T]} \frac{1}{N} \sum_{i=1}^N |X_t^i - \bar{X}_t^i| \leq C_T \int_0^T \mathbf{d}_1(m_{\bar{X}_s}^N, m(s)) ds, \qquad (1.13)$$

where C_T depends on T and $Lip(b)$ (but not on N). Then we can come back to (1.12), use first Gronwall Lemma and then (1.13) to get, for any $T > 0$, and for some (new) constant C_T depending on $Lip(b)$ and which might change from line to line,

$$\begin{aligned}
\sup_{t \in [0,T]} |X_t^i - \bar{X}_t^i| &\leq C_T \int_0^T (\frac{1}{N} \sum_{j=1}^N |X_s^j - \bar{X}_s^j| + \mathbf{d}_1(m_{\bar{X}_s}^N, m(s))) ds \\
&\leq C_T \int_0^T \mathbf{d}_1(m_{\bar{X}_s}^N, m(s)) ds.
\end{aligned}$$

We now take expectation to obtain

$$\mathbb{E}\left[\sup_{t \in [0,T]} |X_t^i - \bar{X}_t^i| \right] \leq C_T \int_0^T \mathbb{E}\left[\mathbf{d}_1(m_{\bar{X}_s}^N, m(s)) ds \right].$$

One can finally check exactly as in the proof of Theorem 1 that the right-hand side tends to 0. □

1.3 The mean field game system

In this section, we focus on the mean field game system of PDEs (henceforth, MFG system) introduced by J.-M. Lasry and P.-L. Lions ([144], [145]). It takes the form of a backward Hamilton-Jacobi equation coupled with a forward Kolmogorov equation

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(x, Du, m) = 0, \\ \partial_t m - \varepsilon \Delta m - \operatorname{div}(m H_p(x, Du, m) = 0, \\ m(0) = m_0, \; u(T) = G(x, m(T)). \end{cases} \tag{1.14}$$

The Hamilton-Jacobi equation formalizes the individual optimization problem and is solved by the value function of each agent, while the Kolmogorov equation describes the evolution of the population density.

We will first derive in a heuristic way the MFG system (1.14). Then we will discuss several PDE methods used to obtain the existence and uniqueness of solutions, in both the diffusive case ($\varepsilon > 0$) and the deterministic case ($\varepsilon = 0$). In order to give a more clear and complete presentation of the PDE approach, we will mostly focus on the simplest form of the system (where the cost of control is separate from the mean-field dependent cost):

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(x, Du) = F(x, m) \\ u(T) = G(x, m(T)) \\ \partial_t m - \varepsilon \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 \\ m(0) = m_0\,, \end{cases} \tag{1.15}$$

where we will distinguish two kind of regimes, depending on the case of smoothing couplings F, G (operators on the space of measures) rather than on the case of local couplings (functions defined on the density of absolutely continuous measures). Sample results of existence and uniqueness will be given in both cases.

For simplicity, we will restrict the analysis of system (1.15) to the periodic case. This means that x belongs to the $d-$dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, and all x-dependent functions are $\mathbb{Z}^d-$periodic in space. We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed as before with the Monge-Kantorovich distance $\mathbf{d}_1$:

$$\mathbf{d}_1(m, m') = \sup_{\phi} \int_{\mathbb{T}^d} \phi \, d(m - m') \qquad \forall m, m' \in \mathcal{P}(\mathbb{T}^d),$$

where the supremum is taken over all $1-$Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$. This distance metricizes the narrow topology on $\mathcal{P}(\mathbb{T}^d)$. Recall that $\mathcal{P}(\mathbb{T}^d)$ is a compact metric space. For $T > 0$, we set $Q_T := (0, T) \times \mathbb{T}^d$.

1.3.1 Heuristic derivation of the MFG system

We describe here the simplest class of mean field games, when the state space is $\mathbb{R}^d$. In this control problem with infinitely many agents, each small agent controls his/her own dynamics:

$$X_s = x + \int_t^s b(X_r, \alpha_r, m(r))dr + \int_t^s \sigma(X_r, \alpha_r, m(r))dB_r, \tag{1.16}$$

where X lives in $\mathbb{R}^d$, α is the control (taking its values in a fixed set A) and B is a given $M-$dimensional Brownian motion. The difference with Section 1.2.1 is the dependence of the drift with respect to the distribution $(m(t))$ of all the players. This (time dependent) distribution $(m(t))$ belongs to the set $\mathcal{P}(\mathbb{R}^d)$ and is, at this stage, supposed to be given: one should think at $(m(t))$ as the *anticipation* made by the agents on their future time dependent distribution. The coefficients $b : \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d \times M}$ are assumed to be smooth enough for the solution (X_t) to exist.

The cost of a small player is given by

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T L(X_s,\alpha_s,m(s))ds + G(X_T,m(T))\right]. \tag{1.17}$$

Here $T > 0$ is the finite horizon of the problem, $L : \mathbb{R}^d \times A \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ and $G : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ are given continuous maps.

If we define the value function u as

$$u(t,x) = \inf_\alpha J(t,x,\alpha),$$

then, at least in a formal way, u solves the Hamilton-Jacobi equation

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x), D^2u(t,x), m(t)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ u(T,x) = G(x,m(T)) & \text{in } \mathbb{R}^d \end{cases}$$

where the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is defined by

$$H(x,p,M,m) := \sup_{a\in A}\left[-L(x,a,m) - p\cdot b(x,a,m) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(x,a,m)M)\right].$$

Let us now introduce $\alpha^*(t,x) \in A$ as a maximum point in the definition of H when $p = Du(t,x)$ and $M = D^2u(t,x)$. Namely

$$\begin{aligned} H(x, Du(t,x), D^2u(t,x), m(t)) = &-L(x,\alpha^*(t,x),m(t)) - Du(t,x)\cdot b(x,\alpha^*(t,x),m(t)) \\ &- \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(x,\alpha^*(t,x))D^2u(t,x),m(t)). \end{aligned} \tag{1.18}$$

Recall from Subsection 1.2.1 that α^* is the optimal feedback for the problem. However, we stress that here u and α^* depend on the time-dependent family of measures $(m(t))$.

We now discuss the evolution of the population density. For this we make the two following assumptions. Firstly we assume that all the agents control the same system (1.16) (although not necessarily starting from the same initial position) and minimize the same cost J. As a consequence, the dynamics at optimum of each player is given by

$$dX_s^* = b(X_s^*, \alpha^*(s,X_s^*), m(s))ds + \sigma(X_s^*, \alpha^*(s,X_s^*), m(s))dB_s.$$

Secondly, we assume that the initial position of the agents and the noise driving their dynamics are independent: in particular, there is no “common noise" impacting all the players. The initial distribution of the agents at time $t = 0$ is denoted by $\bar{m}_0 \in \mathcal{P}(\mathbb{R}^d)$. From the analysis of the mean field limit of Subsection 1.2.3 (in the simple case where the coefficients do not depend on the other agents) the actual distribution $(\tilde{m}(s))$ of all agents at time s is simply given by the law of (X_s^*) with $\mathcal{L}(X_0^*) = m_0$.

Let us now write the equation satisfied by $(\tilde{m}(s))$. By ItÃ´'s formula, we have, for any smooth map $\phi : [0,T)\times\mathbb{R}^d \to \mathbb{R}$ with a compact support:

$$\begin{aligned} 0 = \mathbb{E}\left[\phi(T,X_T^*)\right] &= \mathbb{E}\left[\phi(0,X_0^*)\right] + \int_0^T \mathbb{E}\Big[\partial_t\phi(s,X_s^*) + b(X_s^*,\alpha^*(s,X_s^*),m(s))\cdot D\phi(s,X_s^*) \\ &\qquad + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(X_s^*,\alpha^*(s,X_s^*),m(s))D^2\phi(s,X_s^*))\Big]ds \\ &= \int_{\mathbb{R}^d}\phi(0,x)m_0(dx) + \int_0^T\int_{\mathbb{R}^d}\Big[\partial_t\phi(s,x) + b(x,\alpha^*(s,x),m(s))\cdot D\phi(s,x) \\ &\qquad + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*(x,\alpha^*(s,x),m(s))D^2\phi(s,x))\Big]\tilde{m}(t,dx)ds \end{aligned}$$

After integration by parts, we obtain that $(\tilde{m}(t))$ satisfies, in the sense of distributions,

$$\begin{cases} \partial_t \tilde{m} - \dfrac{1}{2} \sum_{ij} D^2_{ij}(\tilde{m}(t,x) a_{ij}(x, \alpha^*(t,x), m(t))) + \operatorname{div}(\tilde{m}(t,x) b(x, \alpha^*(t,x), m(t))) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \\ \tilde{m}(0) = m_0 \qquad \text{in } \mathbb{R}^d, \end{cases}$$

where $a = \sigma\sigma^*$.

At equilibrium, one expects the anticipation $(m(t))$ made by the agents to be correct: $\tilde{m}(t) = m(t)$. Collecting the above equations leads to the MFG system:

$$\begin{cases} -\partial_t u(t,x) + H(x, Du(t,x), D^2 u(t,x), m(t)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d, \\ \partial_t m - \dfrac{1}{2} \sum_{ij} D^2_{ij}(m(t,x) a_{ij}(x, \alpha^*(t,x), m(t)) + \operatorname{div}(m(t,x) b(x, \alpha^*(t,x), m(s))) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \\ m(0) = m_0, \; u(T,x) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d, \end{cases}$$

where α^* is given by (1.18) and $a = \sigma\sigma^*$.

In order to simplify a little this system, let us assume that $M = d$ and $\sigma = \sqrt{2\varepsilon} I_d$ (where now ε is a constant). We set (warning! abuse of notation!)

$$H(x,p,m) := \sup_{a \in A} \left[-L(x,a,m) - p \cdot b(x,a,m) \right]$$

and note that, by Lemma 1, under suitable assumptions one has (see (1.6))

$$H_p(x, Du(t,x), m(t)) = -b(x, \alpha^*(t,x), m(t)).$$

In this case the MFG system becomes

$$\begin{cases} -\partial_t u(t,x) - \varepsilon \Delta u(t,x) + H(x, Du(t,x), m(t)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d, \\ \partial_t m - \varepsilon \Delta m(t,x) - \operatorname{div}(m(t,x) H_p(x, Du(t,x), m(t)) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^d, \\ m(0) = m_0, \; u(T,x) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d, \end{cases}$$

This system will be the main object of analysis of this chapter. Note that it is not a standard PDE system, since the first equation is backward in time, while the second one is forward in time. As this analysis is not too easy, it will be more convenient to work with periodic boundary condition (namely on the $d-$dimensional torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$).

1.3.2 Second order MFG system with smoothing couplings

We start with the analysis of the MFG system (1.15). Hereafter, we assume that x belongs to the $d-$dimensional torus $\mathbb{T}^d$. In this Section we assume that the coupling functions are smoothing operators defined on the set $C^0([0,T], \mathcal{P}(\mathbb{T}^d))$. To stress that the couplings are operators, we will write their action as $F[m]$ and $G[m]$, so the system will be written as

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(t,x,Du) = F[m] \\ u(T) = G[m(T)] \\ \partial_t m - \varepsilon \Delta m - \operatorname{div}(m \, H_p(t,x,Du)) = 0 \\ m(0) = m_0 \, . \end{cases} \tag{1.19}$$

Definition 1 We say that a pair (u,m) is a *classical* solution to (1.19) if

(i) $m \in C^0([0,T], \mathcal{P}(\mathbb{T}^d))$ and $m(0) = m_0$; $u \in C(\overline{Q_T})$ and $u(T) = G[m(T)]$
(ii) u, m are continuous functions in $(0,T) \times \mathbb{T}^d$, of class C^2 in space and C^1 in time, and the two equations are satisfied pointwise for $x \in \mathbb{T}^d$ and $t \in (0,T)$.

Let us stress that the above definition only requires u, m to be smooth for $t \in (0,T)$, which allows m_0 to be a general probability measure. Of course, the smoothness can extend up to $t = 0$ and/or $t = T$ in case m_0 and/or $G[\cdot]$ are sufficiently smooth. We also notice that the above definition requires $H_p(x,p)$ to be differentiable, in order for m to be a classical solution. It is often convenient to use a weaker notion as well: we will simply say that (u,m) is a solution of (1.19) (without using the adjective *classical*) if (u,m) satisfy point (i) in the above Definition, m and Du are locally bounded and if both equations are satisfied in the sense of distributions in $(0,T)$, i.e. against test functions $\varphi \in C^1((0,T)\times\mathbb{T}^d)$ with compact support in $(0,T)$.

The smoothing character of the couplings F, G is assumed in the following conditions:

$$F : C^0([0,T],\mathcal{P}(\mathbb{T}^d)) \to C^0(\overline{Q_T}) \quad \text{is continuous with range into a bounded set of } L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d)) \tag{1.20}$$

and similarly

$$G : \mathcal{P}(\mathbb{T}^d)\cap L^1(\mathbb{T}^d) \to C^0(\mathbb{T}^d) \quad \text{is continuous with range into a bounded set of } W^{1,\infty}(\mathbb{T}^d). \tag{1.21}$$

We notice that functions $F(t,x,m)$ which are (locally) Lipschitz continuous on $Q_T \times \mathcal{P}(\mathbb{T}^d)$ naturally provide with corresponding operators given by $F[m] = F(t,x,m(t))$; the above assumption (1.20) is satisfied if, for instance, F is Lipschitz in x uniformly for $m \in \mathcal{P}(\mathbb{T}^d)$. Examples of such smoothing operators are easily obtained by convolution.

As is now well-known in the theory, a special case occurs if the operators F, G are *monotone*. This can be understood as an extension of the standard monotonicity for L^2 operators (indeed, F and G are defined in $L^2(Q_T)$ and $L^2(\mathbb{T}^d)$ respectively). For instance, F is said to be monotone if

$$\int_0^T\int_{\mathbb{T}^d}(F[m_1]-F[m_2])d(m_1-m_2) \geq 0 \qquad \forall m_1, m_2 \in C^0([0,T],\mathcal{P}(\mathbb{T}^d))$$

and a similar definition applies to G. Let us observe that the monotonicity condition on F, G is satisfied for a restricted class of convolution operators, but one can take for instance $F[m] = f(k\star m)\star k$, where f is a nondecreasing function and $k \geq 0$ a smooth symmetric kernel.

We start by giving one of the early results by J.-M. Lasry and P.-L. Lions.

Theorem 4 ([144],[145]) *Let F, G satisfy conditions (1.20), (1.21). Assume that $H \in C^1(Q_T \times \mathbb{R}^d)$ is convex with respect to p and satisfies at least one of the two following assumptions:*

$$\exists\ c_0 > 0\ :\ |H_p(t,x,p)| \leq c_0(1+|p|) \quad \forall (t,x,p) \in Q_T\times\mathbb{R}^d \tag{1.22}$$

$$\exists\ c_1 > 0\ :\ H_x(t,x,p)\cdot p \geq -c_1(1+|p|^2) \quad \forall (t,x,p) \in Q_T\times\mathbb{R}^d \tag{1.23}$$

Then, for every $m_0 \in \mathcal{P}(\mathbb{T}^d)$, there exist $u \in L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))\cap C^0(\overline{Q_T})$ and $m \in C^0([0,T],\mathcal{P}(\mathbb{T}^d))$ such that (u,m) is a solution to (1.19).

In addition, let F, G be monotone operators. If one of the two following conditions hold:

$$\begin{cases} \int_0^T\int_{\mathbb{T}^d}(F[m_1]-F[m_2])d(m_1-m_2) = 0 \Rightarrow\ F[m_1] = F[m_2] \\ \int_{\mathbb{T}^d}(G[m_1]-G[m_2])d(m_1-m_2) = 0 \Rightarrow\ G[m_1] = G[m_2] \end{cases} \tag{1.24}$$

$$H(t,x,p_1) - H(t,x,p_2) - H_p(t,x,p_2)(p_1-p_2) = 0 \Rightarrow\ H_p(t,x,p_1) = H_p(t,x,p_2) \quad \forall p_1,p_2 \in \mathbb{R}^d \tag{1.25}$$

then (u,m) is unique in the above class.

Finally, if in addition $H_p \in C^1(Q_T\times\mathbb{R}^d)$ and F is a bounded map in the space of Hölder continuous functions, then (u,m) is a classical solution.

Proof. We start by assuming that m_0 is Hölder continuous in $\mathbb{T}^d$. We set

$$X := \{m \in C^0([0,T],L^1(\mathbb{T}^d))\ :\ m \geq 0\,,\ \int_{\mathbb{T}^d} m(t) = 1\ \forall t \in [0,T]\}\,,$$

and we define the following operator: for $\mu \in X$, if u_μ denotes the unique solution to

$$\begin{cases} -\partial_t u_\mu - \varepsilon \Delta u_\mu + H(t,x,Du_\mu) = F[\mu] \\ u_\mu(T) = G[\mu(T)]\,, \end{cases}$$

then we set $m := \Phi(\mu)$ as the solution to

$$\begin{cases} \partial_t m - \varepsilon \Delta m - \operatorname{div}(m H_p(t,x,Du_\mu)) = 0\,, \\ m(0) = m_0\,. \end{cases}$$

We observe that, for given $\mu \in X$, $F[\mu]$ belongs to $L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$ and $G[\mu]$ is Lipschitz as well. If condition (1.22) is satisfied, then H has at most natural quadratic growth since $|H(t,x,p)| \le c(1+|p|^2)$ for some constant $c>0$. Hence the classical parabolic theory applies (see [142, Chapter V, Thm 3.1]); there exists a constant $K>0$ and $\alpha \in (0,1)$ such that $Du_\mu \in C^\alpha(Q_T)$ and

$$\|Du_\mu\|_\infty \le K\,. \tag{1.26}$$

More precisely, the constant K is independent of μ due to the assumptions on the range of F,G in (1.20)–(1.21).

By contrast, if condition (1.23) holds true, then H has not necessarily natural growth; however, a gradient estimate follows by using the classical Bernstein's method. This means that we look at the equation satisfied by $w := |Du|^2$. Assuming u to be smooth, a direct computation gives

$$\begin{aligned} \partial_t w - \varepsilon \Delta w &= -2|D^2 u|^2 - 2Du \cdot D\left(H(t,x,Du) - F[m]\right) \\ &\le -H_p \cdot Dw - 2H_x(t,x,Du)\cdot Du + 2DuDF[m] \\ &\le -H_p \cdot Dw + 2c_1(1+|Du|^2) + |Du|^2 + \|F[m]\|^2_{W^{1,\infty}} \end{aligned}$$

where we used both assumptions (1.20) and (1.23). Since $\|F[m]\|_{W^{1,\infty}}$ is bounded uniformly with respect to m, we conclude that there exists $C>0$ such that

$$\partial_t w - \varepsilon \Delta w + H_p \cdot Dw \le C(1+w)\,.$$

At time $t=T$ we have $\|w(T)\|_\infty \le \|DG[m]\|^2_\infty \le C$, so we deduce, by maximum principle, that

$$\|Du\|^2_\infty = \|w\|_\infty \le C_T \tag{1.27}$$

for some constant C_T depending on T, c_1, F, G. Therefore, (1.26) holds true under condition (1.23) as well.

Eventually, we conclude that $H_p(t,x,Du_\mu)$ is uniformly bounded for $\mu \in X$. By parabolic regularity (see e.g. [142, Chapter V, Thms 1.1 and 2.1]), this implies that m is uniformly bounded in $C^\alpha(\overline{Q_T})$, for some $\alpha \in (0,1)$. In particular, the operator Φ has bounded range in $C^\alpha(\overline{Q_T})$, so the range of Φ is a compact subset of X. The continuity of Φ is straightforward: if $\mu_n \to \mu$, we have $F[\mu_n] \to F[\mu]$ in $C(\overline{Q_T})$, so u_n converges uniformly to the corresponding solution u_μ, while Du_n converges a.e. to Du_μ, hence $H_p(x,Du_n) \to H_p(x,Du_\mu)$ in $L^p(Q_T)$ for every $p>1$, which entails the convergence of m_n towards $m = \Phi(\mu)$. By Schauder's fixed point theorem (see e.g. [110]) applied to Φ, we deduce the existence of m such that $m = \Phi(m)$, which means a solution (u,m) of (1.19).

For general $m_0 \in \mathcal{P}(\mathbb{T}^d)$, we can proceed by approximation. Given a sequence of smooth functions m_{0n} converging to m_0 in $\mathcal{P}(\mathbb{T}^d)$, the corresponding solutions u_n will satisfy (1.26) uniformly thanks to (1.20)-(1.21). Using as before the parabolic regularity one gets that Du_n is relatively compact in $C^0([a,b]\times\mathbb{T}^d)$ for all compact subsets $[a,b] \subset (0,T)$. Hence $H_p(t,x,Du_n)$ converges in $L^p(Q_T)$ for every $p<\infty$. By standard stability of Fokker-Planck equations, this implies the compactness of m_n in $C^0([0,T];\mathcal{P}(\mathbb{T}^d))$. In particular, we deduce both the initial and the terminal condition (due to the continuity of G). Finally, the limit couple (u,m) satisfies $u \in L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$, $m \in C^0([0,T];\mathcal{P}(\mathbb{T}^d))$ and is a solution of (1.19). In fact, by the parabolic regularity recalled before, this solution satisfies $m, Du \in C^\alpha(Q_T)$ for some $\alpha \in (0,1)$. If F is a bounded map in the space of Hölder continuous functions, then we bootstrap the regularity once more. We have that $F[m]$ is Hölder continuous, so is $H(t,x,Du)$ and therefore by Schauder regularity (see e.g. [142, Chapter IV]) u belongs to $C^{1+\frac{\alpha}{2},2+\alpha}(Q_T)$ for some $\alpha \in (0,1)$. If H_p is C^1, this implies that $div(H_p(t,x,Du))$ is Hölder continuous as well, and we conclude that m is also a classical solution in $(0,T)$.

Uniqueness: Let (u_1,m_1) and (u_2,m_2) be solutions of (1.19) such that $u_i \in L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d)) \cap C(\overline{Q_T})$ and $m_i \in C^0([0,T];\mathcal{P}(\mathbb{T}^d))$. As we already used above, the m_i are locally bounded and Hölder continuous; therefore, $m_1 - m_2$ can be justified as test function in the equation of $u_1 - u_2$ (and viceversa) in any interval (a,b) compactly contained in $(0,T)$. It follows that

$$
\begin{aligned}
-\frac{d}{dt}\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2) = &\int_{\mathbb{T}^d}(F[m_1]-F[m_2])(m_1-m_2) \\
&+\int_{\mathbb{T}^d} m_1\left\{H(t,x,Du_2)-H(t,x,Du_1)-H_p(t,x,Du_1)(Du_2-Du_1)\right\} \\
&+\int_{\mathbb{T}^d} m_2\left\{H(t,x,Du_1)-H(t,x,Du_2)-H_p(t,x,Du_2)(Du_1-Du_2)\right\}
\end{aligned}
\tag{1.28}
$$

where the equality is meant in the weak sense in $(0,T)$. By convexity of H and monotonicity of F, it follows that $\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2)$ is non increasing in time. Moreover, this quantity is continuous in $[0,T]$ because $u_i \in C(\overline{Q_T})$ and $m_i \in C^0([0,T];\mathcal{P}(\mathbb{T}^d))$. By monotonicity of G, this quantity is nonnegative at $t=T$, however it vanishes for $t=0$. We deduce that it vanishes for all $t \in [0,T]$. In particular, the previous equality implies that all terms in the right-hand side are equal to zero. If condition (1.24) holds true, this implies that $F[m_1]=F[m_2]$ and $G[m_1(T)]=G[m_2(T)]$; hence, by uniqueness of the parabolic equation (namely, by maximum principle), we deduce that $u_1=u_2$. This implies $H_p(t,x,Du_1)=H_p(t,x,Du_2)$, and for the Fokker-Planck equation this implies that $m_1=m_2$. Indeed, given a bounded drift $b \in L^\infty(Q_T)$, one can easily verify with a duality argument that if $\mu \in C^0([0,T];\mathcal{P}(\mathbb{T}^d))$ is a weak solution of the equation $\partial_t\mu - \Delta\mu - \mathrm{div}(b\,\mu)=0$ and $\mu(0)=0$, then $\mu \equiv 0$. Alternatively, if (1.25) holds true, then we first obtain that $H_p(t,x,Du_1)=H_p(t,x,Du_2)$, hence we deduce that $m_1=m_2$ and we conclude by uniqueness of u. □

Remark 4 Uniqueness of the solution of (1.19) is not expected in general if Lasry-Lions' monotonicity condition fails. This lack of uniqueness is well-documented in the literature: see for instance [25, 40, 68, 149]. By contrast, it is relatively easy to check that uniqueness holds if the horizon is "short" or if the functions H and G do not "depend too much" on m, see e.g. [25].

The existence part of the above result can easily be extended to more general MFG systems, in which the Hamiltonian has no separate structure:

$$
\begin{cases}
-\partial_t u - \varepsilon\Delta u + H(t,x,Du,m(t)) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\
\partial_t m - \varepsilon\Delta m - \mathrm{div}\,(m\,H_p(t,x,Du,m(t))) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\
m(0)=m_0\,,\; u(T,x)=G[m(T)] & \text{in } \mathbb{T}^d
\end{cases}
\tag{1.29}
$$

The notion of classical solution is given as before. A general existence result in this direction sounds as follows.

Theorem 5 *Assume that $H: Q_T \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is a continuous function, differentiable with respect to p, and such that both H and H_p are C^1 continuous on $Q_T \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)$, and in addition H satisfies the growth condition*

$$
H_x(t,x,p,m)\cdot p \ge -C_0(1+|p|^2) \qquad \forall (t,x,p,m) \in Q_T \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)
\tag{1.30}
$$

for some constant $C_0 > 0$. Assume that G satisfies (1.21) and that $m_0 \in \mathcal{P}(\mathbb{T}^d)$. Then there is at least one classical solution to (1.29).

Remark 5 Of course, the solution found in the above Theorem is smooth up to $t=0$ (respectively $t=T$) if, for some $\beta \in (0,1)$, $m_0 \in C^{2+\beta}(\mathbb{T}^d)$ (respectively, $G[m]$ is bounded in $C^{2+\beta}(\mathbb{T}^d)$ uniformly with respect to $m \in \mathcal{P}(\mathbb{T}^d)$).

The proof is relatively easy and relies on gradient estimates for Hamilton-Jacobi equations (as already used in Theorem 4 above) and on the following estimate on the McKean-Vlasov equation

$$\begin{cases} \partial_t m - \varepsilon \Delta m - \operatorname{div}(m\, b(t,x,m(t))) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ m(0) = m_0 \end{cases} \tag{1.31}$$

To this purpose, it is convenient to introduce the following stochastic differential equation (SDE)

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sqrt{2\varepsilon}\, dB_t, & t \in [0,T] \\ X_0 = Z_0 \end{cases} \tag{1.32}$$

where (B_t) is a standard $d-$dimensional Brownian motion over some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and where the initial condition $Z_0 \in L^1(\Omega)$ is random and independent of (B_t).

We assume that the vector field $b : [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^d$ is continuous in time and Lipschitz continuous in (x,m) uniformly in t. Under the above condition on b, we have proved in Subsection 1.2.3 that there is a unique solution to (1.32). This solution is closely related to equation (1.31).

Lemma 6 *Under the above condition on b, if $\mathcal{L}(Z_0) = m_0$, then $m(t) := \mathcal{L}(X_t)$ is a weak solution of (1.31) and satisfies*

$$\mathbf{d}_1(m(t), m(s)) \le c_0(1 + \|b\|_\infty)|t-s|^{\frac{1}{2}} \qquad \forall s,t \in [0,T] \tag{1.33}$$

for some constant $c_0 = c_0(T)$ independent of $\varepsilon \in (0,1]$.

Proof. The fact that $m(t) := \mathcal{L}(X_t)$ is a weak solution of (1.31) is a straightforward consequence of ItÃ´s formula: if $\varphi : Q_T \to \mathbb{R}$ is smooth, then

$$\varphi(t,X_t) = \varphi(0,Z_0) + \int_0^t [\varphi_t(s,X_s) + D\varphi(s,X_s)\cdot b(s,X_s,m(s)) + \varepsilon\Delta\varphi(s,X_s)]\; ds + \int_0^t D\varphi(s,X_s)\cdot dB_s.$$

Taking the expectation on both sides of the equality, we have, since $\mathbb{E}\left[\int_0^t D\varphi(s,X_s)\cdot dB_s\right] = 0$,

$$\mathbb{E}\left[\varphi(t,X_t)\right] = \mathbb{E}\left[\varphi(0,Z_0) + \int_0^t [\varphi_t(s,X_s) + D\varphi(s,X_s)\cdot b(s,X_s,m(s)) + \varepsilon\Delta\varphi(s,X_s)]\; ds\right] .$$

So by definition of $m(t)$, we get

$$\int_{\mathbb{R}^d} \varphi(t,x)m(t,dx) = \int_{\mathbb{R}^d} \varphi(0,x)m_0(dx) + \int_0^t \int_{\mathbb{R}^d} [\varphi_t(s,x) + D\varphi(s,x)\cdot b(s,x,m(s)) + \varepsilon\Delta\varphi(s,x)]\, m(s,dx)\; ds$$

hence m is a weak solution to (1.31), provided we check that m is continuous in time. This is the aim of the next estimate. Let $\phi : \mathbb{T}^d \to \mathbb{R}$ be 1-Lipschitz continuous and take, for instance, $s < t$. Then, using (1.8), we have

$$\mathbf{d}_1(m(t),m(s)) \le \mathbb{E}\left[|X_t - X_s|\right] \le \mathbb{E}\left[\int_s^t |b(\tau, X_\tau, m(\tau))|\, d\tau + \sqrt{2\varepsilon}\, |B_t - B_s|\right] \le \|b\|_\infty (t-s) + \sqrt{2\varepsilon(t-s)} \tag{1.34}$$

which yields (1.33). □

Remark 6 Observe that not only the estimate (1.33) is independent of the diffusion coefficient ε, but actually the precise form (1.34) shows that, when $\varepsilon \to 0$, the map $m(t)$ becomes Lipschitz in the time variable.

We further notice that an estimate also follows, similarly as in (1.34), for the Wasserstein distance. Indeed, recalling that $\mathbf{d}_2(m_1,m_2) = \inf_{X,Y}\left(\mathbb{E}\left[|X-Y|^2\right]\right)^{1/2}$, proceeding similarly as in (1.34) yields

$$\mathbf{d}_2(m(t),m(s)) \leq \sqrt{|t-s|}\,\mathbb{E}\left[\int_s^t |b(\tau,X_\tau,m(\tau))|^2\,d\tau\right]^{\frac{1}{2}} + o(1) \quad \text{as } \varepsilon\to 0$$
$$\leq \sqrt{|t-s|}\,\left(\int_0^T\int_{\mathbb{T}^d} |b|^2\,m\,dxdt\right)^{\frac{1}{2}} + o(1) \quad \text{as } \varepsilon\to 0.$$

We prove now Theorem 5.

Proof of Theorem 5.

For a large constant C_1 to be chosen below, let $\mathcal{C}$ be the set of maps $\mu\in C^0([0,T],\mathcal{P}(\mathbb{T}^d))$ such that

$$\sup_{s\neq t}\frac{\mathbf{d}_1(\mu(s),\mu(t))}{|t-s|^{\frac{1}{2}}} \leq C_1. \tag{1.35}$$

Then $\mathcal{C}$ is a closed convex subset of $C^0([0,T],\mathcal{P}(\mathbb{T}^d))$. It is actually compact thanks to Ascoli's Theorem and the compactness of the set $\mathcal{P}(\mathbb{T}^d)$. To any $\mu\in\mathcal{C}$ we associate $m=\Psi(\mu)\in\mathcal{C}$ in the following way. Let u be the solution to

$$\begin{cases} -\partial_t u-\varepsilon\Delta u+H(t,x,Du,\mu(t))=0 & \text{in } (0,T)\times\mathbb{T}^d\\ u(T)=G[\mu(T)] \quad \text{in } \mathbb{T}^d \end{cases} \tag{1.36}$$

Then we define $m=\Psi(\mu)$ as the solution of the Fokker-Planck equation

$$\begin{cases} \partial_t m-\varepsilon\Delta m-\operatorname{div}\left(m\,H_p(t,x,Du,\mu(t))\right)=0 & \text{in } (0,T)\times\mathbb{T}^d\\ m(0)=m_0 \quad \text{in } \mathbb{T}^d\,. \end{cases} \tag{1.37}$$

Let us check that Ψ is well-defined and continuous. We start by assuming that H is globally Lipschitz continuous. Then, by standard parabolic theory (see e.g. [142, Chapter V, Thm 6.1]), equation (1.36) has a unique classical solution u. Moreover, u is of class $C^{1+\alpha/2,2+\alpha}(Q_T)$ where the constant α do not depend on μ. In addition, the Bernstein gradient estimate (1.27) holds exactly as in Theorem 4, which means that

$$\|Du\|_\infty \leq K$$

for some constant K only depending on T, $\|DG[\mu]\|_\infty$ and on the constant C_0 in (1.30). Due to (1.21), the constant K is therefore independent of μ. We see now that the global Lipschitz condition on H can be dropped: indeed, it is enough to replace $H(t,x,p,m)$ with $\tilde{H}(t,x,p,m)=\zeta(p)H(t,x,p,m)+(1-\zeta(p))|p|$ where $\zeta:\mathbb{R}^d\to[0,1]$ is a smooth function such that $\zeta(p)\equiv 1$ for $|p|\leq 2K$ and $\zeta(p)\equiv 0$ for $|p|>2K+1$. Thanks to the gradient estimate, solving the problem for $\tilde{H}$ is the same as for H.

Next we turn to the Fokker-Planck equation (1.37). Since the drift $b(t,x):=-H_p(t,x,Du(x),\mu(t))$ belongs to $L^\infty(Q_T)$, there is a unique solution $m\in C^0([0,T];\mathcal{P}(\mathbb{T}^d))$; moreover, since b is bounded independently of μ, say by a constant C_2, from Lemma 6 we have the following estimates on m:

$$\mathbf{d}_1(m(t),m(s)) \leq c_0(1+\|H_p(\cdot,Du,\mu)\|_\infty)|t-s|^{\frac{1}{2}} \leq c_0(1+C_2)|t-s|^{\frac{1}{2}} \qquad \forall s,t\in[0,T]\,.$$

So if we choose C_1 so large that $C_1\geq c_0(1+C_2)$, then m belongs to $\mathcal{C}$. Moreover, if we write the equation in the form

$$\partial_t m-\varepsilon\Delta m-Dm\cdot H_p(t,x,Du,\mu(t))-m\,\operatorname{div}H_p(t,x,Du,\mu(t))=0$$

then we observe that m is a classical solution in $(0,T)$. Indeed, since $u\in C^{1+\alpha/2,2+\alpha}(Q_T)$ and $t\to\mu(t)$ is Holder continuous, the maps $(t,x)\to H_p(t,x,Du,\mu(t))$ and $(t,x)\to\operatorname{div}H_p(t,x,Du,\mu(t))$ belong to $C^\alpha(Q_T)$, so that the solution m belongs to $C^{1+\alpha/2,2+\alpha}(Q_T)$ ([142, Chapter IV, Thm 10.1]).

We have just proved that the mapping $\Psi:\mu\to m=\Psi(\mu)$ is well-defined from $\mathcal{C}$ into itself. The continuity of Ψ can be proved exactly as in Theorem 4. We conclude by Schauder fixed point Theorem that the continuous map

$\mu \to m = \Psi(\mu)$ has a fixed point $\bar\mu$ in $\mathcal{C}$. Let $\bar u$ be associated to $\bar\mu$ as above. Then $(\bar u, \bar\mu)$ is a solution of our system (1.29). In addition $(\bar u, \bar\mu)$ is a classical solution in view of the above estimates. □

Let us mention that there are no general criteria for the uniqueness of solutions to (1.29) in arbitrary time horizon T, except for the Lasry-Lions' monotonicity condition (1.24) for the case of separate Hamiltonian treated in Theorem 4. In case of local dependence of $H(t, x, p, m)$ with respect to the density m of the measure, a structure condition on H ensuring the uniqueness was given by P.-L. Lions in [149] (Lessons 5-12/11 2010) and will be discussed later in Theorem 13, in the subsection devoted to local couplings.

Otherwise, the uniqueness of solutions to (1.29) can be proved for short time horizon, e.g. using directly the Banach fixed point theorem for contraction mappings, in order to produce both existence and uniqueness of solutions (as in the papers by Caines, Huang and MalhamÃ© [132], under a smallness assumption on the coefficients or on the time interval).

1.3.3 Application to games with finitely many players

In this subsection, we show how to apply the previous results on the MFG system to study a $N-$player differential game in which the number N of players is "large".

The $N-$player game

The dynamic of player i (where $i \in \{1, \dots, N\}$) is given by

$$dX_t^i = b^i(X_t^1, \dots, X_t^N, \alpha_t^1, \dots, \alpha_t^N)dt + \sqrt{2}dB_t^i, \qquad X_0^i = Z^i,$$

where (B_t^i) is a $d-$dimensional Brownian motion[1]. The initial condition X_0^i for this system is also random and has for law $\tilde m_0 \in \mathcal{P}_1(\mathbb{R}^d)$, and we assume that all Z^i and all the Brownian motions (B_t^i) $(i = 1, \dots, N)$ are independent. Player i can choose his bounded control α^i with values in $\mathbb{R}^d$ and adapted to the filtration $(\mathcal{F}_t = \sigma\{X_0^j,\ B_s^j,\ s \leq t,\ j = 1, \dots, N\})$. We make the structure assumption that the drift b^i of player i depends only on his/her own control and position and on the distribution of the other players. Namely:

$$b^i(x^1, \dots, x^N, \alpha^1, \dots, \alpha^N) = b(x^i, \alpha^i, \pi\sharp m_{\mathbf{x}}^{N,i}), \qquad \text{where } \mathbf{x} = (x^1, \dots, x^N) \in (\mathbb{R}^d)^N \text{ and } m_{\mathbf{x}}^{N,i} = \sum_{j \neq i} \delta_{x^j},$$

where $b : \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}^d$ is a globally Lipschitz continuous map. We have denoted by $\pi : \mathbb{R}^d \to \mathbb{T}^d$ the canonical projection and by $\pi\sharp\tilde m_0$ the image of the measure $\tilde m_0$ by the map π. The fact that the players interact through the projection over $\mathbb{T}^d$ of the empirical measure $m_{\mathbf{x}}^{N,i}$ is only a simplifying assumption related to the fact that we have so far led our analysis of the MFG system on the torus. Indeed, here we systematically see maps defined on $\mathbb{T}^d$ as $\mathbb{Z}^d-$periodic maps on $\mathbb{R}^d$.

The cost of player i is then given by

$$\mathcal{J}_i^N(\alpha^1, \dots, \alpha^N) = \mathcal{J}_i^N(\alpha^i, (\alpha^j)_{j\neq i}) = \mathbb{E}\left[\int_0^T L^i(X_t^1, \dots, X_t^N, \alpha_t^1, \dots, \alpha_t^N)dt + G^i(X_T^1, \dots, X_T^N)\right].$$

Here again we make the structure assumption that the running cost L^i of player i depends only on his/her own control and position and on the distribution of the other players' positions, while the terminal cost depends only on his/her position and on the distribution of the other players' positions:

[1] In order to avoid the (possible but) cumbersome definition of stochastic processes on the torus but, at the same time, be able to use the results of the previous parts, we work here with diffusions in $\mathbb{R}^d$ with periodic coefficients and assume that the mean field dependence of the data is always through the projection of the measures over the torus.

$$L^i(x^1,\dots,x^N,\alpha^1,\dots,\alpha^N) = L(x^i,\alpha^i,\pi\sharp m_{\mathbf{x}}^{N,i}), \qquad G^i(x^1,\dots,x^N) = G(x^i,\pi\sharp m_{\mathbf{x}}^{N,i}),$$

where $L : \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ and $G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are continuous maps, with

$$|L(x,\alpha,m) - L(x',\alpha,m')| + |G(x,m) - G(x',m')| \le K(|x-x'| + \mathbf{d}_1(m,m')),$$

the constant K being independent of α. In addition, we need a coercivity assumption on L with respect to α:

$$L(x,\alpha,m) \ge C^{-1}|\alpha| - C, \tag{1.38}$$

where C is independent of $(x,m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$. These assumptions are a little strong in practice, but allow us to avoid several (very) technical points in the proofs.

In that setting, a natural notion of equilibrium is the following.

Definition 2 We say that a family $(\bar\alpha^1,\dots,\bar\alpha^N)$ of bounded open-loop controls is an $\varepsilon-$Nash equilibrium of the $N-$player game (where $\varepsilon > 0$) if, for any $i \in \{1,\dots,N\}$ and any bounded open-loop control α^i,

$$\mathcal{J}_i^N(\bar\alpha^i,(\bar\alpha^j)_{j\neq i}) \le \mathcal{J}_i^N(\alpha^i,(\bar\alpha^j)_{j\neq i}) + \varepsilon.$$

The MFG system and the $N-$player game.

Our aim is to understand to what extent the MFG system can provide an $\varepsilon-$Nash equilibrium of the $N-$player game, at least if N is large enough. For this, we set

$$H(x,p,m) = \sup_{\alpha\in\mathbb{R}^d}\{-b(x,\alpha,m)\cdot p - L(x,\alpha,m)\}$$

and we assume that H, G and $m_0 := \pi\sharp\tilde m_0$ satisfy the assumptions of Theorem 5.

Hereafter, we fix (u,m) a classical solution to (1.29) (here with $\varepsilon = 1$). Following the arguments of Subsection 1.3.1, we recall the interpretation of the MFG system. In the mean-field approach, a generic player controls the solution to the SDE

$$X_t = X_0 + \int_0^t b(X_s,\alpha_s,m(s))ds + \sqrt{2}B_s,$$

and faces the minimization problem

$$\inf_\alpha \mathcal{J}(\alpha) \qquad \text{where} \qquad \mathcal{J}(\alpha) = \mathbb{E}\left[\int_0^T L(X_s,\alpha_s,m(s))\,ds + G\left(X_T,m(T)\right)\right].$$

In the above dynamics we assume that X_0 is a fixed random initial condition with law $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and the control α is adapted to some filtration $(\mathcal{F}_t)$. We also assume that (B_t) is a $d-$dimensional Brownian motion adapted to $(\mathcal{F}_t)$ and that X_0 and (B_t) are independent.

Then, given a solution (u,m) of (1.29), $u(0)$ is the optimal value and the feedback strategy α^* such that $b(x,\alpha^*(t,x),m(t)) := -H_p(x,Du(t,x),m(t))$ is optimal for the single player. Namely:

Lemma 7 *Let $(\bar X_t)$ be the solution of the stochastic differential equation*

$$\begin{cases} d\bar X_t = b(\bar X_t,\alpha^*(t,\bar X_t),m(t))dt + \sqrt{2}dB_t \\ \bar X_0 = X_0 \end{cases}$$

and set $\bar\alpha_t = \alpha^*(t,\bar X_t)$. Then

$$\inf_\alpha \mathcal{J}(\alpha) = \mathcal{J}(\bar\alpha) = \int_{\mathbb{T}^d} u(0,x)\,m_0(dx)\ .$$

Our goal now is to show that the strategy given by the mean field game is almost optimal for the $N-$player problem. We assume that the feedback $\alpha^*(t,x)$ defined above is continuous in (t,x) and globally Lipschitz continuous in x uniformly in t. With the feedback strategy α^* one can associate the open-loop control $\bar{\alpha}^i$ obtained by solving the system of SDEs:

$$d\bar{X}^i_t = b(\bar{X}^i_t, \alpha^*(t,\bar{X}^i_t), m^{N,i}_{\mathbf{X}_t})dt + \sqrt{2}dB^i_t,\ X^i_0 = Z^i \qquad (\text{where } \mathbf{X}_t = (X^1_t,\dots,X^N_t)), \tag{1.39}$$

and setting $\bar{\alpha}^i_t = \alpha^*(t,\bar{X}^i_t)$. We are going to show that the controls $\bar{\alpha}^i$ realize an approximate Nash equilibrium for the N-player game.

Theorem 6 *Assume that (Z^i) are i.i.d. random variables on $\mathbb{R}^d$ such that $\mathbb{E}[|Z^1|^q] < +\infty$ for some $q > 4$. There exists a constant $C > 0$ such that, for any $N \in \mathbb{N}^*$, the symmetric strategy $(\bar{\alpha}^1,\dots,\bar{\alpha}^N)$ is a $C\varepsilon_N-$Nash equilibrium in the game $\mathcal{J}^N_1,\dots,\mathcal{J}^N_N$ where*

$$\varepsilon_N := \begin{cases} N^{-1/2} & \text{if } d < 4 \\ N^{-1/2}\ln(N) & \text{if } d = 4 \\ N^{-2/d} & \text{if } d > 4 \end{cases}$$

Namely, for any $i \in \{1,\dots,N\}$ and for any control α^i adapted to the filtration $(\mathcal{F}_t)$,

$$\mathcal{J}^N_i(\bar{\alpha}^i, (\bar{\alpha}^j)_{j\neq i}) \le \mathcal{J}^N_i(\alpha^i, (\bar{\alpha}^j)_{j\neq i}) + C\varepsilon_N.$$

The Lipschitz continuity assumption on H and G with respect to m allows us to quantify the error. If H and G are just continuous with respect to m, one can only say that, for any $\varepsilon > 0$, there exists N_0 such that the symmetric strategy $(\bar{\alpha}^1,\dots,\bar{\alpha}^N)$ is an $\varepsilon-$Nash equilibrium in the game $\mathcal{J}^N_1,\dots,\mathcal{J}^N_N$ for any $N \ge N_0$.

Before starting the proof, we need the following result on product measures which can be found in [68] (Theorem 5.8. See also the references therein):

Lemma 8 *Assume that (Z_i) are i.i.d. random variables on $\mathbb{R}^d$ of law μ such that $\mathbb{E}[|Z_0|^q] < +\infty$ for some $q > 4$. Then there is a constant C, depending only on d, q and $\mathbb{E}[|Z_0|^q]$, such that*

$$\mathbb{E}\left[\mathbf{d}_2(m^N_Z,\mu)\right] \le \begin{cases} CN^{-1/2} & \text{if } d < 4 \\ CN^{-1/2}\ln(N) & \text{if } d = 4 \\ CN^{-2/d} & \text{if } d > 4 \end{cases}$$

Proof of Theorem 6. From the symmetry of the problem, it is enough to show that

$$\mathcal{J}^N_1(\bar{\alpha}^1, (\bar{\alpha}^j)_{j\ge 2}) \le \mathcal{J}^N_1(\alpha^1, (\bar{\alpha}^j)_{j\neq 2}) + C\varepsilon_N \tag{1.40}$$

for any control α, as soon as N is large enough. We note for this that the map $\tilde{b}(t,x,m) := b(x,\alpha^*(t,x),\pi\sharp m)$ is globally Lipschitz continuous in (x,m) uniformly in t thanks to our assumptions on b and α^*. Following the proof of Theorem 3 in Subsection 1.2.3, we have therefore that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N |X^i_t - \bar{X}^i_t|\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |X^i_t - \bar{X}^i_t|\right] \le C_T \int_0^T \mathbb{E}\left[\mathbf{d}_1(m^N_{\bar{X}_s}, m(s))ds\right], \tag{1.41}$$

where $\bar{X}_t = (\bar{X}^1,\dots,\bar{X}^N)$ solves

$$d\bar{X}^i_t = b(\bar{X}^i_t, \alpha^*(t,\bar{X}^i_t), \pi\sharp\mathcal{L}(\bar{X}^i_t))dt + \sqrt{2}dB^i_t, \qquad \bar{X}^i_0 = Z^i.$$

By uniqueness in law of the solution of the McKean-Vlasov equation, we have that the $\bar{X}^i_s$ are i.i.d. with a law $\tilde{m}(s)$, where $\tilde{m}(s)$ solves

$$\partial_t\tilde{m} - \Delta\tilde{m} - \operatorname{div}(\tilde{m}b(x,\alpha^*(t,x),\pi\sharp\tilde{m}(t)) = 0, \qquad \tilde{m}(0) = \tilde{m}_0.$$

In view of the assumption on b, it is easy to check that

$$\mathbb{E}[\sup_{t\in[0,T]} |\bar{X}_t^1|^q] \le C(1+\mathbb{E}[|Z^1|^q]) < +\infty.$$

Therefore using Lemma 8, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}_1(m^{N,1}_{\bar{X}_t}, \tilde{m}(t))\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}_2(m^{N,1}_{\bar{X}_t}, \tilde{m}(t))\right] \le C\varepsilon_N.$$

Note also that, by the uniqueness of the solution of the McKean-Vlasov equation (this time in $\mathbb{T}^d$), we have that $\pi\sharp\tilde{m} = m$ since both flows solve the same equation. Hence

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}_1(\pi\sharp m^{N,1}_{\bar{X}_t}, m(t))\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}_1(m^{N,1}_{\bar{X}_t}, \tilde{m}(t))\right] \le C\varepsilon_N.$$

Using (1.41) we obtain therefore

$$\mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N |X_t^i - \bar{X}_t^i|\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |X_t^i - \bar{X}_t^i|\right] \le C\varepsilon_N.$$

In particular, by Lemma 7 and the local Lipschitz continuity of L and G, we get

$$\begin{aligned}
\mathcal{J}_1^N(\bar{\alpha}^1, (\bar{\alpha}^j)_{j\ge 2}) &= \mathbb{E}\left[\int_0^T L(X_s^1, \bar{\alpha}_s^1, \pi\sharp m^{N,i}_{X_s})\, ds + G(X_T^1, \pi\sharp m^{N,i}_{X_T})\right] \\
&\le \mathbb{E}\left[\int_0^T L(\bar{X}_s^1, \bar{\alpha}_s^1, m(s))\, ds + G(\bar{X}_T^1, m(T))\right] + C\varepsilon_N \\
&\le \int_{\mathbb{T}^d} u(0,x) m_0(dx)dx + C\varepsilon_N.
\end{aligned} \tag{1.42}$$

Let now α^1 be a bounded control adapted to the filtration $(\mathcal{F}_t)$ and (Y_t^i) be the solution to

$$dY_t^1 = b(Y_t^1, \alpha_t^1, m^{N,1}_{\mathbf{Y_t}})dt + \sqrt{2}dB_t^1,\ Y_0^1 = Z^1,$$

and

$$dY_t^i = b(Y_t^i, \bar{\alpha}_t^i, m^{N,i}_{\mathbf{Y_t}})dt + \sqrt{2}dB_t^i,\ Y_0^i = Z^i.$$

We first note that we can restrict our analysis to the case where $\mathbb{E}[\int_0^T |\alpha_s^1|ds] \le A$, for A large enough. Indeed, if $\mathbb{E}[\int_0^T |\alpha_s^1|ds] > A$, we have by assumption (1.38) and inequality (1.42), as soon as A is large enough (independent of N) and N is large enough:

$$\begin{aligned}
\mathcal{J}_1^N(\alpha^1, (\bar{\alpha}^j)_{j\ge 2}) &\ge C^{-1}\mathbb{E}\left[\int_0^T |\alpha_s^1|ds\right] - CT \ge C^{-1}A - CT \\
&\ge \int_{\mathbb{T}^d} u(0,x)m_0(dx)dx + 1 \ge \mathcal{J}_1^N(\bar{\alpha}^1, (\bar{\alpha}^j)_{j\ge 2}) + 1 - C\varepsilon_N \\
&\ge \mathcal{J}_1^N(\bar{\alpha}^1, (\bar{\alpha}^j)_{j\ge 2}) - (C/2)\varepsilon_N.
\end{aligned}$$

From now on we assume that $\displaystyle\int_0^T |\alpha_s^1|ds \le A$. Let us first estimate $\mathbf{d}_1(m^{N,1}_{\mathbf{Y_s}}, m^{N,1}_{\mathbf{X_s}}))$. Note that we have, by Lipschitz continuity of b,

$$|Y_t^1 - X_t^1| \le C \int_0^t (|Y_s^1 - X_s^1| + |\alpha_s^1 - \bar{\alpha}_s^1| + \mathbf{d}_1(m^{N,1}_{\mathbf{Y_s}}, m^{N,1}_{\mathbf{X_s}}))ds$$
$$\le C \int_0^t (|Y_s^1 - X_s^1| + |\alpha_s^1 - \bar{\alpha}_s^1| + \frac{1}{N-1}\sum_{j=2}^N |Y_s^j - X_s^j|)ds$$

while for $i \in \{2, \ldots, N\}$ we have, arguing in the same way,

$$|Y_t^i - X_t^i| \le C \int_0^t (|Y_s^i - X_s^i| + \frac{1}{N-1}\sum_{j\neq i}^N |Y_s^j - X_s^j|)ds.$$

So

$$\frac{1}{N}\sum_{i=1}^N |Y_t^i - X_t^i| \le C \int_0^t (\frac{1}{N}|\alpha_s^1 - \bar{\alpha}_s^1| + \frac{1}{N}\sum_{i=1}^N |Y_s^i - X_s^i|)ds.$$

By Gronwall lemma we obtain therefore

$$\frac{1}{N}\sum_{i=1}^N |Y_t^i - X_t^i| \le \frac{C}{N}\int_0^t |\alpha_s^1 - \bar{\alpha}_s^1| ds.$$

So

$$\sup_{t\in[0,T]} \mathbf{d}_1(m^{N,1}_{\mathbf{Y_s}}, m^{N,1}_{\mathbf{X_s}})) \le \sup_{t\in[0,T]} \frac{1}{N-1}\sum_{i=2}^N |Y_t^i - X_t^i|$$
$$\le \sup_{t\in[0,T]} \frac{1}{N-1}\sum_{i=1}^N |Y_t^i - X_t^i| \le \frac{C}{N}\int_0^T |\alpha_s^1 - \bar{\alpha}_s^1| ds.$$

As $\mathbb{E}[\int_0^T |\alpha_s^1 - \bar{\alpha}_s^1| ds] \le A$, this shows that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}_1(\pi\sharp m^{N,1}_{\mathbf{Y_s}}, m(s))\right] \le \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{d}_1(m^{N,1}_{\mathbf{Y_s}}, m^{N,1}_{\mathbf{X_s}})\right] + \left[\sup_{t\in[0,T]} \mathbf{d}_1(\pi\sharp m^{N,1}_{\mathbf{X_s}}, m(s))\right]$$
$$\le \frac{CA}{N} + C\varepsilon_N \le C_A \varepsilon_N,$$

where C_A depends also on A. Therefore, using again the Lipschitz continuity of L and G with respect to m, we get

$$\begin{aligned}\mathcal{J}_1^N(\alpha^1, (\bar{\alpha}^j)_{j\neq 2}) &= \mathbb{E}\left[\int_0^T L(Y_s^1, \alpha_s^1, \pi\sharp m^{N,i}_{\mathbf{Y_s}})\, ds + G(Y_T^1, \pi\sharp m^{N,i}_{\mathbf{Y_T}})\right] \\ &\ge \mathbb{E}\left[\int_0^T L(Y_s^1, \alpha_s^1, m(s))\, ds + G(X_T^1, m(T))\right] - C_A\varepsilon_N \\ &\ge \int_{\mathbb{T}^d} u(0,x) m_0(dx) - C_A\varepsilon_N,\end{aligned}$$

where the last inequality comes from the optimality of $\bar{\alpha}$ in Lemma 7. Recalling (1.42) proves the result. □

We conclude this subsection by recalling that the use of the MFG system to obtain $\varepsilon-$*Nash equilibria* (Theorem 6) has been initiated—in a slightly different framework—in a series of papers due to Caines, Huang and MalhamÃ©: see in particular [131] (for linear dynamics) and [132] (for nonlinear dynamics). In these papers, the dependence with respect of the empirical measure of dynamics and payoff occurs through an average, so that the CTL implies that the error term is of order $N^{-1/2}$. The genuinely non linear version of the result given above is a variation on a result by

Carmona and Delarue [67] (see also [68], Section 6 in Vol. II). Many variations and extensions of these results have followed since then: we refer to [68] and the references therein.

We discuss below, in Section 1.4.4, the reverse statement: to what extent the MFG system pops up as the limit of Nash equilibria. Let us just underline at this stage that this latter problem is much more challenging.

1.3.4 The vanishing viscosity limit and the first order system with smoothing couplings.

We now analyze the vanishing viscosity limit for system (1.19) and the corresponding existence and uniqueness of solutions for the deterministic problem

$$\begin{cases} -\partial_t u + H(t,x,Du) = F[m] \\ u(T) = G[m(T)] \\ \partial_t m - \operatorname{div}(m\, H_p(t,x,Du)) = 0 \\ m(0) = m_0\,. \end{cases} \tag{1.43}$$

To this purpose we strengthen the assumptions on F, G, H. Namely, we assume that

$$\begin{aligned} F : C^0([0,T];\mathcal{P}(\mathbb{T}^d)) \to C^0(\overline{Q_T}) \quad &\text{is continuous with range into a bounded set of } L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d)) \\ &\text{and } F \text{ is a bounded map from } C^\alpha([0,T];\mathcal{P}(\mathbb{T}^d)) \text{ into } C^\alpha(\overline{Q_T}), \text{ for any } \alpha\in(0,1). \end{aligned} \tag{1.44}$$

Similarly, we assume that

$$G : \mathcal{P}(\mathbb{T}^d) \to C^0(\mathbb{T}^d) \quad \text{is continuous with range into a bounded set of } W^{2,\infty}(\mathbb{T}^d). \tag{1.45}$$

Moreover we assume that $H \in C^2(\overline{Q}_T \times \mathbb{R}^d)$ and satisfies

$$\exists\, c_0 > 0 \;: \qquad c_0^{-1}\, I_d \le H_{pp}(t,x,p) \le c_0\, I_d \qquad \forall (t,x,p) \in Q_T \times \mathbb{R}^d \tag{1.46}$$

and one between (1.23) or the following condition:

$$\exists\; c_1 > 0 \;: \qquad |H_{xx}(t,x,p)| \le c_1(1+|p|^2)\,, \qquad |H_{xp}(t,x,p)| \le c_1(1+|p|) \qquad \forall (t,x,p) \in Q_T \times \mathbb{R}^d \tag{1.47}$$

Under the above smoothing conditions on the couplings F, G, it will be possible to consider u as a viscosity solution of the Hamilton-Jacobi equation and to make use of several regularity results already known from the standard viscosity solutions' theory. Hence, the notion of solution which is the most suitable here is the following one.

Definition 3 A couple (u,m) is a solution to (1.43) if $u \in C^0(\overline{Q_T}) \cap L^\infty(0,T,W^{1,\infty}(\mathbb{T}^d))$, $m \in C^0([0,T];\mathcal{P}(\mathbb{T}^d))$, u is a viscosity solution of the Hamilton-Jacobi equation, with $u(T) = G[m(T)]$, and m is a distributional solution of the continuity equation such that $m(0) = m_0$.

Assumptions (1.44) and (1.45), together with the uniform convexity of the Hamiltonian ((1.46)), are crucial here in order to guarantee an estimate of semiconcavity for the function u. This is usually a fundamental regularity property of solutions of first order equations, but this is most relevant here because of the properties inherited by the drift term $H_p(t,x,Du)$ in the continuity equation. Let us recall the definition and some properties of semi-concavity. Proofs and references can be found, for instance, in the monograph [50].

Definition 4 A map $w : \mathbb{R}^d \to \mathbb{R}$ is semi-concave if there is some $C > 0$ such that one of the following equivalent conditions is satisfied:

1. the map $x \to w(x) - \frac{C}{2}|x|^2$ is concave in $\mathbb{R}^d$,
2. $w(\lambda x + (1-\lambda)y) \ge \lambda w(x) + (1-\lambda)w(y) - C\lambda(1-\lambda)|x-y|^2$ for any $x, y \in \mathbb{R}^d,\ \lambda \in [0,1]$,
3. $D^2 w \le C\, I_d$ in the sense of distributions,

4. $(p-q)\cdot(x-y)\le C|x-y|^2$ for any $x,y\in\mathbb{R}^d$, $p\in D^+w(x)$ and $q\in D^+w(y)$, where D^+w denotes the super-differential of w, namely

$$D^+w(x)=\left\{p\in\mathbb{R}^d\ ;\ \limsup_{y\to x}\frac{w(y)-w(x)-(p,y-x)}{|y-x|}\le 0\right\}.$$

We will use later the following main consequences of semi-concavity (see e.g. [50]).

Lemma 9 *Let $w:\mathbb{R}^d\to\mathbb{R}$ be semi-concave. Then w is locally Lipschitz continuous in $\mathbb{R}^d$, and it is differentiable at x if and only if $D^+w(x)$ is a singleton.*

Moreover $D^+w(x)$ is the closed convex hull of the set $D^*w(x)$ of reachable gradients defined by

$$D^*w(x)=\left\{p\in\mathbb{R}^d\ ,\ \exists x_n\to x \text{ such that } Dw(x_n) \text{ exists and converges to } p\right\}.$$

In particular, for any $x\in\mathbb{R}^d$, $D^+w(x)$ is a compact, convex and non empty subset of $\mathbb{R}^d$.

Finally, if (w_n) is a sequence of uniformly semi-concave maps on $\mathbb{R}^d$ which pointwisely converges to a map $w:\mathbb{R}^d\to\mathbb{R}$, then the convergence is locally uniform, $Dw_n(x)$ converges to $Dw(x)$ for a.e. $x\in\mathbb{R}^d$ and w is semi-concave. Moreover, for any $x_n\to x$ and any $p_n\in D^+w_n(x_n)$, the set of cluster points of (p_n) is contained in $D^+w(x)$.

The following theorem is given in [145], and some details also appeared in [59]. Here we give a slightly more general version of the result.

Theorem 7 *Let $m_0\in L^\infty(\mathbb{T}^d)$. Assume (1.44)-(1.46) and that at least one between the conditions (1.23) and (1.47) holds true. Let $(u^\varepsilon,m^\varepsilon)$ be a solution of (1.19). Then there exists a subsequence, not relabeled, and a couple $(u,m)\in W^{1,\infty}(Q_T)\times L^\infty(Q_T)$ such that*

$$u^\varepsilon\to u \qquad \text{in } C(\overline{Q_T}),\qquad m^\varepsilon\to m\qquad \text{in } L^\infty(Q_T)-\text{ weak}^*,$$

and (u,m) is a solution of (1.43) in the sense of Definition 3.

Proof.

Step 1. (bounds for $u^\varepsilon,m^\varepsilon$) Let us recall that, by Theorem 4, u^ε and m^ε are classical solutions in $(0,T)$. First of all, by maximum principle and (1.44)-(1.45), it follows that u^ε is uniformly bounded in Q_T. The next key point consists in proving a semi concavity estimate for u^ε. To this purpose, let ξ be a direction in $\mathbb{R}^d$. We drop the index ε for simplicity and we set $u_{\xi\xi}(t,x)=D^2u(t,x)\xi\cdot\xi$. Then we look at the equation satisfied by $w:=u_{\xi\xi}+\lambda(u+M)^2$ where λ,M are positive constants to be fixed later. Straightforward computations give the following:

$$\begin{aligned}-\partial_t w-\varepsilon\Delta w+H_p(t,x,Du)\cdot Dw+H_{\xi\xi}(t,x,Du)+2H_{\xi p}(t,x,Du)\cdot Du_\xi+H_{pp}(t,x,Du)Du_\xi\cdot Du_\xi\\ =(F[m])_{\xi\xi}-2\lambda(u+M)\left(H(t,x,Du)-F[m]\right)-2\lambda\varepsilon|Du|^2\,.\end{aligned}$$

We choose $M=\|u\|_\infty+1$, and we use the coercivity of H which satisfies, from (1.46), $H(t,x,p)\ge\frac12 c_0^{-1}|p|^2-c$ for some constant $c>0$. Therefore we estimate

$$\begin{aligned}-\partial_t w-\varepsilon\Delta w+&H_p(t,x,Du)\cdot Dw+H_{\xi\xi}(t,x,Du)+2H_{\xi p}(t,x,Du)\cdot Du_\xi+H_{pp}(t,x,Du)Du_\xi\cdot Du_\xi\\ &\le (F[m])_{\xi\xi}-\lambda\, c_0^{-1}|Du|^2+c\,\lambda\,(1+\|u\|_\infty)(1+\|F[m]\|_\infty)\,.\end{aligned}$$

Now we estimate the terms with the second derivatives of H, using condition (1.46) and one between (1.23) and (1.47). To this purpose, we notice that, if (1.23) holds true, then we already know that Du^ε is uniformly bounded in Q_T, see (1.27) in Theorem 4. Then the bounds assumed in (1.47) come for free because H is a C^2 function and the arguments (t,x,Du) live in compact sets. Therefore, we can proceed using (1.47) in both cases. Thanks to Young's inequality, we estimate

$$H_{\xi\xi}(t,x,Du)+2H_{\xi p}(t,x,Du)\cdot Du_\xi+H_{pp}(t,x,Du)Du_\xi\cdot Du_\xi\ge\frac12 c_0^{-1}|Du_\xi|^2-c(1+|Du|^2)$$

hence we deduce that

$$-\partial_t w - \varepsilon\Delta w + H_p(t,x,Du)\cdot Dw + \frac{1}{2}c_0^{-1}|Du_\xi|^2 \le c(1+|Du|^2) + \|D^2_{xx}F[m]\|_\infty - \lambda\, c_0^{-1}|Du|^2 + c\,\lambda\,(1+\|F[m]\|_\infty)$$

where we used that $\|u\|_\infty$ is bounded and we denote by c any generic constant independent of ε. The terms given by $F[m]$ are uniformly bounded due to (1.44). Thus, by choosing λ sufficiently large we deduce that, at an internal maximum point (t,x) of w, we have $|Du_\xi|^2 \le C$ for a constant C independent of ε. Since $|Du_\xi| \ge |u_{\xi\xi}| \ge |w| - c\|u\|_\infty^2$, this gives an upper bound at the maximum point of $w(t,x)$, whenever it is attained for $t < T$. By the way, if the maximum of w is reached at T, then $\max w \le \|G[m]\|_{W^{2,\infty}} + c\,\|u\|_\infty^2$. We conclude an estimate for $\max w$, and therefore an upper bound for $u_{\xi\xi}$. The bound being independent of ξ, we have obtained so far that

$$D^2 u^\varepsilon(t,x) \le C \qquad \forall (t,x) \in Q_T$$

for a constant C independent of ε. Since u^ε is $\mathbb{Z}^d$-periodic, this also implies a uniform bound for $\|Du^\varepsilon\|_\infty$.

At this stage, let us observe that the above estimate has been obtained as if $u_{\xi\xi}$ was a smooth function, but this is a minor point: indeed, since $u \in C^{1,2}(Q_T)$ and $H \in C^2$, we have $w \in C^0(Q_T)$ and the above computation shows that $-\partial_t w - \varepsilon\Delta w + H_p(x,Du)\cdot Dw$ is itself a continuous function; so the estimate for w follows applying the maximum principle for continuous solutions.

Now we easily deduce an upper bound on m^ε as well. Indeed, m^ε satisfies, for some constant $K > 0$

$$\partial_t m - \varepsilon\Delta m - Dm\cdot H_p(t,x,Du) = m\,\mathrm{Tr}\left(H_{pp}(t,x,Du)D^2u\right) \le K\,m$$

thanks to the semi concavity estimate and the upper bound of H_{pp} given in (1.46). We deduce that m^ε satisfies

$$\|m^\varepsilon\|_\infty \le e^{Kt}\,\|m_0\|_\infty\,. \tag{1.48}$$

Step 2. (compactness) From the previous step we know that Du^ε is uniformly bounded. Lemma 6 then implies that the map $t \to m^\varepsilon(t)$ is Hölder continuous in $\mathcal{P}(\mathbb{T}^d)$, uniformly in ε. This implies that m^ε is relatively compact in $C^0([0,T],\mathcal{P}(\mathbb{T}^d))$. Moreover, from the uniform bound (1.48), m^ε is also relatively compact in the weak$-*$ topology of $L^\infty(Q_T)$. Therefore, up to subsequences m^ε converges in $L^\infty-$weak$*$ and in $C^0([0,T],P(\mathbb{T}^d))$ towards some $m \in L^\infty(Q_T)\cap C^0([0,T],P(\mathbb{T}^d))$. In particular, $m(0) = m_0$. In order to pass to the limit in the equation of m^ε, we observe that the uniform semi concavity bound implies that, up to subsequences, $H_p(t,x,Du^\varepsilon)$ almost everywhere converges towards $H_p(t,x,Du)$ (see Lemma 9), and then it converges strongly in $L^1(Q_T)$ by Lebesgue theorem. This allows us to pass to the limit in the product $m^\varepsilon H_p(t,x,Du^\varepsilon)$ and deduce that m is a distributional solution of the continuity equation.

We conclude now with the compactness of u^ε. Since Du^ε is bounded, we only need to check the uniform continuity of u^ε in time, which is done with a standard time-translation argument. First we observe that, as $G[m^\varepsilon(T)] \in W^{2,\infty}(\mathbb{T}^d)$, the maps $w^+(x,t) = G[m^\varepsilon(T)] + C_1(T-t)$ and $w^-(x,t) = G[m^\varepsilon(T)] - C_1(T-t)$ are, respectively, a super and sub solution of the equation of u^ε, for C_1 sufficiently large (but not depending of ε). Hence, by comparison principle

$$\|u^\varepsilon(t) - G[m^\varepsilon(T)]\|_\infty \le C_1(T-t)\,. \tag{1.49}$$

For $h > 0$, we consider $u^\varepsilon_h(x,t) = u^\varepsilon(x,t-h)$ in (h,T), which satisfies

$$-\partial_t u^\varepsilon_h - \varepsilon\Delta u^\varepsilon_h + H(t-h,x,Du^\varepsilon_h) = F[m^\varepsilon](t-h)\,.$$

Because of the uniform Hölder regularity of the map $t \to m^\varepsilon(t)$ in $P(\mathbb{T}^d)$ and the assumption (1.44) (with $\alpha = \frac{1}{2}$), we have

$$\sup_{t\in[h,T]} \|F[m^\varepsilon](t-h) - F[m^\varepsilon](t)\|_\infty \le C\sqrt{h}$$

and since H is locally Lipschitz and Du^ε is uniformly bounded we also have

$$\sup_{t\in[h,T]} |H(t-h,x,Du_h^\varepsilon) - H(t,x,Du_h^\varepsilon)| \le C\,h\,.$$

For the terminal condition we also have, using (1.49),

$$\|u_h^\varepsilon(T) - u^\varepsilon(T)\|_\infty = \|u^\varepsilon(T-h) - G[m^\varepsilon(T)]\|_\infty \le C_1\,h\,.$$

By the L^∞ stability (say, the comparison principle) we deduce that

$$\|u^\varepsilon(t-h) - u^\varepsilon(t)\|_\infty \le C(T-t)\,\sqrt{h} + C_1 h.$$

This proves the equi-continuity of u^ε in time, and so we conclude that u^ε is relatively compact in $C^0(\overline{Q_T})$.

By continuity assumptions on F and G, we know that $G[m^\varepsilon(T)]$ converges to $G[m(T)]$ in $C^0(\mathbb{T}^d)$, and $F[m^\varepsilon]$ converges to $F[m]$ in $C^0(\overline{Q_T})$. It is now possible to apply the classical stability results for viscosity solutions and we deduce, as $\varepsilon \to 0$, that u is a viscosity solution of the HJ equation $-\partial_t u + H(t,x,Du) = F[m]$, with $u(T) = G[m(T)]$. Note that, as $H(t,x,Du)$ and $F[m]$ are bounded, by standard results in viscosity solutions' theory it turns out that u is Lipschitz continuous in time as well. □

Let us now turn to the question of uniqueness of solutions to (1.43). On one hand, the uniqueness will again rely on the monotonicity argument introduced by Lasry and Lions; on another hand, the new difficulty lies in showing the uniqueness of m from the continuity equation; this step is highly non trivial and we will detail it later.

Theorem 8 *Let $m_0 \in L^\infty(\mathbb{T}^d)$, and let H and F satisfy the conditions of Theorem 7. In addition, assume that F, G are monotone (nondecreasing) operators, i.e.*

$$\int_0^T\int_{\mathbb{T}^d} (F[m_1] - F[m_2])d(m_1 - m_2) \ge 0 \qquad \forall m_1, m_2 \in C^0([0,T], \mathcal{P}(\mathbb{T}^d))$$

and

$$\int_{\mathbb{T}^d} (G[m_1] - G[m_2])d(m_1 - m_2) \ge 0 \qquad \forall m_1, m_2 \in \mathcal{P}(\mathbb{T}^d)\,.$$

Then (1.43) *admits a unique solution (in the sense of Definition 3) (u,m) such that $m \in L^\infty(Q_T)$.*

Proof. We first observe that the Lasry-Lions monotonicity argument works perfectly in the setting of solutions given above. Indeed, let (u^1, m^1) and (u^2, m^2) be two solutions of (1.43) in the sense of Definition 3, with the additional property that $m^1, m^2 \in L^\infty(Q_T)$. We recall that for $m = m^i$, we have the weak formulation

$$\int_0^T \int_{\mathbb{T}^d} (-m\,\partial_t\varphi + m\,H_p(t,x,Du)\cdot D\varphi) = 0 \qquad \forall \varphi \in C_c^1((0,T)\times\mathbb{T}^d)\,. \tag{1.50}$$

Since $m \in L^\infty(Q_T)$, by an approximation argument it is easy to extend this formulation to hold for every $\varphi \in W^{1,\infty}(Q_T)$ with compact support in $(0,T)$. We recall here that $u = u^i$ belongs to $L^\infty(0,T,W^{1,\infty}(\mathbb{T}))$ by definition and then, by properties of viscosity solutions, it is also Lipschitz in time. Hence $u \in W^{1,\infty}(Q_T)$. In particular, u is almost everywhere differentiable in Q_T and, by definition of viscosity solutions, it satisfies

$$-\,\partial_t u + H(t,x,Du) = F[m] \qquad \text{a.e. in } Q_T. \tag{1.51}$$

Let here $\xi = \xi(t)$ be a function in $W^{1,\infty}(0,T)$ with compact support. Using (1.50) with $m = m^i$ and $\varphi = u^j\,\xi$ and (1.51) for u^j, $i,j = 1,2$, we obtain the usual equality (1.28) in the weak form

$$\int_0^T\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2)\partial_t\xi = \int_0^T\int_{\mathbb{T}^d}\xi\,(F[m_1]-F[m_2])(m_1-m_2)$$
$$+\int_0^T\int_{\mathbb{T}^d}\xi\, m_1\left\{H(t,x,Du_2)-H(t,x,Du_1)-H_p(t,x,Du_1)(Du_2-Du_1)\right\}$$
$$+\int_0^T\int_{\mathbb{T}^d}\xi\, m_2\left\{H(t,x,Du_1)-H(t,x,Du_2)-H_p(t,x,Du_2)(Du_1-Du_2)\right\}\,. \tag{1.52}$$

Now we take $\xi=\xi_\varepsilon(t)$ such that ξ is supported in $(\varepsilon,T-\varepsilon)$, $\xi\equiv 1$ for $t\in(2\varepsilon,T-2\varepsilon)$ and ξ is linear in $(\varepsilon,2\varepsilon)$ and in $(T-2\varepsilon,T-\varepsilon)$. Of course we have $\xi_\varepsilon\to 1$ and all integrals in the right-hand side of (1.52) converge by Lebesgue theorem. The boundary layers terms give

$$\int_0^T\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2)\partial_t\xi_\varepsilon=\frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon}\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2)-\frac{1}{\varepsilon}\int_{T-2\varepsilon}^{T-\varepsilon}\int_{\mathbb{T}^d}(u_1-u_2)(m_1-m_2)$$

where we can pass to the limit because $m^i\in C^0([0,T],\mathcal{P}(\mathbb{T}^d))$ and $u^i\in C(\overline{Q_T})$. Therefore letting $\varepsilon\to 0$ in (1.52), and using the same initial condition for m^1,m^2 we conclude that

$$\int (G[m_1(T)]-G[m_2(T)])\,d(m_1(T)-m_2(T))+\int_0^T\int_{\mathbb{T}^d}(F[m_1]-F[m_2])(m_1-m_2)$$
$$+\int_0^T\int_{\mathbb{T}^d} m_1\left\{H(t,x,Du_2)-H(t,x,Du_1)-H_p(t,x,Du_1)(Du_2-Du_1)\right\}$$
$$+\int_0^T\int_{\mathbb{T}^d} m_2\left\{H(t,x,Du_1)-H(t,x,Du_2)-H_p(t,x,Du_2)(Du_1-Du_2)\right\}=0\,.$$

Thanks to the monotonicity condition on F,G, and to the strict convexity of H, given by (1.46), this implies that $Du^1=Du^2$ a.e. in $\{m^1>0\}\cup\{m^2>0\}$. In particular, m^1 and m^2 solve the same Kolmogorov equation: m^1 and m^2 are both solutions to

$$\partial_t m-\operatorname{div}(mD_pH(t,x,Du^1(t,x)))=0,\qquad m(0)=m_0.$$

We admit for a while the (difficult) fact that this entails the equality $m^1=m^2$ (see Lemma 10 below). Then u^1 and u^2 are two viscosity solutions of the same equation with the same terminal condition; by comparison, they are therefore equal. □

In order to complete the above proof, we are left with the main task, which is the content of the following result.

Lemma 10 *Assume that $u\in C(\overline{Q_T})$ is a viscosity solution to*

$$-\partial_t u+H(t,x,Du)=F(t,x),\qquad u(T,x)=u_T(x), \tag{1.53}$$

where $H:[0,T]\times\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{R}$ satisfies the conditions of Theorem 7 and $F\in C(\overline{Q_T})\cap L^\infty(0,T;W^{2,\infty}(\mathbb{T}^d))$, $u_T\in W^{2,\infty}(\mathbb{T}^d)$.

Then, for any $m_0\in L^\infty(\mathbb{T}^d)$, the transport equation

$$\partial_t m-\operatorname{div}\left(mH_p(t,x,Du)\right)=0,\qquad m(0,x)=m_0(x) \tag{1.54}$$

possesses at most one weak solution in L^∞.

The proof of the Lemma is delicate and is the aim of the rest of the section. The difficulty comes from the fact that the vector field $H_p(t,x,Du)$ is not smooth: it is actually discontinuous in general. The analysis of transport equations with non smooth vector fields has attracted a lot of attention since the Di Perna-Lions seminal paper [98]. We rely here on Ambrosio's approach [16, 17], in particular for the "superposition principle" (see Theorem 9 below). A key point will be played by the semi concavity property of u. In particular, this implies that $H_p(t,x,Du)$ has bounded variation; nevertheless, this does not seem to be enough to apply previous results on the continuity equation, where the vector field is usually supposed to have a non singular divergence. We will overcome this problem by using the optimal control representation of $H_p(t,x,Du)$ and the related properties of the characteristic curves.

Let us first point out some basic properties of the solution u of (1.53). Henceforth, for simplicity (and without loss of generality) we assume that $F=0$ in (1.53), which is always possible up to defining a new Hamiltonian $\tilde{H}=H-F$.

We already know that u is unique and Lipschitz continuous, and it is obtained by viscous approximation. Therefore, one can check (exactly as in Theorem 7) that u is semiconcave in space for any t, with a modulus bounded independently of t. Moreover, we will extensively use the fact that u can be represented as the value function of a problem of calculus of variations:

$$u(t,x)=\inf_{\gamma:\ \gamma(t)=x}\int_t^T L(s,\gamma(s),\dot\gamma(s))ds+u_T(\gamma(T)) \tag{1.55}$$

where $L(t,x,\xi)=\sup_{p\in\mathbb{R}^d}[-\xi\cdot p-H(t,x,p)]$ and where $\gamma\in AC([0,T];\mathbb{T}^d)$ are absolutely continuous curves in $[0,T]$. For $(t,x)\in[0,T)\times\mathbb{T}^d$ we denote by $\mathcal{A}(t,x)$ the set of optimal trajectories for the control problem (1.55). One easily checks that, under the above assumptions on H, such set is nonempty, and that, if $(t_n,x_n)\to(t,x)$ and $\gamma_n\in\mathcal{A}(t_n,x_n)$, then, up to some subsequence, γ_n weakly converges in H^1 to some $\gamma\in\mathcal{A}(t,x)$.

We need to analyze precisely the connection between the differentiability of u with respect to the x variable and the uniqueness of the minimizer in (1.55). The following properties are well-known in the theory of optimal control and Hamilton-Jacobi equations (see e.g. [50, Chapter 6]), but we will give the proofs for the reader's convenience.

Lemma 11 (Regularity of u along optimal solutions) *Let $(t,x)\in[0,T]\times\mathbb{T}^d$ and $\gamma\in\mathcal{A}(t,x)$. Then*

1. *(Uniqueness of the optimal control along optimal trajectories) for any $s\in(t,T]$, the restriction of γ to $[s,T]$ is the unique element of $\mathcal{A}(s,\gamma(s))$.*
2. *(Uniqueness of the optimal trajectories) $Du(t,x)$ exists if and only if $\mathcal{A}(t,x)$ is reduced to a singleton. In this case, $\dot\gamma(t)=-H_p(t,x,Du(t,x))$ where $\mathcal{A}(t,x)=\{\gamma\}$.*

Remark 7 In particular, if we combine the above statements, we see that, for any $\gamma\in\mathcal{A}(t,x)$, $u(s,\cdot)$ is always differentiable at $\gamma(s)$ for $s\in(t,T)$, with $\dot\gamma(s)=-H_p(s,\gamma(s),Du(s,\gamma(s)))$.

Proof. We recall that, since H is C^2 and strictly convex in p, then L is also C^2 and strictly convex in ξ, which ensures the regularity of the minimizers. So if $\gamma\in\mathcal{A}(t,x)$, then γ is of class $\mathcal{C}^2$ on $[t,T]$ and satisfies the Euler-Lagrange equation

$$\frac{d}{dt}\,L_\xi(s,\gamma(s),\dot\gamma(s))=L_x(s,\gamma(s),\dot\gamma(s)) \qquad \forall s\in[t,T] \tag{1.56}$$

with the trasversality condition

$$Du_T(\gamma(T))=-L_\xi(T,\gamma(T),\dot\gamma(T)). \tag{1.57}$$

Let $\gamma_1\in\mathcal{A}(s,\gamma(s))$. For any $h>0$ small we build some $\gamma_h\in\mathcal{A}(t,x)$ in the following way:

$$\gamma_h(\tau)=\begin{cases}\gamma(\tau) & \text{if } \tau\in[t,s-h)\\ \gamma(s-h)+(\tau-(s-h))\dfrac{\gamma_1(s+h)-\gamma(s-h)}{2h} & \text{if } \tau\in[s-h,s+h)\\ \gamma_1(\tau) & \text{if } \tau\in[s+h,T]\end{cases}$$

Since $\gamma_{|_{[s,T]}}$ and γ_1 are optimal for $u(s,\gamma(s))$, the concatenation γ_0 of $\gamma_{|_{[t,s]}}$ and γ_1 is also optimal for $u(t,x)$. So, comparing the payoff for γ_0 (which is optimal) and the payoff for γ_h we have

$$\int_{s-h}^{s}L(\tau,\gamma(\tau),\dot\gamma(\tau))d\tau+\int_s^{s+h}L(\tau,\gamma_1(\tau),\dot\gamma_1(\tau))d\tau-\int_{s-h}^{s+h}L(\tau,\gamma_h(\tau),\frac{\gamma_1(s+h)-\gamma(s-h)}{2h})d\tau\le 0\ .$$

We divide this inequality by h and let $h\to 0^+$ to get

$$L(s,\gamma(s),\dot\gamma(s))+L(s,\gamma(s),\dot\gamma_1(s))-2L(s,\gamma(s),\frac{1}{2}(\dot\gamma(s)+\dot\gamma_1(s)))\leq 0$$

since $\lim_{h\to 0,\ s\in[s-h,s+h]}\gamma_h(s)=\gamma(s)=\gamma_1(s)$. By strict convexity of L with respect to the last variable, we conclude that $\dot\gamma(s)=\dot\gamma_1(s)$. Since we also have $\gamma(s)=\gamma_1(s)$, and since both $\gamma(\cdot)$ and $\gamma_1(\cdot)$ satisfy on the time interval $[s,T]$ the second order equation (1.56), we conclude that $\gamma(\tau)=\gamma_1(\tau)$ on $[s,T]$. This means that the optimal solution for $u(s,\gamma(s))$ is unique.

Next we show that, if $Du(t,x)$ exists, then $\mathcal{A}(t,x)$ is a reduced to a singleton and $\dot\gamma(t)=-H_p(t,x,Du(t,x))$ where $\mathcal{A}(t,x)=\{\gamma\}$. Indeed, let $\gamma\in\mathcal{A}(t,x)$. Then, for any $v\in\mathbb{R}^d$,

$$u(t,x+v)\leq\int_t^T L(s,\gamma(s)+v,\dot\gamma(s))ds+u_T(\gamma(T)+v)\ .$$

Since equality holds for $v=0$ and since left- and right-hand sides are differentiable with respect to v at $v=0$ we get by (1.56)-(1.57):

$$\begin{aligned} Du(t,x)\ &=\int_t^T L_x(s,\gamma(s),\dot\gamma(s))ds+Du_T(\gamma(T))\\ &=\int_t^T \frac{d}{dt}\,L_\xi(s,\gamma(s),\dot\gamma(s))+Du_T(\gamma(T))\ =-L_\xi(t,x,\dot\gamma(t))\ . \end{aligned}$$

By definition of L, this means that $\dot\gamma(t)=-H_p(t,x,Du(t,x))$ and therefore $\gamma(\cdot)$ is the unique solution of the Euler-Lagrange equation with initial conditions $\gamma(t)=x$ and $\dot\gamma(t)=-H_p(t,x,Du(t,x))$. This shows the claim.

Conversely, let us prove that, if $\mathcal{A}(t,x)$ is a singleton, then $u(t,\cdot)$ is differentiable at x. For this we note that, if p belongs to $D^*u(t,x)$ (the set of reachable gradients of the map $u(t,\cdot)$), then the solution to (1.56), with initial conditions $\gamma(t)=x$, $\dot\gamma(t)=-H_p(t,x,p)$, is optimal. Indeed, by definition of p, there is a sequence $x_n\to x$ such that $u(t,\cdot)$ is differentiable at x_n and $Du(t,x_n)\to p$. Now, since $u(t,\cdot)$ is differentiable at x_n, we know by what proved before that the unique solution $\gamma_n(\cdot)$ to (1.56) with initial conditions $\gamma_n(t)=x_n$, $\dot\gamma_n(t)=-H_p(t,x_n,Du(t,x_n))$, is optimal. Passing to the limit as $n\to+\infty$ implies (by the stability of optimal trajectories), that $\gamma(\cdot)$, which is the uniform limit of the $\gamma_n(\cdot)$, is also optimal.

Now, from our assumptions, there is a unique optimal curve in $\mathcal{A}(t,x)$. Therefore $D^*u(t,x)$ has to be reduced to a singleton, which implies, since $u(t,\cdot)$ is semi-concave, that $u(t,\cdot)$ is differentiable at x (Lemma 9). □

We now turn the attention to the solutions of the differential equation

$$\begin{cases}\dot\gamma(s)=-H_p(s,\gamma(s),Du(s,\gamma(s))) & \text{a.e. in } [t,T]\\ \gamma(t)=x\ . \end{cases}\tag{1.58}$$

Here we fix a Borel representative of $Du(t,x)$ (e.g. a measurable selection of $D^+u(t,x)$), so that the vector field $H_p(t,x,Du(t,x))$ is defined everywhere in Q_T. In what follows, we say that γ is a solution to (1.58) if $\gamma\in AC([0,T];\mathbb{T}^d)$, if $u(s,\cdot)$ is differentiable at $\gamma(s)$ for a.e. $s\in(t,T)$ and if

$$\gamma(s)=x-\int_t^s H_p(\tau,\gamma(\tau),Du(\tau,\gamma(\tau)))d\tau\qquad\forall s\in[t,T]\,.$$

We already know (see Remark 7) that, if $\gamma\in\mathcal{A}(t,x)$, then γ is a solution to (1.58); now we show that the converse is also true.

Lemma 12 (Optimal synthesis) *Let $(t,x)\in[0,T)\times\mathbb{T}^d$ and $\gamma(\cdot)$ be a solution to (1.58). Then the trajectory γ is optimal for $u(t,x)$.*

In particular, if $u(t,\cdot)$ is differentiable at x, then equation (1.58) has a unique solution, corresponding to the optimal trajectory.

Proof. We first note that $\gamma(\cdot)$ is Lipschitz continuous because so is u. Let $s\in(t,T)$ be such that equation (1.58) holds (in particular $u(s,\cdot)$ is differentiable at $\gamma(s)$) and the Lipschitz continuous map $s\to u(s,\gamma(s))$ has a derivative at s.

Since u is Lipschitz continuous, Lebourg's mean value Theorem [94, Th. 2.3.7] states that, for any $h > 0$ small, there is some $(s_h, y_h) \in [(s, \gamma(s)), (s+h, \gamma(s+h))]$ and some $(\xi_t^h, \xi_x^h) \in \mathrm{Co}D^*_{t,x}u(s_h, y_h)$ with

$$u(s+h, \gamma(s+h)) - u(s, \gamma(s)) = \xi_t^h h + \xi_x^h \cdot (\gamma(s+h) - \gamma(s)) \ , \tag{1.59}$$

(where $\mathrm{Co}D^*_{t,x}u(s,y)$ stands for the closure of the convex hull of the set of reachable gradients $D^*_{t,x}u(s,y)$). From CarathÃ©odory Theorem, there are $(\lambda^{h,i}, \xi_t^{h,i}, \xi_x^{h,i})_{i=1,\ldots,d+2}$ such that $\lambda^{h,i} \geq 0$, $\sum_i \lambda^{h,i} = 1$, with $(\xi_t^{h,i}, \xi_x^{h,i}) \in D^*_{t,x}u(s_h, y_h)$ and

$$(\xi_t^h, \xi_x^h) = \sum_i \lambda^{h,i}(\xi_t^{h,i}, \xi_x^{h,i}) \ .$$

Note that the $\xi_x^{h,i}$ converge to $Du(s, \gamma(s))$ as $h \to 0$ because, from Lemma 9, any cluster point of the $\xi_x^{h,i}$ must belong to $D^+u(s, \gamma(s))$, which is reduced to $Du(s, \gamma(s))$ since $u(s, \cdot)$ is differentiable at $\gamma(s)$. In particular, $\xi_x^h = \sum_i \lambda^{h,i}\xi_x^{h,i}$ converges to $Du(s, \gamma(s))$ as $h \to 0$.

Since u is a viscosity solution of (1.53) and $(\xi_t^{h,i}, \xi_x^{h,i}) \in D^*_{t,x}u(s_h, y_h)$, we have

$$-\xi_t^{h,i} + H(s_h, y_h, \xi_x^{h,i}) = 0 \ .$$

Therefore $\xi_t^h = \sum_i \lambda^{h,i}\xi_t^{h,i} = \sum_i \lambda^{h,i} H(s_h, y_h, \xi_x^{h,i})$ converges to $H(s, \gamma(s), Du(s, \gamma(s))$ as $h \to 0$.

Then, dividing (1.59) by h and letting $h \to 0^+$ we get

$$\frac{d}{ds}u(s, \gamma(s)) = H(s, \gamma(s), Du(s, \gamma(s)) + Du(s, \gamma(s)) \cdot \dot{\gamma}(s) \ .$$

Since $\dot{\gamma}(s) = -H_p(s, \gamma(s), Du(s, \gamma(s)))$, this implies that

$$\frac{d}{ds}u(s, \gamma(s)) = -L(s, \gamma(s), \dot{\gamma}(s)) \text{ a.e. in } (t, T) \ .$$

Integrating the above inequality over $[t, T]$ we finally obtain, since $u(T, y) = u_T(y)$,

$$u(x, t) = \int_t^T L(s, \gamma(s), \dot{\gamma}(s))\, ds + u_T(\gamma(T)) \ .$$

which means that γ is optimal. The last statement of the Lemma is a direct consequence of Lemma 11-(2). $\square$

The next step is a key result by Ambrosio (the so-called superposition principle) on the probabilistic representation of weak solutions to the continuity equation

$$\partial_t \mu + \mathrm{div}(\mu b(t, x)) = 0 \,. \tag{1.60}$$

For this let us define for any $t \in [0, T]$ the map $e_t : C^0([0, T], \mathbb{T}^d) \to \mathbb{T}^d$ by $e_t(\gamma) = \gamma(t)$ for $\gamma \in C^0([0, T], \mathbb{T}^d)$.

Theorem 9 ([16]) *Let $b : [0, T] \times \mathbb{T}^d \to \mathbb{R}^d$ be a given Borel vector field and μ be a solution to* (1.60) *such that $\int_{Q_T} |b|^2 \, d\mu < \infty$. Then there exists a Borel probability measure η on $C^0([0, T], \mathbb{T}^d)$ such that $\mu(t) = e_t \sharp \eta$ for any t and, for $\eta-$a.e. $\gamma \in C^0([0, T], \mathbb{T}^d)$, γ is a solution to the ODE*

$$\begin{cases} \dot{\gamma}(s) = b(s, \gamma(s)) & \text{a.e. in } [0, T] \\ \gamma(0) = x \,. \end{cases} \tag{1.61}$$

We will also need the notion of disintegration of a measure and the following well-known disintegration theorem, see for instance [19, Thm 5.3.1].

Theorem 10 *Let X and Y be two Polish spaces and λ be a Borel probability measure on $X \times Y$. Let us set $\mu = \pi_X \sharp \lambda$, where π_X is the standard projection from $X \times Y$ onto X. Then there exists a μ-almost everywhere uniquely determined family of Borel probability measures (λ_x) on Y such that*

1. *the function $x \mapsto \lambda_x$ is Borel measurable, in the sense that $x \mapsto \lambda_x(B)$ is a Borel-measurable function for each Borel-measurable set $B \subset Y$,*
2. *for every Borel-measurable function $f : X \times Y \to [0, +\infty]$,*

$$\int_{X\times Y} f(x,y) d\lambda(x,y) \;=\; \int_X \int_Y f(x,y)\, \mathrm{d}\lambda_x(y) \mathrm{d}\mu(x).$$

We are finally ready to prove the uniqueness result:

Proof of Lemma 10. Let m be a solution of the transport equation (1.54). We set $\Gamma := C^0([0,T], \mathbb{T}^d)$. From Ambrosio superposition principle, there exists a Borel probability measure η on Γ such that $m(t) = e_t \sharp \eta$ for any t and, for $\eta-$a.e. $\gamma \in \Gamma$, γ is a solution to the ODE $\dot\gamma = -H_p(t, \gamma(t), Du(t, \gamma(t)))$. We notice that, since $m \in L^1(Q_T)$, for any subset $E \subset Q_T$ of zero measure we have

$$\int_0^T \int_\Gamma 1_{\{\gamma(t)\in E\}} d\eta = \int_0^T \int_{\mathbb{T}^d} 1_E \, dm_t = 0$$

which means that $\gamma(t) \in E^c$ for a.e. $t \in (0, T)$ and $\eta-$a.e. $\gamma \in \Gamma$. In particular, since u is a.e. differentiable, this implies that $u(t, \cdot)$ is differentiable at $\gamma(t)$ for a.e. $t \in (0, T)$ and $\eta-$a.e. $\gamma \in \Gamma$. As $m_0 = e_0 \sharp \eta$, we can disintegrate the measure η into $\eta = \int_{\mathbb{T}^d} \eta_x dm_0(x)$, where $\gamma(0) = x$ for η_x-a.e. γ and m_0-a.e. $x \in \mathbb{T}^d$. Therefore, since m_0 is absolutely continuous, for m_0-a.e. $x \in \mathbb{T}^d$, η_x-a.e. map γ is a solution to the ODE starting from x. By Lemma 12 we know that such a solution γ is optimal for the calculus of variation problem (1.55). As, moreover, for a.e. $x \in \mathbb{T}^d$ the solution of this problem is reduced to a singleton $\{\bar\gamma_x\}$, we can conclude that $d\eta_x(\gamma) = \delta_{\bar\gamma_x}$ for m_0-a.e. $x \in \mathbb{T}^d$. Hence, for any continuous map $\phi : \mathbb{T}^d \to \mathbb{R}$, one has

$$\int_{\mathbb{T}^d} \phi(x) m(t,x)) dx = \int_{\mathbb{T}^d} \phi(\bar\gamma_x(t)) m_0(x) dx$$

which defines m uniquely. □

1.3.5 Second order MFG system with local couplings

We now consider the case that the coupling functions F, G depend on the local density of the measure $m(t, x)$. Thus we assume that $F \in C^0(\overline{Q}_T \times \mathbb{R})$ and $G \in C^0(\mathbb{T}^d \times \mathbb{R})$ and we consider the system

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(t,x,Du) = F(t,x,m(t,x)) \\ u(T) = G(x, m(T,x)) \\ \partial_t m - \varepsilon\Delta m - \mathrm{div}(m\, H_p(t,x,Du)) = 0 \\ m(0) = m_0\,. \end{cases} \tag{1.62}$$

We assume that both nonlinearities are bounded below:

$$\exists\, c_0 \in \mathbb{R} : \quad F(t,x,m) \geq c_0\,, \qquad G(x,m) \geq c_0 \quad \forall (t,x,m) \in \overline{Q}_T \times \mathbb{R}_+ \tag{1.63}$$

where $\mathbb{R}_+ = [0, \infty)$. We observe that F, G could be allowed to be measurable with respect to t and x, and bounded when the real variable m lies in compact sets. However, we simplify here the presentation by assuming continuity with respect to all variables.

Existence and uniqueness of solutions

For local couplings, there are typically two cases where the existence of solutions can be readily proved, namely whenever F, G are bounded, or $H(t,x,p)$ is globally Lipschitz in p. We give a sample result in this latter case. We warn the reader that, in the study of system (1.62) with local couplings, the notion of solution to be used may strongly depend on the regularity of (u,m) which is available. As a general framework, both equations will be understood in distributional sense, and a basic notion of weak solution will be discussed later.

In this first result that we give, assuming m_0 and $H_p(t,x,p)$ to be bounded, the function m turns out to be globally bounded and regular for $t>0$. Then (u,m) is a solution of (1.62) in the sense that $m \in L^\infty(Q_T) \cap L^2(0,T;H^1(\mathbb{T}^d))$ is a weak solution of the Fokker-Planck equation, with $m \in C^0([0,T],L^1(\mathbb{T}^d))$ and $m(0)=m_0$, whereas $u \in C(\overline{Q_T}) \cap L^2(0,T;H^1(\mathbb{T}^d))$, with $u(T)=G(x,m(T))$, and is a weak solution of the first equation.

Theorem 11 *Let $m_0 \in L^\infty(\mathbb{T}^d)$, $m_0 \geq 0$ with $\int_{\mathbb{T}^d} m_0 = 1$. Assume that $H(t,x,p)$ is a Carathéodory function such that H is convex and differentiable with respect to p and satisfies*

$$\exists\, \beta > 0 : \quad |H_p(t,x,p)| \leq \beta \qquad \forall (t,x,p) \in Q_T \times \mathbb{R}^d\,. \tag{1.64}$$

Then there exists a solution (u,m) to (1.62) with $Du, m \in C^\alpha(Q_T)$ for some $\alpha \in (0,1)$. If $F(t,x,m)$ is a locally Hölder continuous function and $H(t,x,p), H_p(t,x,p)$ are of class C^1, then (u,m) is a classical solution in $(0,T)$.

Finally, if $F(t,x,\cdot)$, $G(x,\cdot)$ are nondecreasing, then the solution is unique.

Proof. For simplicity, we fix the diffusion coefficient $\varepsilon = 1$. We set

$$K = \{m \in C^0([0,T];L^2(\mathbb{T}^d)) \cap L^\infty(Q_T) : \ \|m\|_\infty \leq L\} \tag{1.65}$$

where L will be fixed later. For any $\mu \in K$, defining $u_\mu \in L^2(0,T;H^1(\mathbb{T}^d))$ the (unique) bounded solution to

$$\begin{cases} -\partial_t u_\mu - \Delta u_\mu + H(t,x,Du_\mu) = F(t,x,\mu) \\ u_\mu(T) = G(x,\mu(T))\,, \end{cases}$$

one sets $m := \Phi(\mu)$ as the solution to

$$\begin{cases} \partial_t m - \Delta m - \operatorname{div}(m H_p(t,x,Du_\mu)) = 0\,, \\ m(0) = m_0\,. \end{cases}$$

Due to the global bound on H_p, there exists $L>0$, depending only on β and $\|m_0\|_\infty$, such that $\|m\|_\infty \leq L$. This fixes the value of L in (1.65), so that K is an invariant convex subset of $C^0([0,T];L^2(\mathbb{T}^d))$. Continuity and compactness of Φ are an easy exercise, so Schauder's fixed point theorem applies which yields a solution. By parabolic regularity for Fokker-Planck equations with bounded drift, m is $C^\alpha(Q_T)$ for some $\alpha>0$, and so is Du from the first equation. Finally, if the nonlinearity F preserves the Hölder regularity of m, and if H, H_p are of class C^1, then the Schauder's theory can be applied exactly as in Theorem 4, so u and m will belong to $C^{1+\frac{\alpha}{2},2+\alpha}(Q_T)$ for some $\alpha \in (0,1)$ and they will be classical solutions.

The uniqueness follows by the time monotonicity estimate (1.28), which still holds for any two possible solutions $(u_1,m_1), (u_2,m_2)$ of system (1.62), because they are bounded. The convexity of H and the monotonicity of F, G imply that $F(t,x,m_1) = F(t,x,m_2)$ and $G(x,m_1(T)) = G(x,m_2(T))$. This readily yields $u_1 = u_2$ by standard uniqueness of the Bellman equation with Lipschitz Hamiltonian and bounded solutions. Since $H_p(t,x,Du_1) = H_p(t,x,Du_2)$, from the Fokker-Planck equation we deduce $m_1 = m_2$. □

Let us stress that the global Lipschitz bound for H implies a global L^∞ bound for m and Du, which is independent of the time horizon T as well. We will come back to that in Section 1.3.6.

Remark 8 The existence of solutions would still hold assuming the minimal condition that the initial distribution $m_0 \in \mathcal{P}(\mathbb{T}^d)$. The proof remains essentially the same up to using the smoothing effect in the Fokker-Planck equation, where $\|m(t)\|_\infty \leq Ct^{-\frac{d}{2}}$ for some C only depending on the constant β in (1.64). However, it is unclear how to prove uniqueness when m_0 is just a probability measure, unless some restriction is assumed on the growth of the coupling F. Of course one can combine the growth of F with respect to m and the integrability assumption of m_0 in order to get uniqueness results for some class of unbounded initial data, but this is not surprising.

Remark 9 The monotonicity condition on F and G can be slightly relaxed, depending on the diffusive coefficient ε and on $\|m_0\|_\infty$. In particular, if H satisfies (1.64) and is locally uniformly convex with respect to p, there exists a positive value γ, depending on H, F, ε and $\|m_0\|_\infty$, such that (1.62) admits a unique solution whenever $F(x,m) + \gamma\, m$ is nondecreasing in m. The value of γ tends to zero if $\|m_0\|_\infty \to \infty$ or if $\varepsilon \to 0$. Indeed, this is an effect of diffusivity, which could be understood in the theory as the impact of the independent noise in the players' dynamics against a mild aggregation cost. This phenomenon was observed first in [149] and recently addressed in [91] in relation with the long-time stabilization of the system.

When the Hamiltonian has not linear growth in the gradient, the existence and uniqueness of solutions with local couplings is no longer a trivial issue. The main problem is that solutions can hardly be proved to be smooth unless the growth of the coupling functions F, G or the growth of the Hamiltonian are restricted (see Remark 12 below).

On one hand, unbounded solutions of the Bellman equation may be not unique. On another hand, if the drift $H_p(t,x,Du)$ has not enough integrability, the standard parabolic estimates (including boundedness and strict positivity of the solution) are not available for the Fokker-Planck equation. This kind of questions are discussed in [165], where a theory of existence and uniqueness of weak solutions is developed using arguments from renormalized solutions and L^1-theory. We give a sample result of this type, assuming here that the Hamiltonian $H(t,x,p)$ satisfies the following coercivity and growth conditions in $Q_T \times \mathbb{R}^d$:

$$H(t,x,p) \geq \alpha |p|^2 - \gamma \tag{1.66}$$

$$|H_p(t,x,p)| \leq \beta\,(1 + |p|) \tag{1.67}$$

$$H_p(t,x,p) \cdot p - H(t,x,p) \geq \alpha\, |p|^2 - \gamma \tag{1.68}$$

for some constants $\alpha, \beta, \gamma > 0$.

We stress that, under conditions (1.66)-(1.68), and for couplings F, G with general growth, the existence of smooth solutions is not known.

Definition 5 Assume (1.66)-(1.68). A couple (u,m) is a *weak solution* to system (1.62) if

- $F(t,x,m) \in L^1(Q_T)$, $G(x,m(T)) \in L^1(\mathbb{T}^d)$ and $u \in L^2(0,T;H^1(\mathbb{T}))$ is a distributional solution of

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(t,x,Du) = F(t,x,m(t,x)) \\ u(T) = G(x, m(T,x)) \end{cases}$$

- $m \in C^0([0,T];L^1(\mathbb{T}^d))$, $m\,|Du|^2 \in L^1(Q_T)$ and m is a distributional solution of

$$\begin{cases} \partial_t m - \varepsilon \Delta m - \operatorname{div}(m\, H_p(t,x,Du)) = 0 \\ m(0) = m_0\,. \end{cases}$$

Let us stress that the terminal condition for u is understood in $L^1(\mathbb{T}^d)$, because $u \in C^0([0,T];L^1(\mathbb{T}^d))$ as a consequence of the equation itself.

The following result is essentially taken from [165], although the uniqueness statement that we give below generalizes the original result, by establishing that the uniqueness of u always holds m-almost everywhere. This seems to be the most general well-posedness result available so far for system (1.62), in terms of the conditions allowed on H and F, G. Later we discuss the issue of smooth solutions, some special cases, and several related results, including other possible approaches to weak solutions.

Theorem 12 *[[165]] Assume that $H(t,x,p)$ is convex in p and satisfies conditions (1.66)-(1.68), and that F, G satisfy (1.63) and $G(x,\cdot)$ is nondecreasing. Then, for any $m_0 \in L^\infty(\mathbb{T}^d)$, there exists a weak solution to (1.62).*

If we assume in addition that $F(x,\cdot)$ is nondecreasing, then $F(x,m) = F(x,\tilde m)$ and $G(x,m(T)) = G(x,\tilde m(T))$ for any two couples of weak solutions $(u,m),(\tilde u,\tilde m)$. Moreover, if at least one of the following two assumptions holds:
(i) $F(x,\cdot)$ is increasing
(ii) $H(t,x,p) - H(t,x,q) - H_p(t,x,q)\cdot(p-q) = 0 \Rightarrow H_p(t,x,p) = H_p(t,x,q) \quad \forall p,q \in \mathbb{R}^d$
then $m = \tilde m$ and $u = \tilde u$ m-almost everywhere.

In particular, there is at most one weak solution (u,m) with $m>0$ and, if $m_0>0$ and $\log(m_0)\in L^1(\mathbb{T}^d)$, there exists one and only one weak solution.

Remark 10 We stress that if (u,m) is a weak solution such that $u,m \in L^\infty(Q_T)$, then both u and m belong to $L^2(0,T;H^1(\mathbb{T}^d))$ and the two equations hold in the usual formulation of finite energy solutions, e.g. against test functions $\varphi \in L^2(0,T;H^1(\mathbb{T}^d)) \cap L^\infty(Q_T)$ with $\partial_t\varphi \in L^2(0,T;H^1(\mathbb{T}^d)') + L^1(Q_T)$. This fact can be deduced, for instance, from the characterization that weak solutions in the sense of Definition 5 are also renormalized solutions (see [165, Lemma 4.2]).

In addition, if (u,m) are bounded weak solutions, further results in the literature can be applied: since $F(x,m)$ is bounded and H has at most quadratic growth, it turns out that Du is also bounded for $t<T$, which is enough to ensure that $m \in C^\alpha(Q_T)$ and $m(t)>0$ for $t>0$. In other words, bounded weak solutions are regularized with standard bootstrap arguments. In particular, under the assumptions of Theorem 12, *bounded weak solutions are unique*.

The existence part of Theorem 12 requires many technical tools which we will only sketch here, referring to [165] for the details. It is instructive first to recall the basic a priori estimates of the system (1.62), which explain the natural framework of *weak solutions*. We stress that the estimates below are independent of the diffusion constant ε.

Lemma 13 *Assume that (u,m) are bounded weak solutions to system (1.62) and F,G are continuous functions satisfying (1.63). There exists a constant K, independent on ε, such that*

$$\int_0^T\!\!\int_{\mathbb{T}^d} m\{H_p(t,x,Du)Du - H(t,x,Du)\} + \int_0^T\!\!\int_{\mathbb{T}^d} H(t,x,Du) + \int_0^T\!\!\int_{\mathbb{T}^d} F(t,x,m)m + \int_{\mathbb{T}^d} G(x,m)m \le K\,. \tag{1.69}$$

The constant K depends on $\|m_0\|_\infty, T, \|H(t,x,0)\|_\infty, c_0$ and $\sup\limits_{m\le 2\|m_0\|_\infty}[F(t,x,m)+G(x,m)]$.

Proof. We omit the dependence on t of the nonlinearities, which plays no role. Since u and m are bounded, they can be used as test functions in the usual weak formulations of both equations. This yields the energy equality

$$\int_{\mathbb{T}^d} G(x,m(T))m(T) + \int_0^T\!\!\int_{\mathbb{T}^d} F(x,m)m + \int_0^T\!\!\int_{\mathbb{T}^d} m\,[H_p(x,Du)\cdot Du - H(x,Du] = \int_{\mathbb{T}^d} m_0\,u(0) \tag{1.70}$$

which implies

$$\begin{aligned}&\int_{\mathbb{T}^d} G(x,m(T))m(T) + \int_0^T\!\!\int_{\mathbb{T}^d} F(x,m)m + \int_0^T\!\!\int_{\mathbb{T}^d} m\,[H_p(x,Du)\cdot Du - H(x,Du] \le \|m_0\|_\infty \int_{\mathbb{T}^d} u(0)_+ \\ &\le \|m_0\|_\infty \left\{\int_0^T\!\!\int_{\mathbb{T}^d} (F(x,m)) + \int_{\mathbb{T}^d} G(x,m(T)) - \int_0^T\!\!\int_{\mathbb{T}^d} H(x,Du)\right\} + \|m_0\|_\infty \int_{\mathbb{T}^d} u(0)_-\end{aligned}$$

where we used that $\int_{\mathbb{T}^d} u(0) = \int_0^T\!\int_{\mathbb{T}^d} F(x,m) + \int_{\mathbb{T}^d} G(x,m(T)) - \int_0^T\!\int_{\mathbb{T}^d} H(x,Du)$.

Now we estimate the right-hand side of the previous inequality. From assumption (1.63) and the maximum principle, we have that u is bounded below by a constant depending on c_0 and the L^∞- bound of $H(x,0)$, so last term is bounded. We also have $F(x,m) \le \frac{1}{2\|m_0\|_\infty} F(x,m)m + C$, for some constant C depending on $\sup\limits_{m\le 2\|m_0\|_\infty} F(x,m)$. Similarly we estimate $G(x,m)$. Therefore, we conclude that (1.69) holds true. □

Proof of Theorem 12 (sketch).
Without loss of generality, we fix the diffusion coefficient $\varepsilon = 1$.

Existence. To start with, one can build a sequence of smooth solutions, e.g. by defining $F^n(t,x,m) = \rho^n \star F(t,\cdot,\rho^n \star m))(x)$, $G^n(x,m) = \rho^n \star G(\cdot,\rho^n \star m))(x)$, where $\star$ denotes the convolution with respect to the spatial variable and ρ^n is a standard symmetric mollifier, i.e. $\rho^n(x) = n^N \rho(nx)$ for a nonnegative function $\rho \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \rho(x)dx = 1$.

The existence of a bounded solution u^n, m^n is given, for instance, by Theorem 4. From assumption (1.63) and the maximum principle, we have that u^n is bounded from below. Due to the a priori estimates (1.69), applied to (u^n, m^n), and thanks to (1.68) and (1.66), we have that

$$u^n \text{ is bounded in } L^2((0,T); H^1(\mathbb{T}^d), \text{ and } m^n\,|Du^n|^2 \text{ is bounded in } L^1(Q_T).$$

In addition, we have that

$$F(t,x,m^n), G(x,m^n(T)) \quad \text{are bounded and equi-integrable in } L^1(Q_T) \text{ and } L^1(\mathbb{T}^d), \text{ respectively.} \tag{1.71}$$

The heart of the existence proof consists then in considering both the stability properties of the viscous HJ equation

$$-\partial_t u^n - \Delta u^n + H(t,x,Du^n) = f^n \tag{1.72}$$

for some f^n converging in $L^1(Q_T)$, and the compactness of the Fokker-Planck equation

$$\partial_t m^n - \Delta m^n - \mathrm{div}(m^n\, b^n) = 0 \tag{1.73}$$

where $m^n\,|b^n|^2$ is bounded (or eventually, converging) in $L^1(Q_T)$.

Indeed, as a first step one uses (1.73) to show that m^n is relatively compact in $L^1(Q_T)$, as well as in $C^0([0,T]; W^{-1,q}(\mathbb{T}^d))$ for some dual Sobolev space $W^{-1,q}(\mathbb{T}^d)$, and for every t we have that $m^n(t)$ is relatively compact in $\mathcal{P}(\mathbb{T}^d)$. Using the extra estimates (1.71), $m^n(T)$ is relatively compact in the weak L^1 topology and $F(t,x,m^n)$ is compact in $L^1(Q_T)$. If we turn the attention to the Bellman equation (1.72), the L^1 convergence of f^n is enough to ensure that u^n, Du^n are relatively compact in $L^1(Q_T)$ and, thanks to existing results of the L^1-theory for divergence form operators, one concludes that $u^n \to u$ which solves

$$-\partial_t u - \Delta u + H(t,x,Du) = F(t,x,m)\,.$$

The convergence of Du^n now implies that $m^n\,H_p(t,x,Du^n) \to m\,H_p(t,x,Du)$ in $L^1(Q_T)$ and m can be proved to be a weak solution of the limit equation. The proof of the existence would be concluded if not for the coupling in the terminal condition $u(T)$; in fact, to establish that $u(T) = G(x,m(T))$ some extra work is needed, and this can be achieved by using the monotonicity of $G(x,\cdot)$. For the full proof of this stability argument, we refer to [165][Thm 4.9].

Uniqueness. To shortness notations, we omit here the dependence on t of the nonlinearities H, F. A key point for uniqueness is to establish that both u and m are renormalized solutions of their respective equations (see [165, Lemma 4.2]). This means that if (u,m) is any weak solution, then u satisfies

$$-\partial_t S_h(u) - \Delta S_h(u) + S_h'(u)H(x,Du) = F(x,m)S_h'(u) - S_h''(u)|Du|^2 \tag{1.74}$$

where $S_h(r)$ is the sequence of functions (an approximation of the identity function) defined as

$$S_h(r) = h\,S\left(\frac{r}{h}\right)\,,\text{ where } S(r) = \int_0^r S'(r)dr, \quad S'(r) = \begin{cases} 1 & \text{if } |s| \le 1 \\ 2 - |s| & \text{if } 1 < |s| \le 2 \\ 0 & \text{if } |s| > 2 \end{cases} \tag{1.75}$$

Notice that $S_h(r) \to r$ as $h \to \infty$ and since S_h' has compact support the *renormalized equation* (1.74) is restricted to a set where u is bounded. Similarly m is also a renormalized solution, in particular it satisfies

$$\partial_t S_n(m) - \Delta S_n(m) - \mathrm{div}(S_n'(m)m\,H_p(x,Du)) = \omega_n\,, \qquad \text{for some } \omega_n \overset{n\to\infty}{\longrightarrow} 0 \text{ in } L^1(Q_T). \tag{1.76}$$

We recall that the renormalized formulations are proved to hold for all weak solutions, since $F(x,m) \in L^1(Q_T)$ and since $m|Du|^2 \in L^1(Q_T)$. In addition, it is proved in [165, Lemma 4.6] that, for any couple of weak solutions (u,m) and $(\tilde u, \tilde m)$, the following *crossed regularity* holds: $m|D\tilde u|^2, \tilde m |Du|^2 \in L^1(Q_T)$. This is what is needed in order to perform first the Lasry-Lions' argument on the renormalized formulations, and then letting $n \to \infty$ and subsequently $h \to \infty$ to conclude the usual monotonicity inequality:

$$\int_0^T\int_{\mathbb{T}^d} (F(x,m)-F(x,\tilde m))(m-\tilde m) + \int_{\mathbb{T}^d} [G(x,m(T))-G(x,\tilde m(T))][m(T)-\tilde m(T)] \\ + \int_0^T\int_{\mathbb{T}^d} m\,[H(x,D\tilde u)-H(x,Du)-H_p(x,Du)(D\tilde u - Du)] \\ + \int_0^T\int_{\mathbb{T}^d} \tilde m\,[H(x,Du)-H(x,D\tilde u)-H_p(x,D\tilde u)(Du-D\tilde u)] \le 0\,.$$

This implies, because $F(x,\cdot), G(x,\cdot)$ are nondecreasing,

$$F(x,m) = F(x,\tilde m)\,, \quad G(x,m(T)) = G(x,\tilde m(T)) \tag{1.77}$$

and, from the convexity of $H(x,\cdot)$,

$$\begin{aligned} H(x,D\tilde u)-H(x,Du) &= H_p(x,Du)(D\tilde u - Du) \qquad \text{in } \{(t,x):\ m(t,x)>0\} \\ H(x,Du)-H(x,D\tilde u) &= H_p(x,D\tilde u)(Du-D\tilde u) \qquad \text{in } \{(t,x):\ \tilde m(t,x)>0\}\,. \end{aligned} \tag{1.78}$$

We warn the reader that (1.77) does not imply alone that $u=\tilde u$, because unbounded weak solutions to the Bellman equation may be not unique. So we need to use some extra information.

We first want to show that $m = \tilde m$. This is straightforward if $F(x,\cdot)$ is increasing. Otherwise, suppose that (1.25) holds true. Then we deduce that

$$m\,H_p(x,Du) = m\,H_p(x,D\tilde u) \qquad \tilde m\,H_p(x,Du) = \tilde m\,H_p(x,D\tilde u) \quad \text{a.e. in } Q_T. \tag{1.79}$$

We take now the difference of the renormalized equations of $m, \tilde m$, namely

$$\partial_t\,(S_n(m)-S_n(\tilde m)) - \Delta\,(S_n(m)-S_n(\tilde m)) - \mathrm{div}(S_n'(m)m\,H_p(x,Du) - S_n'(\tilde m)\tilde m\,H_p(x,D\tilde u)) = \omega_n - \tilde\omega_n$$

and we aim at showing that, roughly speaking, $\|m(t)-\tilde m(t)\|_{L^1(\mathbb{T}^d)}$ is time contractive. To do it rigorously, we consider the function $\Theta_\varepsilon(s) = \int_0^r \frac{T_\varepsilon(r)}{\varepsilon}dr$, with $T_\varepsilon(r) = \min(\varepsilon, r)$; then $\Theta_\varepsilon(s)$ approximates $|s|$ as $\varepsilon \to 0$. Using $\frac{T_\varepsilon(S_n(m)-S_n(\tilde m))}{\varepsilon}$ as test function in the previous equality we get

$$\int_{\mathbb{T}^d} \Theta_\varepsilon[S_n(m(t))-S_n(\tilde m(t))] + \frac{1}{\varepsilon}\int_0^T\int_{\mathbb{T}^d} |DT_\varepsilon(S_n(m)-S_n(\tilde m))|^2 \le \|\omega_n\|_{L^1(Q_T)} + \|\tilde\omega_n\|_{L^1(Q_T)} \\ - \frac{1}{\varepsilon}\int_0^T\int_{\mathbb{T}^d} (S_n'(m)m\,H_p(x,Du) - S_n'(\tilde m)\tilde m\,H_p(x,D\tilde u))DT_\varepsilon(S_n(m)-S_n(\tilde m))$$

where we used that the test function is smaller than one. Thanks to Young's inequality we deduce

$$\int_{\mathbb{T}^d} \Theta_\varepsilon[S_n(m(t))-S_n(\tilde m(t))] \le \frac{1}{4\varepsilon}\int_0^T\int_{\mathbb{T}^d} |S_n'(m)m\,H_p(x,Du) - S_n'(\tilde m)\tilde m\,H_p(x,D\tilde u)|^2\, 1_{\{|S_n(m)-S_n(\tilde m)|<\varepsilon\}} \\ + \|\omega_n\|_{L^1(Q_T)} + \|\tilde\omega_n\|_{L^1(Q_T)}. \tag{1.80}$$

Now we use (1.79): if one between $m,\tilde m$ is positive, then $H_p(x,Du) = H_p(x,D\tilde u)$, so

$$\int_0^T\int_{\mathbb{T}^d} |S_n'(m)m\, H_p(x,Du) - S_n'(\tilde m)\tilde m\, H_p(x,D\tilde u)|^2\, 1_{\{|S_n(m)-S_n(\tilde m)|<\varepsilon\}}$$
$$= \int_0^T\int_{\mathbb{T}^d} |S_n'(m)m - S_n'(\tilde m)\tilde m|^2\, |H_p(x,Du)|^2\, 1_{\{|S_n(m)-S_n(\tilde m)|<\varepsilon\}}$$

and since

$$|S_n'(m)m - S_n'(\tilde m)\tilde m|^2\, |H_p(x,Du)|^2\, 1_{\{|S_n(m)-S_n(\tilde m)|<\varepsilon\}} \le C\,\varepsilon\,(m+\tilde m)(1+|Du|^2)$$

we can let $n\to\infty$ using Lebesgue's theorem since $m|Du|^2, \tilde m|Du|^2 \in L^1(Q_T)$. Therefore, letting $n\to\infty$, from (1.80) we obtain (for a.e. $\varepsilon>0$):

$$\begin{aligned}\int_{\mathbb{T}^d} \Theta_\varepsilon[m(t)-\tilde m(t)] &\le \frac{1}{4\varepsilon}\int_0^T\int_{\mathbb{T}^d} |m-\tilde m|^2\, |H_p(x,Du)|^2\, 1_{\{m-\tilde m|<\varepsilon\}} \\ &\le \frac14 \int_0^T\int_{\mathbb{T}^d} |m-\tilde m|\, |H_p(x,Du)|^2 1_{\{m-\tilde m|<\varepsilon\}}\,. \end{aligned}\tag{1.81}$$

Last term converges to zero as $\varepsilon\to 0$ (using again Lebesgue's theorem), whereas the first integral converges to $\|m(t)-\tilde m(t)\|_{L^1(\mathbb{T}^d)}$. Hence, by letting $\varepsilon\to0$ we get $\|m(t)-\tilde m(t)\|_{L^1(\mathbb{T}^d)} = 0$. This concludes the proof of the uniqueness of m.

Now we show that u is unique $m-$a.e.; to this purpose, we are going to show that

$$\int_{\mathbb{T}^d} m(t)(u-\tilde u)_+(t) \le 0 \qquad \text{for a.e. } t<T. \tag{1.82}$$

To prove (1.82), we subtract the renormalized formulations (1.74) for $u, \tilde u$. By using the convexity of H, we have

$$\begin{aligned}&-\partial_t(S_h(u)-S_h(\tilde u)) - \Delta(S_h(u)-S_h(\tilde u)) + S_h'(u)\, H_p(x,D\tilde u)D(u-\tilde u) + (S_h'(u)-S_h'(\tilde u))H(x,D\tilde u)\\ &\quad\le F(x,m)(S_h'(u)-S_h'(\tilde u)) - S_h''(u)|Du|^2 + S_h''(\tilde u)|D\tilde u|^2\,.\end{aligned}$$

We multiply this equation by $\varphi_\varepsilon := \frac{T_\varepsilon(S_h(u)-S_h(\tilde u))_+}{\varepsilon}$; denoting as before Θ_ε the primitive of $\frac{T_\varepsilon(t)}{t}$, using that $0\le\varphi_\varepsilon\le1$ we get, in weak sense,

$$\begin{aligned}&-\partial_t\Theta_\varepsilon[(S_h(u)-S_h(\tilde u))_+] - \Delta\Theta_\varepsilon[(S_h(u)-S_h(\tilde u))_+] + \varphi_\varepsilon S_h'(u)\, H_p(x,D\tilde u)D(u-\tilde u)\\ &\quad\le |S_h'(u)-S_h'(\tilde u)|\,|H(x,D\tilde u)+F(x,m)| + |S_h''(u)|\,|Du|^2 + |S_h''(\tilde u)|\,|D\tilde u|^2\,.\end{aligned}$$

Now we multiply by $S_n(m)$ this equation, we integrate in (t,T), we use that $u(T)=\tilde u(T)$ and (1.76). We obtain

$$\begin{aligned}&\int_{\mathbb{T}^d} S_n(m(t))\Theta_\varepsilon[(S_h(u(t))-S_h(\tilde u(t)))_+] - \int_t^T\int_{\mathbb{T}^d} S_n'(m)\, mH_p(x,Du)\varphi_\varepsilon D(S_h(u)-S_h(\tilde u))\\ &\quad+\int_t^T\int_{\mathbb{T}^d} S_n(m)\,\varphi_\varepsilon S_h'(u)\, H_p(x,D\tilde u)D(u-\tilde u) \le \int_t^T\int_{\mathbb{T}^d} S_n(m)\,|S_h'(u)-S_h'(\tilde u)|\,|H(x,D\tilde u)+F(x,m)|\\ &\quad+\int_t^T\int_{\mathbb{T}^d} S_n(m)[|S_h''(u)|\,|Du|^2 + |S_h''(\tilde u)|\,|D\tilde u|^2] + \int_t^T\int_{\mathbb{T}^d} \Theta_\varepsilon[(S_h(u(t))-S_h(\tilde u(t)))_+]\,\omega_n\end{aligned}$$

where we used that $D\Theta_\varepsilon[(S_h(u(t))-S_h(\tilde u(t)))_+] = \varphi_\varepsilon\, D(S_h(u)-S_h(\tilde u))$. Now we let $n,h\to\infty$, which is allowed using that $F(x,m)m, m|Du|^2, m|D\tilde u|^2\in L^1(Q_T)$. First we let $n\to\infty$, so that we can use the L^1-convergence to zero of ω_n (whereas $S_h(u), S_h(\tilde u)$ are bounded functions). Once n has gone to infinity, we let $h\to\infty$, so that $S_h'\to1$; using dominated convergence in each term and Fatou's lemma in the first integral, we get

$$\int_{\mathbb{T}^d} m(t)\Theta_\varepsilon[(u(t)-\tilde u(t))_+] - \int_t^T\int_{\mathbb{T}^d} mH_p(x,Du)\varphi_\varepsilon D(u-\tilde u) + \int_t^T\int_{\mathbb{T}^d} m\,\varphi_\varepsilon\, H_p(x,D\tilde u)D(u-\tilde u) \le 0$$

for a.e. $t \in (0,T)$. Since $m\,H_p(x,Du)D(u-\tilde{u}) = m\,H_p(x,D\tilde{u})D(u-\tilde{u})$ from (1.78) (where now $\tilde{m} = m$), we deduce that $\int_{\mathbb{T}^d} m(t)\Theta_\varepsilon[(u(t)-\tilde{u}(t))_+] \le 0$. Letting $\varepsilon \to 0$ yields (1.82). Reversing the roles of $u, \tilde{u}$, we conclude that $u = \tilde{u}$ m-a.e.

Finally, it is proved in [165] that, if $\log m_0 \in L^1(\mathbb{T}^d)$, then we have $m > 0$ a.e., in which case we deduce that $u = \tilde{u}$ almost everywhere. □

Several comments and remarks are in order as far as the previous result and MFG systems with local couplings are concerned.

Remark 11 (*extensions of Theorem 12*)

(i) The result of Theorem 12 also holds with homogeneous Dirichlet or Neumann boundary conditions; this extension already appears in [165]. Let us stress that this is one of the main advantage for the use of renormalized solutions, which are well adapted to boundary conditions. Indeed, through the use of renormalization one wishes to approximate a weak solution with its own truncations, which often preserve natural boundary conditions. By contrast, the approximation of weak solutions through mollification introduces many technical problems when dealing with boundary conditions.
Results on the whole space $\mathbb{R}^d$ are also available in [166], assuming $m_0 \in L^1(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$; in that case u belongs to $L^\infty((0,T)\times\mathbb{R}^d) + L^\infty(0,T;L^1(\mathbb{R}^d))$ and $m \in L^\infty(0,T;L^1(\mathbb{R}^d))$.

(ii) Similar results hold by assuming the Hamiltonian coercive with q-growth, namely replacing $|p|^2$ with $|p|^q$ in (1.66), (1.68) and a $q-1$-growth for H_p, where $1 < q < 2$. However, general uniqueness results in this case have been proved so far only for the periodic case or for the whole space ([165], [166]).

(iii) The same results hold for more general diffusion coefficients, namely if the Laplacian is replaced by the divergence form operator $\mathrm{div}(A(t,x)D(\cdot))$ with $A(t,x) \in L^\infty(0,T;W^{1,\infty})$. In particular, this includes the case where Δu is replaced by $\mathrm{Tr}(\sigma(t,x)\sigma^*(t,x)D^2u)$ for a bounded, Lipschitz and elliptic matrix $\sigma(t,\cdot)$, modeling diffusion processes associated to stochastic dynamics with Lipschitz diffusion coefficients.

Remark 12 (*smoothness of solutions*) Solutions of system (1.62) with local couplings can be proved to be more regular under growth restrictions on $H(t,x,p)$ and/or on $F(t,x,m)$.

An easy case occurs when

$$|H_p(t,x,p)| \le C(1+|p|^{q-1}) \qquad \text{with } q < \tfrac{d+2}{d+1}. \tag{1.83}$$

Indeed, since $F(t,x,m) \in L^1(Q_T)$ (regardless of the growth of F, see estimate (1.69)) then any weak solution u belongs to $L^s(0,T;W^{1,s}(\mathbb{T}))$ for any $s < \frac{d+2}{d+1}$. If (1.83) holds, this implies that $H_p(t,x,Du) \in L^r$ for some $r > d+2$; in turn, by standard parabolic results, the solution of the Fokker-Planck equation becomes bounded in this case, and actually even Hölder continuous. One concludes that u is bounded as well, from the first equation, and actually Du is Hölder continuous as well. Smoothness up to C^2 regularity then follows according to the smoothness required on the coefficients.

A somewhat similar situation occurs if F has restricted growth, namely if

$$|F(t,x,m)| \le C(1+m^\gamma)$$

with $\gamma < \frac{2}{d}$; in this case estimate (1.69) implies that $F(t,x,m) \in L^r(Q_T)$ for some $r > \frac{2+d}{2}$, and the standard parabolic regularity immediately gives the boundedness, and then smoothness, of u, m.

The above two situations are straightforward applications, using parabolic regularity, of the a priori estimates (1.69); in particular they do not require any smoothness in the x-dependence of the nonlinearities, and directly apply to weak solutions in order to obtain their boundedness. We recall that *proving boundedness of weak solutions is enough to show that they are unique*, see Remark 10.

However, in order to get smooth solutions, one can go beyond the above conditions up to using refined estimates on the system. This was addressed first by P.-L. Lions, who showed that $F(x,m) \simeq m^\gamma$ with $\gamma < \frac{2}{d-2}$ was enough to ensure smoothness of solutions, standing on second order estimates which further exploit the monotonicity of the coupling F. This issue has been extensively investigated later in a series of papers by D. Gomes and co-workers (see e.g. [117], [118]; most results are encoded in the book [119]), coupling the second order estimates with regularity estimates for the Fokker-Planck equation obtained through the adjoint method introduced by L.C. Evans. In this series of contributions, some growth conditions on H and F have been given which allow to have smooth solutions, both for sub quadratic and for super quadratic Hamiltonians. They are specially important for the case that $H(x,p)$ grows superquadratically in p, because in that situation the approach through weak solutions as developed in Theorem 12 cannot be used. It must be said that the aforementioned regularity results usually require smoothness of the Hamiltonian and periodic setting, and the smoothness of solutions remains largely open under general growth assumptions.

Remark 13 (*quadratic Hamiltonian and Hopf-Cole reduction*) In the special case that $H(t,x,p) = \frac{1}{2}|p|^2 + b(t,x)\cdot p$, the system (1.62) can be transformed into a system of semi linear equations. By introducing the two new unknowns: $w = e^{-\frac{u}{2}}$ and $\varphi = m\, e^{\frac{u}{2}}$, then (1.62) (with $\varepsilon = 1$) is equivalent to the system

$$\begin{cases} -\partial_t w - \Delta w + b\cdot Dw + \frac{1}{2} w F(t,x,\varphi\, w) = 0 \\ \partial_t \varphi - \Delta\varphi - \mathrm{div}(b\,\varphi) + \frac{1}{2}\varphi F(t,x,\varphi\, w) = 0 \\ w(T) = e^{-G(x,\varphi(T)w(T))/2}\,, \quad \varphi(0) = \frac{m_0}{w(0)} \end{cases} \tag{1.84}$$

Notice that the system (1.84) appears to be simplified, compared to (1.62), but the initial-terminal conditions are both coupled. The initial condition at $t = 0$ makes sense because $w > 0$ by strong maximum principle. Still by maximum principle, the function φ is positive as well. Assuming $G(x,\cdot)$ to be nondecreasing, the condition $w(T) = e^{-G(x,\varphi(T)w(T))/2}$ defines $w(T)$ implicitly as a function of $\varphi(T)$; hence the final condition reads as $w(T) = \psi(x,\varphi(T))$ for a function ψ defined by the implicit relation $\psi(r) - e^{-\frac{1}{2}G(x,r\,\psi(r))} = 0$, for $r \geq 0$.

When $b = 0$ and G only depends on x, it is proved in [60] that weak solutions to (1.84) are bounded. The proof uses a Moser iteration scheme, and it can be easily verified that the proof still holds, without additional difficulty, for the case that $b \in L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))$ and $G(x,\cdot)$ is monotone. The equivalence between (1.84) and (1.62) (for this special H) is easy to verify for bounded solutions, since the maximum principle gives $\varphi, w > 0$, so $u = -2\log w$, $m = w\,\varphi$ defines (u,m) back from (1.84). Once solutions are shown to be bounded, then they are smooth ($m, Du \in C^\alpha(Q_T)$) by standard bootstrap arguments, and they are classical solutions in Q_T if F is locally Hölder continuous. Therefore, system (1.62) possesses regular, and even classical, solutions in the special case that $H(t,x,Du) = b(t,x)\cdot Du + \frac{1}{2}|Du|^2$, with b Lipschitz continuous in x, and this holds true without any growth restriction on the couplings F, G.

We mention here further related results on weak solutions and on systems with local coupling.

- Under general assumptions, essentially the conditions of Theorem 12 above, it has been proved that the discrete solutions of finite difference schemes, as defined in [5], converge to weak solutions as the numerical scheme approximates the continuous equation, i.e. as the mesh size tends to zero. This result is proved in [9] and provides an independent, alternative proof of the existence of weak solutions.
- A different notion of weak solution was introduced in [104] relying on the theory of motonone operators. In particular, if F, G are nondecreasing, then problem (1.62) can be rephrased as $\mathcal{A}(m,u) = 0$ where $\mathcal{A}$ is a monotone operator (on the couple (m,u)) defined as

$$\mathcal{A}(m,u) := \begin{pmatrix} \partial_t u + \Delta u + f(m) - H(x,Du) \\ \partial_t m - \Delta m - \mathrm{div}(m H_p(x,Du)) \end{pmatrix}$$

Since $\langle \mathcal{A}(m,u) - \mathcal{A}(\mu,v), (m-\mu, u-v)\rangle \geq 0$, where the duality is meant in distributional sense, $\mathcal{A}$ defines a monotone operator. Then the Minty-Browder theory of monotone operators suggests the possibility to define a notion of weak solution (u,m) as a couple satisfying

$$\langle \mathcal{A}(\varphi,v), (m-\varphi, u-v)\rangle \geq 0 \qquad \forall (\varphi,v) \in C^2(\overline{Q_T})^2\,. \tag{1.85}$$

This notion requires even less regularity on (m,u) than in Definition 5, and of course the existence of a couple (m,u) satisfying (1.85) is readily proved by weak stability and monotonicity, as in Minty-Browder's theory. However, the uniqueness of a solution of this kind is unclear, and has not been proved so far.

- The study of non monotone couplings $F(x,m)$ in (1.62) leads to different kind of questions and results. This direction has been mostly exploited for the stationary system ([81], [87]) and in special examples for the evolution case. We refer the reader to [92].
  In a different direction, it is worth pointing out that the assumption that $F(x,m)$ be bounded from below could be relaxed by allowing $F(x,m) \to -\infty$ as $m \to 0^+$ as in the model case $F(x,m) \sim \log m$ for $m \to 0$. Results on this model can be found e.g. in [116], [127].

We conclude this Section by mentioning the case of a general Hamiltonian function $H(t,x,Du,m)$, as in problem (1.29), where now $H : Q_T \times \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$ is a continuous function depending locally on the density m. In his courses at Collége de France, P.-L. Lions introduced structure conditions in order to have uniqueness of solutions (u,m) to the local MFG system:

$$\begin{cases} -\partial_t u - \varepsilon\Delta u + H(t,x,Du,m) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ \partial_t m - \varepsilon\Delta m - \operatorname{div}(m\, H_p(t,x,Du,m)) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ m(0) = m_0,\ u(x,T) = G(x,m(T))) & \text{in } \mathbb{T}^d \end{cases} \tag{1.86}$$

Assuming $H(t,x,p,m)$ to be C^1 in m and C^2 in p, the condition introduced by P.-L. Lions can be stated as the requirement that the following matrix be positive semi-definite:

$$\begin{pmatrix} m\,\partial^2_{pp} H & \frac{1}{2} m\,\partial^2_{pm} H \\ \\ \frac{1}{2} m\,(\partial^2_{pm} H)^T & -\partial_m H \end{pmatrix} \geq 0 \qquad \forall (t,x,p,m)\,. \tag{1.87}$$

Notice that condition (1.87) implies that H is convex with respect to p and nonincreasing with respect to m. In particular, when H has a separate form: $H = \tilde H(t,x,p) - f(x,m)$, condition (1.87) reduces to $\tilde H_{pp} \geq 0$ and $f_m \geq 0$. As usual, this condition needs to be taken in a strict form, so that Lions' result would state as follows in terms of smooth solutions.

Theorem 13 *Assume that $G(x,m)$ is nondecreasing in m and that $H = H(t,x,p,m)$ is a C^1 function satisfying (we omit the (t,x) dependence for simplicity)*

$$(H(p_2,m_2) - H(p_1,m_1))(m_2 - m_1) - (m_2 H_p(p_2,m_2) - m_1 H_p(p_1,m_1)) \cdot (p_2 - p_1) \leq 0\ , \tag{1.88}$$

with equality if and only if $(m_1,p_1) = (m_2,p_2)$. Then system (1.86) *has at most one classical solution.*

Proof. The proof is a straightforward extension of the usual monotonicity argument. Let (u_1,m_1) and (u_2,m_2) be solutions to (1.86). We set

$$\tilde m = m_2 - m_1,\ \tilde u = u_2 - u_1,\ \tilde H = H(t,x,Du_2,m_2) - H(t,x,Du_1,m_1)\,.$$

Then, subtracting the two equations we get

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{T}^d} &(u_2(t) - u_1(t))(m_2(t) - m_1(t)) \\ &= \int_{\mathbb{T}^d} (-\varepsilon\Delta\tilde u + \tilde H)\tilde m + \tilde u(\varepsilon\Delta\tilde m + \operatorname{div}(m_2 H_p(t,x,Du_2,m_2) - m_1 H_p(t,x,Du_1,m_1))) \\ &= \int_{\mathbb{T}^d} \tilde H\,\tilde m - (m_2 H_p(t,x,Du_2,m_2) - m_1 H_p(t,x,Du_1,m_1)) \cdot D\tilde u \ \leq\ 0 \end{aligned}$$

by condition (1.88). Since $\int_{\mathbb{T}^d}(u_2(t) - u_1(t))(m_2(t) - m_1(t))$ vanishes at $t = 0$ and is nonnegative at $t = T$ (by monotonicity of $G(,\cdot)$), integrating the above equality between 0 and T gives

$$\int_0^T \int_{\mathbb{T}^d} \tilde H\,\tilde m - (m_2 H_p(t,x,Du_2,m_2) - m_1 H_p(t,x,Du_1,m_1)) \cdot D\tilde u \ =\ 0.$$

Since (1.88) is assumed in strict form, this implies that $D\tilde{u} = 0$ and $\tilde{m} = 0$, so that $m_1 = m_2$ and $u_1 = u_2$. □

Remark 14 It is immediate to check that if the matrix in (1.87) is positive definite, then (1.88) holds in the strict form. Indeed, set $\tilde{p} = p_2 - p_1$, $\tilde{m} = m_2 - m_1$ and, for $\theta \in [0, 1]$, $p_\theta = p_1 + \theta(p_2 - p_1)$, $m_\theta = m_1 + \theta(m_2 - m_1)$. Let

$$I(\theta) = (H(x, p_\theta, m_\theta) - H(x, p_1, m_1))\tilde{m} - \tilde{p} \cdot (m_\theta H_p(x, Du_\theta, m_\theta) - m_1 \, .H_p(x, Du_1, m_1))$$

Then

$$I'(\theta) = -\begin{pmatrix} \tilde{p}^T & \tilde{m} \end{pmatrix} \begin{pmatrix} m_\theta \, \partial^2_{pp} H & \frac{1}{2} m_\theta \, \partial^2_{pm} H \\ \frac{1}{2} m_\theta \, (\partial^2_{pm} H)^T & -\partial_m H \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{m} \end{pmatrix}.$$

If condition (1.87) holds with a strict sign, then the function $I(\theta)$ is decreasing and, for $(p_1, m_1) \neq (p_2, m_2)$, one has

$$I(0) = 0 > I(1) = (H(p_2, m_2) - H(p_1, m_1))(m_2 - m_1) - (m_2 H_p(p_2, m_2) - m_1 H_p(p_1, m_1)) \cdot (p_2 - p_1).$$

We stress that another way of formulating (1.88) is exactly the requirement that $I(\theta)$ be decreasing for every $(p_1, m_1) \neq (p_2, m_2)$.

The main example of Hamiltonian satisfying (1.88) is given by the so-called congestion case.

Example 1 Assume that H is of the form: $H(x, p, m) = \frac{1}{2} \frac{|p|^2}{(\sigma + m)^\alpha}$, where $\sigma, \alpha > 0$. Then condition (1.87) holds if and only if $\alpha \in [0, 2]$. Notice that H is convex in p and nonincreasing in m if $\alpha \geq 0$. Checking condition (1.87) we find

$$\begin{aligned} \left(-\frac{\partial_m H}{m}\right) \partial^2_{pp} H - \frac{1}{4} \partial^2_{pm} H \otimes \partial^2_{pm} H &= \frac{\alpha |p|^2}{2m^{\alpha+2}} \frac{I_d}{m^\alpha} - \frac{\alpha^2}{4} \frac{p \otimes p}{m^{2\alpha+2}} \\ &= \frac{2\alpha |p|^2 I_d}{4m^{2\alpha+2}} - \frac{\alpha^2}{4} \frac{p \otimes p}{m^{2\alpha+2}} \end{aligned}$$

which is positive if and only if $\alpha \leq 2$.

This example (in the generalized version $H = \frac{|p|^q}{(\sigma+m)^\alpha}$ with $\alpha \leq \frac{4(q-1)}{q}$) was introduced by P.-L. Lions in [149] (Lesson 18/12 2009) as a possible mean field game model for crowd dynamics. In this case, the associated Lagrangian cost of the agents takes the form of $L(x, q) = \frac{1}{2}(\sigma + m)^\alpha |q|^2$, where q represents the velocity chosen by the controllers; the cost being higher in areas of higher density models the impact of the crowd in the individual motion. The case $\sigma = 0$ is also meaningful in this example and was treated by P.-L. Lions as well, even if it leads to a singular behavior of the Hamiltonian for $m = 0$.

As explained before, the existence of classical solutions with local couplings only holds in special cases, and this of course remains true for the general problem (1.86) (see e.g. [113], [121], [123] for a few results on smooth solutions of the congestion model). Therefore, the statement of Theorem 13 is of very little use. However, a satisfactory result of existence and uniqueness is proved in [10] for general Hamiltonians $H(t, x, Du, m)$ which include the congestion case (including the singular model with $\sigma = 0$). This is so far the unique general well-posedness result which exists for the local problem (1.86).

1.3.6 The long time ergodic behavior and the turnpike property of solutions

It is a natural question to investigate the behavior of the MFG system (1.62) as the horizon T tends to infinity. Here we fix the diffusion constant $\varepsilon = 1$ and we consider nonlinearities F, H independent of t. As explained by Lions in [149] (see e.g. Lesson 20/11 2009), the limit of the MFG system, as the time horizon T tends to infinity, is given by the stationary ergodic problem

$$\begin{cases} \lambda - \Delta u + H(x, Du) = F(x, m) & \text{in } T^d \\ -\Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } \mathbb{T}^d \\ \int_{\mathbb{T}^d} m = 1\,, \ \int_{\mathbb{T}^d} u = 0 \end{cases} \tag{1.89}$$

This system has also been introduced by Lasry and Lions in [143] as the limit, when the number of players tends to infinity, of Nash equilibria in ergodic differential games. Here the unknowns are (λ, u, m), where $\lambda \in \mathbb{R}$ is the so-called ergodic constant. The interpretation of the system is the following: each player wants to minimize his/her ergodic cost

$$\mathcal{J}(x, \alpha) := \inf_{\alpha} \limsup_{T\to+\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \{H^*(X_t, -\alpha_t) + F(X_t, m(X_t))\}\, dt\right]$$

where (X_t) is the solution to

$$\begin{cases} dX_t = \alpha_t dt + \sqrt{2} dB_t \\ X_0 = x \end{cases}$$

It turns out that, if (λ, u, m) is a classical solution to (1.89), then the optimal strategy of each player is given by the feedback $\alpha^*(x) := -H_p(x, Du(x))$ and, if X_t is the solution to

$$\begin{cases} dX_t = \alpha^*(X_t) dt + \sqrt{2} dB_t \\ X_0 = x \end{cases} \tag{1.90}$$

then $m(\cdot)$ is the invariant measure associated with (1.90) and, setting $\bar\alpha_t := \alpha^*(X_t)$, then $\mathcal{J}(x, \bar\alpha) = \lambda$ is independent of the initial position.

The "convergence" of the MFG system in $(0, T)$ towards the stationary ergodic system (1.89) was analyzed in [60], [61] when the Hamiltonian is purely quadratic (i.e. $H(x,p) = |p|^2$), in [65] where the long time behavior is completely described in case of smoothing coupling and uniformly convex Hamiltonian, and in [167] for the case of local couplings and globally Lipschitz Hamiltonian. The case of discrete time, finite states system is analyzed in [114].

The "long time stability" takes the form of a *turnpike pattern* for solutions (u^T, m^T) of system (1.62); namely, the solutions become nearly stationary for most of the time, which is related to the so-called turnpike property of optimality systems (see e.g. [170]). This pattern is clearly shown in numerical simulations as one can see in the contribution by Achdou & Lauriere in this volume. The strongest way to state this kind of behavior is through the proof of the exponential estimate

$$\|m^T(t) - \bar m\|_\infty + \|Du^T(t) - D\bar u\|_\infty \le K(e^{-\omega t} + e^{-\omega(T-t)}) \qquad \forall t \in (0, T) \tag{1.91}$$

for some $K, \omega > 0$, where $(\bar u, \bar m)$ is a solution to (1.89).

Notice that a weakest statement is also given by the time-average convergence (which is a consequence of (1.91), if this holds true)

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_{\mathbb{T}^d} \left(|Du^T - D\bar u|^2 + |m^T - \bar m|^2\right)\, dxdt = 0\,.$$

Of course this kind of convergence occurs provided the Lasry-Lions monotonicity condition holds true: in general, the behavior of the time-dependent problem can be much more complex. For instance it can exhibit periodic solutions. On that topic, see in particular [63, 88, 90, 153].

In this Section we give a new proof of the turnpike property of solutions, by showing how it is possible to refine the usual fixed point argument in order to build directly the solution (u, m) embedded with the turnpike estimate (1.91). For simplicity, we develop this approach in the case of local couplings and globally Lipschitz Hamiltonian although, roughly speaking, a similar method would work for any case in which a global (in time) Lipschitz estimate is available for u.

Let us first remark that the stationary system (1.89) is well-defined.

Proposition 2 *Assume that (1.63) and (1.64) hold true and $F(x,m)$ is nondecreasing in m. Then system (1.89) has a unique classical solution $(\bar\lambda,\bar u,\bar m)$, and moreover $\bar m>0$.*

The proof can be established by usual fixed point arguments, very similar as in Theorem 11, so we omit it.

Now we prove the exponential turnpike estimate for locally Lipschitz couplings F (without any growth restriction) and for globally Lipschitz, locally uniformly convex Hamiltonian.

Theorem 14 *Let $m_0\in\mathcal{P}(\mathbb{T}^d)$. Assume that $F(x,m)$ is a Carathéodory function which is nondecreasing with respect to m and satisfies*

$$\forall\,K>0\,,\ \exists\,c_K\,,\,\ell_K>0\,:\,\begin{cases}|F(x,m)|\le c_K\,,\\ |F(x,m)-F(x,m')|\le \ell_K|m-m'|\end{cases}\qquad \forall x\in\mathbb{T}^d\,,m,m'\in\mathbb{R}\,:\,|m|,|m'|\le K\,. \tag{1.92}$$

Assume that $p\mapsto H(x,p)$ is a C^2 function which is globally Lipschitz (i.e. it satisfies (1.64)) and locally uniformly convex:

$$\forall K>0\,,\ \exists\,\alpha_K,\beta_K>0:\quad \alpha_K\,I\le H_{pp}(x,p)\le\beta_K I\qquad \forall (x,p)\in\mathbb{T}^d\times\mathbb{R}^d\,:\,|p|\le K\,. \tag{1.93}$$

Then there exists $\omega,M>0$ (independent of T) such that any solution (u^T,m^T) of problem (1.62) (with $\varepsilon=1$) satisfies

$$\|m^T(t)-\bar m\|_\infty+\|Du^T(t)-D\bar u\|_\infty\le M(e^{-\omega t}+e^{-\omega(T-t)})\qquad \forall t\in(1,T-1)\,. \tag{1.94}$$

This kind of result is proved in [167] with a strategy based on the stabilization properties of the linearized system; an approach which explains the exact exponential rate ω in (1.94) in terms of a Riccati feedback operator. Here we give a new direct proof of (1.94), mostly based on ideas in [65]. This approach is less precise in the rate ω but requires less demanding assumptions and avoids the formal use of the linearized system, though some form of linearization appears in obtaining the following a priori estimate.

Lemma 14 *Under the assumptions of Theorem 14, let $(\bar\lambda,\bar u,\bar m)$ be the unique solution of (1.89).*

For $\sigma\in[0,1]$, $m_0\in L^\infty(\mathbb{T}^d)\cap\mathcal{P}(\mathbb{T}^d)$, $v_T\in C^{1,\alpha}(\mathbb{T}^d)$ for some $\alpha\in(0,1)$, let (μ,v) be a solution of the system

$$\begin{cases}-\partial_t v-\Delta v+H(x,D\bar u+Dv)-H(x,D\bar u)=F(x,\bar m+\mu)-F(x,\bar m)\\ v(T)=v_T\\ \partial_t\mu-\Delta\mu-\operatorname{div}(\mu\,H_p(x,D\bar u+Dv))=\sigma\operatorname{div}(\bar m\,[H_p(x,D\bar u+Dv)-H_p(x,D\bar u)])\\ \mu(0)=\sigma(m_0-\bar m)\,.\end{cases} \tag{1.95}$$

Then there exist constants $\omega,K>0$ such that

$$\|\mu(t)\|_2+\|Dv(t)\|_2\le K(e^{-\omega t}+e^{-\omega(T-t)})[\|m_0-\bar m\|_2+\|Dv_T\|_2]\,. \tag{1.96}$$

Proof. We first notice that, using the equation satisfied by $\bar m$, we can derive the equation satisfied by $\mu+\sigma\bar m$ and we deduce immediately that $\mu+\sigma\bar m\ge0$. Since $\int_{\mathbb{T}^d}\mu(t)=0$ for every t, this implies that $\|\mu(t)\|_{L^1(\mathbb{T}^d)}\le2\sigma$ for every $t>0$. Since H_p is globally bounded, and $\bar m\in L^\infty(\mathbb{T}^d)$, by standard (local in time) regularizing effect in the Fokker-Planck equation, we have $\|\mu(t)\|_\infty\le C\|\mu(t-1)\|_{L^1(\mathbb{T}^d)}$ for every $t>1$ (see e.g. [142, Chapter V]). In addition, since $m_0\in L^\infty(\mathbb{T}^d)$, we have that $\|\mu(t)\|_\infty$ is bounded for $t\in(0,1)$ as well. From the global L^1 bound, we conclude therefore that $\|\mu(t)\|_\infty$ is bounded uniformly, for every $t\in(0,T)$, by a constant independent of the horizon T. Due to (1.92), this means that the function $F(x,\cdot)$ in the first equation can be treated as uniformly bounded and Lipschitz. The global bound on the right-hand side, together with the global Lipschitz character of the Hamiltonian, and the fact that $v_T\in C^{1,\alpha}(\mathbb{T}^d)$ for some $\alpha\in(0,1)$, allow us to deduce the existence of a constant L, independent of T, such that $\|Dv(t)\|_\infty\le L$ for every $t\in(0,T)$. Due to (1.93), this means that $H(x,\cdot)$ can be treated as uniformly convex. Therefore, if we set

$$\begin{aligned}
h(x,p) &:= H(x, D\bar u(x)+p) - H(x, D\bar u(x))\\
f(x,\mu) &:= F(x, \bar m(x)+\mu) - F(x, \bar m(x))\\
B(x,p) &:= \bar m(x)\left[H_p(x, D\bar u(x)+p) - H_p(x, D\bar u(x))\right] ,
\end{aligned}$$

we have that (v,μ) solves the system

$$\begin{cases}
-\partial_t v - \varDelta v + h(x, Dv) = f(x,\mu)\\
v(T) = v_T\\
\partial_t \mu - \varDelta \mu - \mathrm{div}(\mu\, h_p(x,Dv)) = \sigma\, \mathrm{div}(B(x,Dv))\\
\mu(0) = \sigma\, \mu_0
\end{cases} \tag{1.97}$$

where $\mu_0 = m_0 - \bar m$, and where $h(x,p), f(x,s), B(x,p)$ satisfy the following conditions for some constants c_0, C_0, C_1, C_2 and for every $s \in \mathbb{R}$, $p \in \mathbb{R}^d$, $x \in \mathbb{T}^d$:

$$h(x,0) = 0\,, \quad |h_p(x,p)| \le c_0\,, \tag{1.98}$$

$$f(x,s)s \ge 0\,, \quad |f(x,s)| \le C_0\,, \quad |f(x,s)| \le C_1\,|s| \tag{1.99}$$

$$B(x,p)\cdot p \ge C_2^{-1}|p|^2\,, \qquad |B(x,p)| \le C_2\,|p|\,. \tag{1.100}$$

In addition, since $\mu(t,x) \ge -\sigma \bar m(x)$, we also have, for some constant γ_0,

$$\begin{aligned}
\sigma B(x,p)\cdot p - \mu(t,x)(h(x,p) - h_p(x,p)\cdot p) &\ge \sigma B(x,p)\cdot p - \sigma \bar m(x)(h_p(x,p)\cdot p - h(x,p))\\
&= \sigma \bar m(x)\left[-H_p(x, D\bar u(x))\cdot p + H(x, D\bar u(x)+p) - H(x, D\bar u(x))\right]\\
&\ge \sigma\,\gamma_0\,|p|^2 \qquad \forall (t,x)\in Q_T, \forall p \in \mathbb{R}^d\ : |p| \le L\,,
\end{aligned} \tag{1.101}$$

where we used the local uniform convexity of H and that $\bar m > 0$.

We now derive the exponential estimate for system (1.97) under conditions (1.98)-(1.101).

Given $T > 0$, $\sigma \in [0,1]$, $\mu_0 \in L^2(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} \mu_0 = 0$, $v_T \in H^1(\mathbb{T}^d)$, we denote by $S(T,\sigma,\mu_0,v_T)$ the solution (μ, v) of system (1.97). We will denote by $\langle v\rangle = \int_{\mathbb{T}^d} v$ and by $\tilde v = v - \langle v \rangle$. We first prove that there exists a constant C, independent of σ, T, μ_0, v_T, such that

$$\|\mu(t)\|_2 + \|Dv(t)\|_2 \le C(\|\mu_0\|_2 + \|Dv_T\|_2) \qquad \forall (\mu, v) \in S(T,\sigma,\mu_0,v_T)\,. \tag{1.102}$$

To prove (1.102), we observe that, due to (1.101),

$$\begin{aligned}
-\frac{d}{dt}\int_{\mathbb{T}^d} \mu(t)v(t) &= \int_{\mathbb{T}^d} f(x,\mu)\mu + \sigma \int_{\mathbb{T}^d} B(x,Dv)Dv - \int_{\mathbb{T}^d} \mu(h(x,Dv) - h_p(x,Dv)\cdot Dv)\\
&\ge \sigma\,\gamma_0 \int_{\mathbb{T}^d} |Dv|^2\,.
\end{aligned} \tag{1.103}$$

From the Fokker-Planck equation we also have (see e.g. [65, Lemma 1.1]) that there exists $\gamma, c > 0$:

$$\begin{aligned}
\|\mu(t)\|_2^2 &\le c\, e^{-\gamma t}\|\mu_0\|_2^2 + c\,\sigma^2 \int_0^t \int_{\mathbb{T}^d} |B(x,Dv)|^2\\
&\le c\, e^{-\gamma t}\|\mu_0\|_2^2 + c\,\sigma \int_0^t \int_{\mathbb{T}^d} |Dv|^2
\end{aligned}$$

where we used (1.100). Here and after we denote by c possibly different constants independent of T, σ, μ_0, v_T. Putting together the above inequalities we get

$$\|\mu(t)\|_2^2 \le c\, e^{-\gamma t}\|\mu_0\|_2^2 + c \int_{\mathbb{T}^d} \mu_0 v(0) - c\int_{\mathbb{T}^d} \mu(T) v_T$$

which implies

$$\sup_{[0,T]} \|\mu(t)\|_2^2 \leq c\,[\|\mu_0\|_2^2 + \|Dv_T\|_2^2] + c\,\|\mu_0\|_2\|\tilde v(0)\|_2\,. \tag{1.104}$$

Since the Hamilton-Jacobi equation implies (using $|f(x,\mu)| \leq c\,|\mu|$ and [65, Lemma 1.2])

$$\|\tilde v(0)\|_2 \leq c\,e^{-\gamma T}\|\tilde v_T\|_2 + c\int_0^T e^{-\gamma s}\|\mu(s)\|_2 ds \leq c[\|\tilde v_T\|_2 + \sup_{[0,T]}\|\mu(t)\|_2]$$

coming back to (1.104) we deduce that

$$\sup_{[0,T]} \|\mu(t)\|_2^2 \leq c\,[\|\mu_0\|_2^2 + \|\tilde v_T\|_2^2]\,.$$

A similar estimate holds for $\sup_{[0,T]}\|\tilde v(t)\|_2$ as well. Finally, using e.g. [65, Lemma 1.2] we have

$$\|\nabla v(t)\|_2^2 \leq c(\|\tilde v(t+1)\|_2^2 + c\int_t^{t+1}[\|\mu(s)\|_2^2 + \tilde v(s)^2])$$

hence

$$\|\nabla v(t)\|_2^2 \leq c\,[\|\mu_0\|_2^2 + \|\tilde v_T\|_2^2] \quad \forall t < T-1.$$

Standard parabolic estimates also imply that

$$\|\nabla v(t)\|_2^2 \leq c\,[\sup_{[T-1,T]}\|\mu(t)\|_2^2 + \|Dv_T\|_2^2] \quad \forall t \in [T-1,T]$$

so that (1.102) is proved. Now we set

$$\rho(\tau) := \sup_{T\geq 2\tau}\ \sup_{\sigma,\mu_0,v_T}\left\{\sup_{t\in[\tau,T-\tau]}\left|\frac{\int_{\mathbb{T}^d} v(t)\,\mu(t)}{[\|\mu_0\|_2 + \|Dv_T\|_2]^2}\right|\,,\quad (\mu,v)\in S(T,\sigma,\mu_0,v_T)\right\}\,.$$

We first remark that, by elementary inclusion property, one has $\rho(\tau+s)\leq\rho(\tau)$ for every $s>0$. Hence $\rho(\cdot)$ is a non increasing function and we can define

$$\rho(\infty) := \lim_{\tau\to\infty}\rho(\tau)\,.$$

As a first step, we shall prove that $\rho_\infty = 0$. To this purpose, we observe that by definition of ρ there exist sequences $\tau_n\to\infty$, $T_n\geq 2\tau_n$, $t_n\in[\tau_n, T_n-\tau_n]$, $\mu_0^n\in L^\infty(\mathbb{T}^d)$, $v_{T_n}^n\in W^{1,\infty}(\mathbb{T})$ and $\sigma_n\in[0,1]$ such that

$$(\mu^n, v^n)\in S(T_n,\sigma_n,\mu_0^n,v_{T_n}^n)\,,\qquad \left|\frac{\int_{\mathbb{T}^d}\mu^n(t_n)v^n(t_n)}{[\|\mu_0^n\|_2 + \|Dv_{T_n}^n\|_2]^2}\right| \geq \rho_\infty - 1/n.$$

We set, for $t\in[-t_n, T_n-t_n]$:

$$\tilde\mu^n(t,x) = \delta_n\mu^n(t_n+t,x),\ \ \tilde v^n(t,x) = \delta_n(v^n(t_n+t,x) - \langle v^n(t_n)\rangle)$$
$$\delta_n := \tfrac{1}{\|\mu_0^n\|_2 + \|Dv_{T_n}^n\|_2}$$

and we notice that $(\tilde\mu^n,\tilde v^n)$ solve the system

$$\begin{cases} -\tilde v_t^n - \Delta\tilde v^n + \delta_n\,h(x,Dv^n) = \delta_n\,f(x,\mu^n) \\ \tilde\mu_t^n - \Delta\tilde\mu^n - \operatorname{div}(\tilde\mu^n\,h_p(x,Dv^n)) = \delta_n\,\sigma_n\operatorname{div}(B(x,Dv^n)) \end{cases}$$

where v^n,μ^n are computed at t_n+t. By estimate (1.102), $\|\tilde\mu^n(t)\|_2$ and $\|D\tilde v^n(t)\|_2$ are uniformly bounded. Hence, due to (1.98) and (1.99), $-\partial_t\tilde v^n - \Delta\tilde v^n$ is uniformly bounded in $L^2(\mathbb{T}^d)$, which implies that $\tilde v^n$ is relatively compact in

$C^0([a,b];L^2(\mathbb{T}^d))$, for every interval $[a,b]$. In particular, there exists $\tilde v\in L^2_{loc}(\mathbb{R};L^2(\mathbb{T}^d))$ such that $\tilde v^n(t)\to\tilde v(t)$ in $L^2(\mathbb{T}^d)$ for every $t\in\mathbb{R}$, and $D\tilde v^n\to Dv$ in $L^2((a,b)\times\mathbb{T}^d)$ for every bounded interval (a,b). Let us call respectively $\tilde\mu,\sigma$ a limit of (a subsequence of) $\tilde\mu^n,\sigma_n$; since $\tilde\mu^n(t)$ weakly converges to $\mu(t)$ in $L^2(\mathbb{T}^d)$, we have that the scalar product $\int_{\mathbb{T}^d}\tilde\mu^n(t)\tilde v^n(t)$ converges for every $t\in\mathbb{R}$. It follows from (1.103) (integrated between t_n+t_1 and t_n+t_2) and from (1.100), that

$$\sigma\,\gamma_0\int_{t_1}^{t_2}\int_{\mathbb{T}^d}|D\tilde v|^2\le\liminf_{n\to\infty}\sigma_n\,\gamma_0\,\delta_n^2\int_{t_n+t_1}^{t_n+t_2}\int_{\mathbb{T}^d}|Dv^n|^2\le\int_{\mathbb{T}^d}\tilde\mu(t_1)\tilde v(t_1)-\int_{\mathbb{T}^d}\tilde\mu(t_2)\tilde v(t_2)\qquad(1.105)$$

for every fixed $t_1,t_2\in\mathbb{R}$. By construction, we also have

$$\rho_\infty-1/n\le\left|\frac{\int_{\mathbb{T}^d}\mu^n(t_n)v^n(t_n)}{[\|\mu_0^n\|_\infty+\|Dv^n_{T_n}\|_\infty]^2}\right|\le\rho(\tau_n)\to\rho_\infty\,,\qquad(1.106)$$

hence

$$\rho_\infty=\lim_{n\to\infty}\left|\int_{\mathbb{T}^d}\tilde\mu^n(0)\tilde v^n(0)\right|=\left|\int_{\mathbb{T}^d}\tilde\mu(0)\tilde v(0)\right|\,.$$

On another hand, for any $t\in\mathbb{R}$ and for n large enough, we have that $t_n+t\in[\tau_n-|t|,T_n-(\tau_n-|t|)]$, so that

$$\left|\int_{\mathbb{T}^d}\tilde\mu(t)\tilde v(t)\right|=\lim_n\left|\frac{\int_{\mathbb{T}^d}\mu^n(t_n+t)v^n(t_n+t)}{[\|\mu_0^n\|_\infty+\|Dv^n_{T_n}\|_\infty]^2}\right|\le\lim_n\rho(\tau_n-|t|)=\rho_\infty.\qquad(1.107)$$

Now suppose that $\rho_\infty>0$ and $\sigma>0$. If $\rho_\infty=\int_{\mathbb{T}^d}\tilde\mu(0)\tilde v(0)>0$; using (1.105) with $t_2=0$ we deduce, due to (1.107), that $\int_{t_1}^0\int_{\mathbb{T}^d}|D\tilde v|^2\le0$. This implies that $|\tilde v(0)|=0$. If $\rho_\infty=-\int_{\mathbb{T}^d}\tilde\mu(0)\tilde v(0)$, we get at the same conclusion by choosing now $t_1=0$ in (1.105). But $\tilde v(0)=0$ is impossible unless $\rho_\infty=0$. It remains the case that $\sigma=0$; this means that $\tilde\mu$ satisfies

$$\partial_t\tilde\mu-\Delta\tilde\mu-\mathrm{div}(\tilde\mu\,b)=0$$

for a bounded drift $b(t,x)$. But this readily leads to $\tilde\mu=0$ (because $\|\tilde\mu(t)\|\le e^{-\omega(t-t_0)}\|\tilde\mu(t_0)\|$ for all t_0,t and $\tilde\mu$ is uniformly bounded), and again this implies $\rho_\infty=0$.

So we proved that $\rho_\infty=0$. We claim now that this implies the existence of t_0 such that

$$\|\mu(t)\|_2+\|Dv(t)\|_2\le\frac12[\|\mu_0\|_2+\|Dv_T\|_2]\qquad\forall t\in[t_0,T-t_0]\,.\qquad(1.108)$$

In fact, using the Fokker-Planck equation and (1.103), for every $t\in[\tau,T-\tau]$ we have

$$\begin{aligned}\|\mu(t)\|_2^2&\le c\,e^{-\gamma(t-\tau)}\|\mu(\tau)\|_2^2+c\,\sigma^2\int_\tau^{T-\tau}\int_{\mathbb{T}^d}|B(x,Dv)|^2\\&\le c\,e^{-\gamma(t-\tau)}[\|\mu_0\|_2^2+\|Dv_T\|_2]^2+c\left\{\left|\int_{\mathbb{T}^d}\mu(\tau)v(\tau)\right|+\left|\int_{\mathbb{T}^d}\mu(T-\tau)v(T-\tau)\right|\right\}\,,\end{aligned}$$

hence

$$\|\mu(t)\|_2^2\le c[\|\mu_0\|_2^2+\|Dv_T\|_2]^2\left(e^{-\gamma(t-\tau)}+\rho(\tau)\right)\,.\qquad(1.109)$$

Similarly we have, using now the estimate for μ,

$$\begin{aligned}\|\tilde v(t)\|_2^2&\le c\,e^{-\gamma(T-\tau-t)}\|\tilde v(T-\tau)\|_2^2+c\int_t^{T-\tau}e^{-\gamma(s-t)}\|\mu(s)\|_2^2ds\\&\le c\,[\|\mu_0\|_2+\|Dv_T\|_2]^2\left(e^{-\gamma(T-\tau-t)}+e^{-\gamma(t-\tau)}+\rho(\tau)\right)\end{aligned}$$

which implies, for every $t\in(\tau,T-\tau-1)$:

$$\begin{aligned}\|\nabla v(t)\|_2^2 &\le c(\|\tilde v(t+1)\|_2^2 + c\int_t^{t+1}[\|\mu(s)\|_2^2 + \tilde v(s)^2]) \\ &\le c\,[\|\mu_0\|_2 + \|Dv_T\|_2]^2 \left(e^{-\gamma(T-\tau-t)} + e^{-\gamma(t-\tau)} + \rho(\tau)\right) .\end{aligned} \tag{1.110}$$

Since $\rho(\tau) \overset{\tau\to\infty}{\to} 0$, from (1.109)-(1.110) we obtain (1.108) by choosing τ and t conveniently. Finally, by iteration of (1.108), we deduce the exponential estimate (1.96). ☐

Proof of Theorem 14. Let us first assume that $m_0 \in C^\alpha(\mathbb{T}^d)$. We set $X = C^0([0,T]; L^2(\mathbb{T}))$ and we introduce the following norm in X:

$$|||u|||_X := \sup_{[0,T]} \left(\frac{\|u(t)\|_{L^2(\mathbb{T}^d)}}{e^{-\omega t} + e^{-\omega(T-t)}} \right)$$

where $\omega > 0$ is given by Lemma 14. It is easy to verify that $(X, |||u|||_X)$ is a Banach space and $|||\cdot|||$ is equivalent to the standard norm $\|u\| = \sup_{[0,T]} \|u(t)\|_{L^2(\mathbb{T}^d)}$.

We define the operator T on X as follows: given $\mu \in X$, let (v, ρ) be the solution to the system

$$\begin{cases} -v_t - \Delta v + H(x, D\bar u + Dv) - H(x, D\bar u) = F(x, \bar m + \mu) - F(x, \bar m) \\ v(T) = G(x, \bar m + \mu(T)) - \bar u \\ \rho_t - \Delta\rho - \operatorname{div}(\rho\, H_p(x, D\bar u + Dv)) = -\operatorname{div}(\bar m\, [H_p(x, D\bar u + Dv) - H_p(x, D\bar u)]) \\ \rho(0) = m_0 - \bar m \end{cases} \tag{1.111}$$

then we set $\rho = T\mu$. Since H_p is globally bounded, and $\bar m, m_0 \in C^{0,\alpha}$, by standard regularity results (see [142, Chapter V, Thms 1.1 and 2.1]) we notice that the range of T is bounded in $C^{\alpha/2,\alpha}(Q_T)$, in particular the range of T lies in a bounded set of L^∞ and its closure is compact in X. As a consequence, due to (1.92), there is no loss of generality if we consider $F(x, \cdot)$ to be globally bounded and Lipschitz. Using now the global bound on μ and proceeding as in Lemma 14, a global bound for $\|Dv(t)\|_2$ follows, and then, by (local) regularizing effect of parabolic equations, we deduce that there exists a constant $L > 0$ such that

$$\|Dv(t)\|_\infty \le L \qquad \forall t \le T-1\,, \qquad \forall \mu \in X\,. \tag{1.112}$$

We now check that the operator T is continuous: if $\mu_n \to \mu$ in X, then $\mu_n(T)$ is strongly convergent in $L^2(\mathbb{T}^d)$, and $F(x, \bar m + \mu_n) - F(x, \bar m)$ converges in $L^2(Q_T)$ as well. By standard parabolic theory, we have that Dv_n converges in $L^2(Q_T)$ to Dv where v is a solution corresponding to μ. The convergence of Dv_n in L^2 and the boundedness of H_p imply that the drift and source terms in the equation of ρ_n converge in $L^p(Q_T)$ for every $p < \infty$. This immediately implies the convergence of ρ_n in $L^2(0,T; H^1(\mathbb{T}^d))$ and then in $C^0([0,T]; L^2(\mathbb{T}^d))$ as well. By uniqueness, we deduce that ρ_n converges to $T\mu$. This concludes the continuity of T. Thus, T is a compact and continuous operator. We are left with the following claim: there exists a constant $M > 0$ such that

$$|||\mu||| \le M \quad \text{for every } \mu \in X \text{ and every } \sigma \in [0,1] \text{ such that } \mu = \sigma T(\mu). \tag{1.113}$$

In order to prove (1.113), we use Lemma 14 in the interval $(0, T-1)$; indeed if $\mu = \sigma T(\mu)$, then (μ, v) is a solution to (1.95) with $v_{T-1} = v(T-1)$. Therefore, there exists $K > 0$ (only depending on $\|m_0\|_\infty$, and $F, H, \bar u, \bar m$) such that

$$\|\mu(t)\|_2 + \|Dv(t)\|_2 \le K(e^{-\omega t} + e^{-\omega(T-t)}) \qquad \forall t \in (0, T-1)\,.$$

Since $\|\mu(t)\|_2$ is uniformly bounded for $t \in (T-1, T)$, we conclude that (1.113) holds true for some $M > 0$. By the Schaefer's fixed point theorem ([110, Thm 11.3]), we conclude that there exists $\mu \in X$ such that $\mu = T\mu$. Setting $m = \bar m + \mu$, $u = \bar u + \bar\lambda(T-t) + v$, we have found a solution of the MFG system (1.62) which satisfies the estimate

$$\|m(t) - \bar m\|_2 + \|Du(t) - D\bar u\|_2 \le C(e^{-\omega t} + e^{-\omega(T-t)}) \qquad \forall t \in (0, T-1)\,.$$

To conclude with the general case, let $m_0 \in \mathcal{P}(\mathbb{T}^d)$ and let (u, m) be any solution to system (1.62). By mass conservation and the global Lipschitz bound (1.64), there exists $\alpha \in (0, 1)$ and a constant C, only depending on β, such that

$$\|m(t)\|_{C^\alpha(\mathbb{T}^d)} \leq C \qquad \forall t \geq \frac{1}{2}\,.$$

In turn, this implies that

$$\|Du(t)\|_\infty \leq C \qquad \forall t \leq T - \frac{1}{2}\,.$$

for a possibly different constant only depending on β, F, G. By monotonicity of $F(x, \cdot)$, (u, m) is the unique solution of the MFG system in $(\frac{1}{2}, T - \frac{1}{2})$ with initial-terminal conditions given by $m(\frac{1}{2})$ and $u(T - \frac{1}{2})$ respectively. By the first part of the proof, we know that this unique MFG solution satisfies the exponential turnpike estimate. Hence there exists $M > 0$ such that

$$\|m^T(t) - \bar{m}\|_2 + \|Du^T(t) - D\bar{u}\|_2 \leq M(e^{-\omega t} + e^{-\omega(T-t)}) \qquad \forall t \in (1/2, T - 1/2)\,.$$

Using the regularizing effect of the two equations, this estimate is upgraded to L^∞-norms and yields (1.94). □

Let us stress that the turnpike estimate (1.94) gives an information in a long intermediate time of the interval $(0, T)$. A stronger result can also be obtained, by showing the convergence of $(u^T(t), m^T(t))$ *at any time scale*, i.e. for every $t \in (0, T)$. More precisely, there exists (u, m) solution of the problem in $(0, \infty)$:

$$\begin{cases} -\partial_t u + \bar{\lambda} - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, \infty) \times \mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, \infty) \times \mathbb{T}^d, \\ m(0) = m_0\,, \qquad Du \in D\bar{u} + L^2((0, \infty) \times \mathbb{T}^d)\,,\ u \text{ bounded} \end{cases} \tag{1.114}$$

such that

$$u^T(t) + \bar{\lambda}(T - t) \overset{T \to \infty}{\to} u(t) \qquad ; \qquad m^T(t) \overset{T \to \infty}{\to} m(t)$$

where the convergence is uniform (locally in time). We notice that, since $F(x, \cdot)$ is nondecreasing, there is a unique m which solves problem (1.114), while u is unique up to addition of a constant. Nevertheless the above convergence holds for the whole sequence $T \to \infty$. i.e. there is a unique solution u of the infinite horizon problem which is selected in the limit of $u^T(t) + \bar{\lambda}(T - t)$. We also point out that $\bar{m}$ is the unique invariant measure of the Fokker-Planck equation (hence $m(t) \to \bar{m}$ as $t \to \infty$). Finally, the same problem (1.114) is obtained as the limit of the discounted MFG problem when the discount factor vanishes. The discounted (infinite horizon) problem is described by the system

$$\begin{cases} -\partial_t u + \delta u - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, \infty) \times \mathbb{T}^d, \\ \partial_t m - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, \infty) \times \mathbb{T}^d, \\ m(0) = m_0\,,\ \ u \text{ bounded} \end{cases}$$

and corresponds to the following minimization problem for each agent:

$$\mathcal{J}(x, \alpha) = \inf_\alpha \mathbb{E}\left[\int_0^{+\infty} e^{-\delta t}\,(H^*(X_t, -\alpha_t) + F(X_t, m(t)))\,dt\right]$$

where $\delta > 0$ is a fixed discount rate. In case of monotone couplings, the limit as $\delta \to 0$ produces a unique solution of (1.114) and is, once more, related to the ergodic behavior of the controlled system. We refer to [65] where all the above mentioned results are proved for smoothing couplings in connection with the long time behavior and the ergodic properties of the master equation. A different proof is also given in [91] for local couplings and Lipschitz Hamiltonians.

We conclude by mentioning that the aforementioned results, and specifically the exponential convergence, mostly rely on the presence of the diffusion term in the equations (the individual noise of the agents). Indeed, in case of first order MFG systems, only partial results are known, even in case of monotone couplings. The typical result proved so far consists in the long time average convergence towards the ergodic first order system, see [52], [53] for the case of, respectively, smoothing and local couplings.

Remark 15 It is well-known that the ergodic behavior of Hamilton-Jacobi equations has strict connections with the study of homogenization problems. To this respect, the study of MFG systems is still largely open. MFG problems with fast oscillation in the space variable (homogenization) are studied in [151], [83]. Interestingly, the monotonicity structure of MFG might be lost after homogenization (although very recent results by Lions show that some structure is preserved).

1.3.7 The vanishing viscosity limit and the first order system with local couplings.

1.3.7.1 Existence and uniqueness of solutions

We now analyze the vanishing viscosity limit of weak solutions. Compared to the case of smoothing couplings, now we cannot rely anymore on the semi concavity estimates for u, and the relaxed solutions obtained for the deterministic problem fall outside the viscosity solutions setting. However, the monotonicity of the couplings, the coercivity of the Hamiltonian and, eventually, the stability properties of the system, will allow us to handle the two equations in a purely distributional sense.

To fix the ideas, we still assume that the Hamiltonian $H(x,p)$ is convex and C^1 with respect to p and satisfies assumptions (1.66)-(1.68). We also assume that $F, G \in C(\mathbb{T}^d \times \mathbb{R}_+)$ are nondecreasing functions of m which verify, for some constants $C_i > 0$,

$$\begin{aligned} &\exists\, f \in C(\mathbb{R}_+, \mathbb{R}_+) \text{ nondecreasing, with } \lim_{s\to+\infty} f(s) = +\infty \text{ and } f(s)s \text{ convex, such that} \\ &\qquad C_0\, f(m) - C_1 \le F(x,m) \le f(m) + C_1\,, \qquad \forall (x,m) \in \mathbb{T}^d \times \mathbb{R}_+ \end{aligned} \tag{1.115}$$

$$\begin{aligned} &\exists\, g \in C(\mathbb{R}_+, \mathbb{R}_+) \text{ nondecreasing, with } \lim_{s\to+\infty} g(s) = +\infty \text{ and } g(s)s \text{ convex, such that} \\ &\qquad C_2\, g(m) - C_3 \le G(x,m) \le g(m) + C_3\,, \qquad \forall (x,m) \in \mathbb{T}^d \times \mathbb{R}_+\,. \end{aligned} \tag{1.116}$$

Of course the simplest example occurs when $f(s)$ and $g(s)$ are power-type functions, as considered e.g. in [58]. Both nonlinearities F and H could also depend (in a measurable way) on t, but this would not add any change in the following, so we omit this dependence to shorten notations.

The key point here is to consider the duality between weak *sub solutions* of Hamilton-Jacobi equation and weak solutions of the continuity equation. This topic has an independent interest for PDEs especially in connection with the theory of optimal transport.

Definition 6 Given $f \in L^1(Q_T)$, $g \in L^1(\mathbb{T}^d)$, a function $u \in L^2(0,T; H^1(\mathbb{T}^d))$ is a weak sub solution of

$$\begin{cases} -\partial_t u + H(x, Du) = f(t,x) \\ u(T,x) = g(x) \end{cases} \tag{1.117}$$

if it satisfies

$$\int_0^T\!\!\int_{\mathbb{T}^d} u\, \partial_t \varphi + \int_0^T\!\!\int_{\mathbb{T}^d} H(x,Du)\, \varphi \le \int_0^T\!\!\int_{\mathbb{T}^d} f\, \varphi + \int_{\mathbb{T}^d} g \varphi(T) \qquad \forall \varphi \in C^1_c((0,T] \times \mathbb{T}^d)\,,\ \varphi \ge 0\,. \tag{1.118}$$

Hereafter, we will shortly write $-\partial_t u + H(x,Du) \le f$ and $u(T) \le g$ to denote the previous inequality.

Let us point out that, since H is bounded below thanks to condition (1.66), any sub solution u according to the above definition is time-increasing up to an absolutely continuous function; in particular, u admits a right-continuous Borel representative and admits one-sided limits at any $t \in [0,T]$. We refer the reader to [158, Section 4.2] for the analysis of trace properties of u. We will use in particular the existence of a trace at time $t = 0$ for u; this trace should be understood in the sense of limits of measurable functions (convergence in measure of $u(t, x)$ as $t \to 0^+$).

Definition 7 Let $m_0 \in \mathcal{P}(\mathbb{T}^d)$. Given a measurable vector field $b : Q_T \to \mathbb{R}$, a function $m \in L^1(Q_T)$ is a weak solution of the continuity equation

$$\begin{cases} \partial_t m - \operatorname{div}(m\, b) = 0 \\ m(0) = m_0 \end{cases} \tag{1.119}$$

if $m \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$, $\int_0^T\!\int_{\mathbb{T}^d} m\, |b|^2 < \infty$ and the distributional equality holds:

$$-\int_0^T\!\!\int_{\mathbb{T}^d} m\, \partial_t \varphi + \int_0^T\!\!\int_{\mathbb{T}^d} m\, b \cdot D\varphi = \int_{\mathbb{T}^d} m_0 \varphi(0) \qquad \forall \varphi \in C_c^1([0, T) \times \mathbb{T}^d)\,. \tag{1.120}$$

Let us recall that the requirement that $\int_0^T\!\int_{\mathbb{T}^d} m\, |b|^2 < \infty$ is very natural in the framework of weak solutions to the continuity equation, and this is related with the fact that $m(t)$ is an absolutely continuous curve in $\mathcal{P}(\mathbb{T}^d)$ with L^2 metric velocity, see [19].

Standing on the above two definitions, we have a weak setting for the deterministic MFG system. For simplicity, we restrict hereafter to the case that $m_0 \in L^1(\mathbb{T}^d)$.

Definition 8 A pair $(u, m) \in L^2(0, T; H^1(\mathbb{T}^d)) \times L^1(Q_T)_+$ is a weak solution to the first order MFG system

$$\begin{cases} -\partial_t u + H(x, Du) = F(x, m) \\ u(T) = G(x, m(T)) \end{cases} \tag{1.121}$$

$$\begin{cases} \partial_t m - \operatorname{div}(m\, H_p(x, Du)) = 0 \\ m(0) = m_0 \end{cases} \tag{1.122}$$

if

(i) $F(x, m)m \in L^1(Q_T)$, $G(x, m(T))m(T) \in L^1(\mathbb{T}^d)$, $m|Du|^2 \in L^1(Q_T)$, and u is bounded below

(ii) u is a weak sub solution of (1.121), $m \in C^0([0, T]; \mathcal{P}(\mathbb{T}^d))$ is a weak solution of (1.122)

(iii) u and m satisfy the following identity:

$$\begin{aligned} \int_{\mathbb{T}^d} m_0\, u(0)\, dx = \int_{\mathbb{T}^d} G(x, m(T))\, m(T)\, dx + \int_0^T\!\!\int_{\mathbb{T}^d} F(t, x, m) m\, dxdt \\ + \int_0^T\!\!\int_{\mathbb{T}^d} m\, \left[H_p(x, Du) \cdot Du - H(x, Du)\right] dxdt \end{aligned} \tag{1.123}$$

A key point is played by the following lemma, which justifies the duality between weak sub solutions of the Hamilton-Jacobi equation and weak solutions of the continuity equation. This also gives sense to the first term in (1.123), where we recall that the value $u(0)$ is the trace of $u(t)$ as explained before.

In the following, for a convex super linear function $\phi : \mathbb{R}^d \to \mathbb{R}$, we denote by ϕ^* its Legendre transform defined as $\phi^*(q) = \sup_{p \in \mathbb{R}^d}[q \cdot p - \phi(p)]$.

Lemma 15 *Let u be a weak sub solution of (1.117) and m be a weak solution of (1.119). Assume that f, g, u are bounded below and there exist convex increasing and superlinear functions ϕ_1, ϕ_2 such that $\phi_1(m), \phi_1^*(f) \in L^1(Q_T)$ and $\phi_2(m(T)), \phi_2^*(g) \in L^1(\mathbb{T}^d)$.*

Then we have $m|Du|^2 \in L^1(Q_T)$, $u(0)m_0 \in L^1(\mathbb{T}^d)$ and

$$\int_{\mathbb{T}^d} m_0\, u(0)\, dx \le \int_{\mathbb{T}^d} g\, m(T)\, dx + \int_0^T\!\!\int_{\mathbb{T}^d} f\, m\, dxdt + \int_0^T\!\!\int_{\mathbb{T}^d} m\, \left[b \cdot Du - H(x, Du)\right] dxdt \tag{1.124}$$

Proof. Let $\rho_\delta(\cdot)$ be a sequence of standard symmetric mollifiers in $\mathbb{R}^d$. We set $m_\delta(t,x) = m(t) \star \rho_\delta$. We also take a sequence of 1-d mollifiers $\xi_\varepsilon(t)$ such that $\mathrm{supp}(\xi_\varepsilon) \subset (-\varepsilon, 0)$, and we set

$$m_{\delta,\varepsilon} := \int_0^T \xi_\varepsilon(s-t)\, m_\delta(s) ds = \int_0^T \int_{\mathbb{R}^N} m(s,y)\xi_\varepsilon(s-t)\rho_\delta(x-y)\, dyds\,.$$

Notice that this function vanishes near $t = 0$, so we can take it as test function in (1.118). We get

$$\int_0^T\int_{\mathbb{T}^d} u\,\partial_t m_{\delta,\varepsilon} + \int_0^T\int_{\mathbb{T}^d} H(x,Du)m_{\delta,\varepsilon} \le \int_0^T\int_{\mathbb{T}^d} f\, m_{\delta,\varepsilon} + \int_{\mathbb{T}^d} g\, m_{\delta,\varepsilon}(T)\,. \tag{1.125}$$

The first integral is equal to $\int_0^T\int_{\mathbb{T}^d} -\partial_s u_{\delta,\varepsilon} m(s,y)\, dsdy$, where $u_{\delta,\varepsilon}(s,y) = \int_0^T\int_{\mathbb{T}^d} u(t,x)\xi_\varepsilon(s-t)\rho_\delta(x-y)\, dtdx$. Notice that this function vanishes near $s = T$ so it can be used as test function in (1.120). Therefore we have

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} u(t,x)\,\partial_t m_{\delta,\varepsilon}(t,x)\, dxdt &= -\int_0^T\int_{\mathbb{T}^d} m_{\delta,\varepsilon}(s,y)\,\partial_s u_{\delta,\varepsilon}(s,y)\, dsdy \\ &= -\int_0^T\int_{\mathbb{T}^d} m(s,y)b(s,y)\cdot D_y u_{\delta,\varepsilon}\, dsdy + \int_{\mathbb{T}^d} m_0(y)u_{\delta,\varepsilon}(0,y)dy\,.\end{aligned}$$

We shift the convolution kernels from u to m in the right-hand side and we use this equality in (1.125). We get

$$\begin{aligned}&-\int_0^T\int_{\mathbb{T}^d} Du\cdot w_{\delta,\varepsilon} + \int_0^T\int_{\mathbb{T}^d} H(x,Du)m_{\delta,\varepsilon} + \int_{\mathbb{T}^d} (m_0\star\rho_\delta)\left(\int_0^T u(t)\xi_\varepsilon(-t)\, dt\right) \\ &\le \int_0^T\int_{\mathbb{T}^d} f\, m_{\delta,\varepsilon} + \int_{\mathbb{T}^d} g\, m_{\delta,\varepsilon}(T)\end{aligned} \tag{1.126}$$

where we denote $w_\delta = [(b\,m)\star\rho_\delta]$ and $w_{\delta,\varepsilon}(t,x) = \int_0^T w_\delta(s,x)\,\xi_\varepsilon(s-t)\, ds$.

Now we let first $\varepsilon \to 0$, and then $\delta \to 0$. Since u is time increasing (up to an absolutely continuous function), we have

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{T}^d} (m_0\star\rho_\delta)\left(\int_0^T u(t)\xi_\varepsilon(-t)\, dt\right) \ge \int_{\mathbb{T}^d} (m_0\star\rho_\delta)\, u(0)$$

and since u is bounded below once we let $\delta \to 0$ we have $m_0\, u(0) \in L^1(\mathbb{T}^d)$ and

$$\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0} \int_{\mathbb{T}^d} (m_0\star\rho_\delta)\left(\int_0^T u(t)\xi_\varepsilon(-t)\, dt\right) \ge \int_{\mathbb{T}^d} m_0\, u(0)\,. \tag{1.127}$$

Using the time continuity of m into $\mathcal{P}(\mathbb{T}^d)$, we have

$$\|m_{\delta,\varepsilon}(T) - (m(T)\star\rho_\delta)\|_\infty \le \|D\rho_\delta\|_\infty \int_0^T \xi_\varepsilon(s-T)\mathbf{d}_1(m(s),m(T))ds \overset{\varepsilon\to 0}{\to} 0\,,$$

so we handle the term at $t = T$:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^d} g\, m_{\delta,\varepsilon}(T) = \int_{\mathbb{T}^d} g(m(T)\star\rho_\delta)\,.$$

Now we can pass to the limit in the last term due to Lebesgue's theorem, since by the assumptions we dominate

$$g(m(T)\star\rho_\delta) \le \phi_2^*(g) + \phi_2(m(T)\star\rho_\delta) \le \phi_2^*(g) + \phi_2(m(T))\star\rho_\delta$$

where we used Jensen's inequality in the last step. Since $\phi_2^*(g), \phi_2(m(T)) \in L^1(\mathbb{T}^d)$, last term strongly converges in $L^1(\mathbb{T}^d)$, and since g is also bounded below we deduce that $|g(m(T)\star\rho_\delta)|$ is dominated in $L^1(\mathbb{T}^d)$. Therefore

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{T}^d} g\, m_{\delta,\varepsilon}(T) = \int_{\mathbb{T}^d} g\, m(T)\,. \tag{1.128}$$

We reason in a similar way for the term with f, which satisfies, for some constant c_0,

$$c_0\, m_{\delta,\varepsilon} \le f\, m_{\delta,\varepsilon} \le \phi_1^*(f) + (\phi_1(m)\star\rho_{\delta,\varepsilon})\,.$$

By dominated convergence again we deduce

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^d} f\, m_{\delta,\varepsilon} = \int_0^T\int_{\mathbb{T}^d} f\, m\,. \tag{1.129}$$

Finally, using (1.66) we have

$$-Du\cdot w_{\delta,\varepsilon} + H(x,Du)m_{\delta,\varepsilon} \ge \frac{\alpha}{2}\, m_{\delta,\varepsilon}|Du|^2 - C\,\left[m_{\delta,\varepsilon} + \frac{|w_{\delta,\varepsilon}|^2}{m_{\delta,\varepsilon}}\right]$$

Now we define the lower semi-continuous function Ψ on $\mathbb{R}^N\times\mathbb{R}$ by

$$\Psi(w,m) = \begin{cases} \frac{|w|^2}{m} & \text{if} \quad m>0,\\ 0 & \text{if} \quad m=0 \text{ and } w=0,\\ +\infty & \text{otherwise,}\end{cases} \tag{1.130}$$

and we observe that Ψ is convex in the couple (w,m). So by Jensen inequality we have $\frac{|w_{\delta,\varepsilon}|^2}{m_{\delta,\varepsilon}} \le (\frac{|w|^2}{m})\star\rho_{\delta,\varepsilon}$. Recalling that $w = b\,m$ in our setting (hence $\frac{|w|^2}{m} = m\,|b|^2$), we deduce that

$$-Du\cdot w_{\delta,\varepsilon} + H(x,Du)m_{\delta,\varepsilon} \ge \frac{\alpha}{2}\, m_{\delta,\varepsilon}|Du|^2 - C\,\left[m_{\delta,\varepsilon} + (|b|^2\, m)\star\rho_{\delta,\varepsilon}\right]\,.$$

From the previous inequality we are allowed to use Fatou's lemma as $\varepsilon,\delta\to 0$, obtaining

$$\liminf_{\delta\to 0}\liminf_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^d}[-Du\cdot w_{\delta,\varepsilon} + H(x,Du)m_{\delta,\varepsilon}] \ge \int_0^T\int_{\mathbb{T}^d} m[-Du\cdot b + H(x,Du)] \tag{1.131}$$

and we also deduce in between that $m|Du|^2\in L^1(Q_T)$. Finally, collecting (1.127), (1.128), (1.129) and (1.131), we obtain (1.124). □

We are now able to discuss the vanishing viscosity limit of the MFG system. In the following, we will make use of the family of Young measures generated by the sequence $\{m^\varepsilon\}$. To this purpose, we recall the fundamental result concerning Young measures ([181]), see e.g. [22], [161]. Here $\mathcal{P}(\mathbb{R})$ denotes the space of probability measures on $\mathbb{R}$.

Proposition 3 *Let Q be a bounded subset in $\mathbb{R}^N$, and let $\{w_n\}$ be a sequence which is weakly converging in $L^1(Q)$. Then there exists a subsequence $\{w_{n_k}\}$ and a weakly* measurable function $\nu : Q\mapsto\mathcal{P}(\mathbb{R})$ such that if $f(y,s)$ is a Carathéodory function and $\{f(y,w_{n_k}(y))\}$ is an equi-integrable sequence in $L^1(Q)$, then*

$$f(y,w_{n_k}(y)) \rightharpoonup \bar f(y) \quad \textit{weakly in } L^1(Q)\textit{, where} \quad \bar f(y) = \int_{\mathbb{R}} f(y,\lambda)d\nu_y(\lambda)\,.$$

□

Theorem 15 *Assume that F,G satisfy (1.115)-(1.116), and that $H(x,p)$ satisfies (1.66)–(1.68). Let $m_0\in L^\infty(\mathbb{T}^d)$ and let $(u^\varepsilon,m^\varepsilon)$ be a solution of (1.62). Then there exists a subsequence, not relabeled, and a couple $(u,m)\in L^2(0,T;H^1(\mathbb{T}))\times L^1(Q_T)$ such that $(u^\varepsilon,m^\varepsilon)\to(u,m)$ in $L^1(Q_T)$, and (u,m) is a weak solution to (1.121)–(1.122) in the sense of Definition 8.*

Proof. By Lemma 13, $(u^\varepsilon, m^\varepsilon)$ satisfy the a priori estimates (1.69). On account of conditions (1.66)-(1.68), this implies that there exists a constant C, independent of ε, such that

$$\int_0^T\int_{\mathbb{T}^d} F(x,m^\varepsilon)m^\varepsilon + \int_{\mathbb{T}^d} G(x,m^\varepsilon(T))m^\varepsilon(T) + \int_0^T\int_{\mathbb{T}^d} m^\varepsilon\,|Du^\varepsilon|^2 + \int_0^T\int_{\mathbb{T}^d} |Du^\varepsilon|^2 \le C\,.$$

Hence there exists a subsequence, not relabeled, and a function $u \in L^2(0,T;H^1(\mathbb{T}^d))$ such that $u^\varepsilon \to u$ weakly in $L^2(0,T;H^1(\mathbb{T}^d))$. Notice that, since $\partial_t u^\varepsilon$ is bounded in $L^2(0,T;(H^1(\mathbb{T}^d))') + L^1(Q_T)$, by standard compactness results the convergence of u^ε to u is strong in $L^2(Q_T)$. Moreover, since F, G are bounded below, by maximum principle we also have that u^ε is bounded below.

As for m^ε, from (1.115) we have that $f(m^\varepsilon)m^\varepsilon$ is bounded in $L^1(Q_T)$. This implies that m^ε is equi-integrable and so, by Dunford-Pettis theorem, it is relatively compact in the weak topology of $L^1(Q_T)$; there exists a subsequence, not relabeled, and a function $m \in L^1(Q_T)$ such that $m^\varepsilon \to m$ weakly in $L^1(Q_T)$. Let us denote by $\nu_{(t,x)}(\cdot)$ the family of Young measures generated by m^ε, according to Proposition 3. Since $F(x,m^\varepsilon)m^\varepsilon$ is bounded in $L^1(Q_T)$, then $F(x,m^\varepsilon)$ is equi-integrable and then we have

$$F(x,m^\varepsilon) \to \bar f \quad \text{weakly in } L^1(Q_T), \text{ where } \quad \bar f = \int_{\mathbb{R}} F(x,\lambda)d\nu_{(t,x)}(\lambda)\,.$$

We notice that the bound on $F(x,m^\varepsilon)m^\varepsilon$ implies that $\int_{\mathbb{R}} F(x,\lambda)\lambda d\nu_{(t,x)}(\lambda) \in L^1(Q_T)$. Indeed, applying Proposition 3 to the function $F(x,m)T_k(m)$, where $T_k(m) = \min(m,k)$, implies

$$\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}} F(x,\lambda)T_k(\lambda)\,d\nu_{(t,x)}(\lambda) = \lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^d} F(x,m^\varepsilon)T_k(m^\varepsilon) \le C\,,$$

and then, by letting $k \to \infty$, by monotone convergence we get

$$\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}} F(x,\lambda)\lambda\,d\nu_{(t,x)}(\lambda) < \infty\,. \tag{1.132}$$

Similarly we reason for the sequence $m^\varepsilon(T)$. This is equi-integrable in $L^1(\mathbb{T}^d)$ and then, up to subsequences, it converges weakly in $L^1(\mathbb{T}^d)$; in addition, denoting $\{\gamma_x(\cdot)\}$ the sequence of Young measures generated by $m^\varepsilon(T)$, we have that $G(x,m^\varepsilon(T))$ is weakly relatively compact in $L^1(\mathbb{T}^d)$ and

$$G(x,m^\varepsilon) \to \bar g \quad \text{weakly in } L^1(Q_T), \text{ where } \quad \bar g = \int_{\mathbb{R}} G(x,\lambda)d\gamma_x(\lambda)\,.$$

In addition, as before, we deduce that $\int_{\mathbb{R}} G(x,\lambda)\lambda d\gamma_x(\lambda) \in L^1(\mathbb{T}^d)$.

We can now pass to the limit in the two equations. As for the HJ equation, since $p \mapsto H(x,p)$ is convex, then by weak lower semi-continuity we deduce that u satisfies

$$\begin{cases} -\partial_t u + H(x,Du) \le \bar f \\ u(T) \le \bar g \end{cases} \tag{1.133}$$

in the sense of Definition 6. As for m^ε, we observe that (1.69) and (1.67) imply that $m^\varepsilon\,|H_p(x,Du^\varepsilon)|^2$ is bounded in $L^1(Q_T)$. It follows (see also Remark 6) that $d_2(m^\varepsilon(t),m^\varepsilon(s)) \le C|t-s|^{\frac12}$, where d_2 is the Wasserstein distance in $\mathcal{P}(\mathbb{T}^d)$. Therefore, $m^\varepsilon(t)$ is equi-continuous and converges uniformly in the weak* topology. This implies that the L^1- weak limit m belongs to $C^0([0,T];\mathcal{P}(\mathbb{T}^d))$, and $m(0) = m_0$. Finally, $m^\varepsilon\,H_p(x,Du^\varepsilon)$ is equi-integrable and therefore weakly converges (up to subsequences) in $L^1(Q_T)$ to some vector field w. If Ψ is defined in (1.130), we deduce

$$\int_0^T\int_{\mathbb{T}^d}\Psi(w,m) \le \liminf_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^d}\Psi(m^\varepsilon,w^\varepsilon) = \int_0^T\int_{\mathbb{T}^d} m^\varepsilon\,|H_p(x,Du^\varepsilon)|^2 \le C$$

hence $\Psi(w,m) \in L^1(Q_T)$. In particular we can set $b := \frac{w}{m} 1_{\{m>0\}}$, then m is a weak solution of (1.119), with $m\,|b|^2 \in L^1(Q_T)$. Eventually, since m^ε weakly converges to m and $f(s)s$ is convex, by lower semicontinuity we deduce that

$$\int_0^T\!\!\int_{\mathbb{T}^d} f(m)m \le \liminf_{\varepsilon\to 0} \int_0^T\!\!\int_{\mathbb{T}^d} f(m^\varepsilon)m^\varepsilon \le \int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)m^\varepsilon + C_0 \le C\,.$$

Similarly we have for $m^\varepsilon(T)$, hence we conclude that

$$f(m)m \in L^1(Q_T)\,, \qquad g(m(T))m(T) \in L^1(\mathbb{T}^d)\,.$$

Now we observe that, using the monotonicity of $F(x,\cdot)$ and condition (1.115) we can estimate

$$\begin{aligned}\bar f &= \int_{\mathbb{R}} F(x,\lambda) d\nu_{(t,x)}(\lambda) \le F(x,s) + \frac{1}{s}\int_{\mathbb{R}} F(x,\lambda)\lambda\, d\nu_{(t,x)}(\lambda)\\ &\le f(s) + C_1 + \frac{1}{s}\int_{\mathbb{R}} F(x,\lambda)\lambda\, d\nu_{(t,x)}(\lambda)\,.\end{aligned}$$

hence

$$[\bar f - C_1]s - f(s)s \le \int_{\mathbb{R}} F(x,\lambda)\lambda\, d\nu_{(t,x)}(\lambda) \qquad \forall s\ge 0\,.$$

Recall that $f(s)s$ is convex and the right-hand side belongs to $L^1(Q_T)$; we deduce from the above inequality that $\phi_1^*(\bar f - C_1) \in L^1(Q_T)$, where ϕ_1^* is the convex conjugate of $\phi_1(s) := f(s)s$. Similarly we reason for $\bar g$, obtaining that $\phi_2^*(\bar g - C_3) \in L^1(Q_T)$ where $\phi_2(s) = g(s)s$. Notice that the addition of constants to $\bar f, \bar g$ in (1.133) is totally innocent up to replacing u with $u + a(T-t) + b$. Collecting all the above properties, we can apply Lemma 15 to u and m and we obtain that the following inequality holds:

$$\int_{\mathbb{T}^d} m_0\, u(0) \le \int_{\mathbb{T}^d} \bar g\, m(T) + \int_0^T\!\!\int_{\mathbb{T}^d} \bar f\, m + \int_0^T\!\!\int_{\mathbb{T}^d} m\,\left[b\cdot Du - H(x,Du)\right]\,. \tag{1.134}$$

Now we conclude by identifying the weak limits $\bar f, \bar g$ and b. We start from the equality (1.70)

$$\int_{\mathbb{T}^d} G(x,m^\varepsilon(T))m^\varepsilon(T) + \int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)m^\varepsilon + \int_0^T\!\!\int_{\mathbb{T}^d} m^\varepsilon\left[H_p(x,Du^\varepsilon)\cdot Du^\varepsilon - H(x,Du^\varepsilon\right] = \int_{\mathbb{T}^d} m_0\, u^\varepsilon(0)\,.$$

We observe that $u^\varepsilon(0)$ is equi-integrable: indeed, $(u^\varepsilon - k)_+$ is a sub solution of the Bellman equation, so that

$$\int_{\mathbb{T}^d} (u^\varepsilon(0)-k)_+ + \int_0^T\!\!\int_{\mathbb{T}^d} H(x,Du^\varepsilon)1_{\{u^\varepsilon>k\}} \le \int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)1_{\{u^\varepsilon>k\}} + \int_{\mathbb{T}^d} (G(x,m^\varepsilon(T))-k)_+\,.$$

Hence the bound from below of H (see (1.66)) and the equi-integrability of $F(x,m^\varepsilon)$, $G(x,m^\varepsilon(T))$ imply that $\int_{\mathbb{T}^d}(u^\varepsilon(0)-k)_+ \to 0$ as $k\to\infty$ uniformly with respect to ε. This implies that $u^\varepsilon(0)$ is equi-integrable, and then it weakly converges in $L^1(\mathbb{T}^d)$ to some function χ. In particular, when we pass to the limit in (1.62), we have

$$\int_{\mathbb{T}^d} \varphi(0)\,\chi + \int_0^T\!\!\int_{\mathbb{T}^d} u\varphi_t + \int_0^T\!\!\int_{\mathbb{T}^d} H(x,Du)\varphi \le \int_0^T\!\!\int_{\mathbb{T}^d} \bar f\,\varphi + \int_{\mathbb{T}^d} \bar g\,\varphi(T) \qquad \forall\varphi\in C^1(\overline{Q_T})\,,\ \varphi\ge 0\,.$$

By choosing a sequence φ_j such that $\varphi_j(0) = 1$ and φ_j approximates the Dirac mass at $t = 0$, we conclude that $\chi \le u(0)$, where $u(0)$ is the trace of u at time $t = 0$ in the sense explained above. Finally, we have

$$\begin{aligned}&\int_{\mathbb{T}^d} G(x,m^\varepsilon(T))m^\varepsilon(T) + \int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)m^\varepsilon + \int_0^T\!\!\int_{\mathbb{T}^d} m^\varepsilon\left[H_p(x,Du^\varepsilon)\cdot Du^\varepsilon - H(x,Du^\varepsilon)\right]\\ &\qquad\qquad \overset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{T}^d} m_0\,\chi \le \int_{\mathbb{T}^d} m_0\, u(0)\end{aligned} \tag{1.135}$$

and using (1.134) we get

$$\limsup_{\varepsilon\to 0}\left\{\int_{\mathbb{T}^d} G(x,m^\varepsilon(T))m^\varepsilon(T)+\int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)m^\varepsilon+\int_0^T\!\!\int_{\mathbb{T}^d} m^\varepsilon\left[H_p(x,Du^\varepsilon)\cdot Du^\varepsilon-H(x,Du^\varepsilon)\right]\right\}$$
$$\le \int_{\mathbb{T}^d}\bar g\, m(T)+\int_0^T\!\!\int_{\mathbb{T}^d}\bar f\, m\;+\int_0^T\!\!\int_{\mathbb{T}^d} m\;[b\cdot Du-H(x,Du)]\;. \tag{1.136}$$

Let us denote $w^\varepsilon := m^\varepsilon\, H_p(x,Du^\varepsilon)$. We have called w its weak limit in $L^1(Q_T)$; since we have

$$\int_0^T\!\!\int_{\mathbb{T}^d} m^\varepsilon\left[H_p(x,Du^\varepsilon)\cdot Du^\varepsilon-H(x,Du^\varepsilon)\right]=\int_0^T\!\!\int_{\mathbb{T}^d} m^\varepsilon\, H^*(x,H_p(x,Du^\varepsilon))=\int_0^T\!\!\int_{\mathbb{T}^d} m^\varepsilon\, H^*\left(x,\frac{w^\varepsilon}{m^\varepsilon}\right)$$

where H^* is the convex conjugate of H, and since $mH^*\left(x,\frac{w}{m}\right)$ is a convex function of (m,w), by weak lower semicontinuity we have

$$\liminf_{\varepsilon\to 0}\int_0^T\!\!\int_{\mathbb{T}^d} m^\varepsilon\left[H_p(x,Du^\varepsilon)\cdot Du^\varepsilon-H(x,Du^\varepsilon)\right]\ge\int_0^T\!\!\int_{\mathbb{T}^d} mH^*\left(x,\frac{w}{m}\right)\;. \tag{1.137}$$

Therefore we deduce from (1.136)

$$\begin{aligned}\limsup_{\varepsilon\to 0}\left\{\int_{\mathbb{T}^d} G(x,m^\varepsilon(T))m^\varepsilon(T)+\int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)m^\varepsilon\right\}&\le\int_{\mathbb{T}^d}\bar g\, m(T)+\int_0^T\!\!\int_{\mathbb{T}^d}\bar f\, m\\ &\quad+\int_0^T\!\!\int_{\mathbb{T}^d} m\;[b\cdot Du-H(x,Du)]-\int_0^T\!\!\int_{\mathbb{T}^d} mH^*\left(x,\frac{w}{m}\right)\\ &\le\int_{\mathbb{T}^d}\bar g\, m(T)+\int_0^T\!\!\int_{\mathbb{T}^d}\bar f\, m\end{aligned} \tag{1.138}$$

where we have used that $w=b\,m$. We use now the monotonicity of F,G to identify their limits. Indeed, denoting $T_k(s)=\min(s,k)$, we have (there is no loss of generality here in assuming F,G positive, which is true up to addition of constants):

$$\begin{aligned}&\int_0^T\!\!\int_{\mathbb{T}^d}\left[F(x,m^\varepsilon)-F(x,m)\right](m^\varepsilon-m)+\int_{\mathbb{T}^d}\left[G(x,m^\varepsilon(T))-G(x,m(T))\right](m^\varepsilon(T)-m(T))\\ &\qquad\le\int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)m^\varepsilon+\int_{\mathbb{T}^d} G(x,m^\varepsilon(T))m^\varepsilon(T)\\ &\quad-\int_0^T\!\!\int_{\mathbb{T}^d} F(x,m^\varepsilon)\,T_k(m)-\int_0^T\!\!\int_{\mathbb{T}^d} T_k(F(x,m))m^\varepsilon+\int_0^T\!\!\int_{\mathbb{T}^d} F(x,m)m\\ &\quad-\int_{\mathbb{T}^d} G(x,m^\varepsilon(T))\,T_k(m(T))-\int_{\mathbb{T}^d} T_k(G(x,m(T)))\,m^\varepsilon(T)+\int_{\mathbb{T}^d} G(x,m(T))m(T)\,.\end{aligned}$$

Hence, using (1.138) and the weak convergences of $m^\varepsilon, F(x,m^\varepsilon), G(x,m^\varepsilon(T))$ we get

$$\limsup_{\varepsilon\to 0}\left\{\int_0^T\!\!\int_{\mathbb{T}^d}[F(x,m^\varepsilon)-F(x,m)]\,(m^\varepsilon-m)+\int_{\mathbb{T}^d}[G(x,m^\varepsilon(T))-G(x,m(T))]\,(m^\varepsilon(T)-m(T))\right\}$$
$$\le \int_0^T\!\!\int_{\mathbb{T}^d}\bar f\,[m-T_k(m)]+\int_0^T\!\!\int_{\mathbb{T}^d}[F(x,m)-T_k(F(x,m))]m$$
$$+\int_{\mathbb{T}^d}\bar g\,[m(T)-T_k(m(T))]+\int_{\mathbb{T}^d}[G(x,m(T))-T_k(G(x,m(T)))]m(T)\,.$$

Letting $k\to\infty$ the right-hand side vanishes due to Lebesgue 's theorem, so we conclude that

$$\limsup_{\varepsilon\to 0}\left\{\int_0^T\!\!\int_{\mathbb{T}^d}[F(x,m^\varepsilon)-F(x,m)]\,(m^\varepsilon-m)+\int_{\mathbb{T}^d}[G(x,m^\varepsilon(T))-G(x,m(T))]\,(m^\varepsilon(T)-m(T))\right\}=0\,.$$

This means that $[F(x,m^\varepsilon)-F(x,m)]\,(m^\varepsilon-m)\to 0$ in $L^1(Q_T)$, and almost everywhere in Q_T up to subsequences. In particular, we deduce that $F(x,m^\varepsilon)\to F(x,m)$ a.e. in Q_T up to subsequences, hence $\bar f=F(x,m)$ and the convergence (of both $F(x,m^\varepsilon)$ and $F(x,m^\varepsilon)m^\varepsilon$) is actually strong in $L^1(Q_T)$. Similarly we reason for $G(x,m^\varepsilon(T))$, which implies that $\bar g=G(x,m(T))$. If we come back to (1.138), now the limit of the left-hand side coincides with the right-hand side, and trapped in between we deduce that

$$\int_0^T\!\!\int_{\mathbb{T}^d} m\,\left[\frac{w}{m}\cdot Du-H(x,Du)-H^*\left(x,\frac{w}{m}\right)\right]=0$$

which yields that $\frac{w}{m}=H_p(x,Du)\;m-$ a.e. in Q_T. This implies that $w=m\,H_p(x,Du)$. Finally, all the weak limits are identified. Coming back to (1.135), now we know that $F(x,m^\varepsilon)m^\varepsilon\to F(x,m)m$ and $G(x,m^\varepsilon(T))m^\varepsilon(T)\to G(x,m(T))m(T)$, and in addition (1.137) holds with $w=m\,H_p(x,Du)$. Therefore, we have

$$\int_{\mathbb{T}^d}G(x,m(T))m(T)+\int_0^T\!\!\int_{\mathbb{T}^d}F(x,m)m+\int_0^T\!\!\int_{\mathbb{T}^d}m\,[H_p(x,Du)\cdot Du-H(x,Du]$$
$$\le\liminf_{\varepsilon\to 0}\left\{\int_{\mathbb{T}^d}G(x,m^\varepsilon(T))m^\varepsilon(T)+\int_0^T\!\!\int_{\mathbb{T}^d}F(x,m^\varepsilon)m^\varepsilon+\int_0^T\!\!\int_{\mathbb{T}^d}m^\varepsilon\,[H_p(x,Du^\varepsilon)\cdot Du^\varepsilon-H(x,Du^\varepsilon]\right\}$$
$$\le\int_{\mathbb{T}^d}u(0)\,m_0\,.$$

Combining this information with (1.134), where $\bar f,\bar g,b$ are now identified, yields the energy equality (1.123). Thus, we conclude that (u,m) is actually a weak solution of the MFG system in the sense of Definition 8. □

Now we conclude the analysis with a uniqueness result. To this purpose, we need a refined version of Lemma 15, as follows.

Lemma 16 *Assume that F,G satisfy (1.115)-(1.116), and that $H(x,p)$ satisfies (1.66)-(1.68). Let (u,m) be a weak solution to (1.121)-(1.122). Then $u(t)m(t)\in L^1(\mathbb{T}^d)$ for a.e. $t\in(0,T)$, and the following equality holds:*

$$\int_{\mathbb{T}^d}m(t)\,u(t)\,dx=\int_{\mathbb{T}^d}G(x,m(T))\,m(T)\,dx+\int_t^T\int_{\mathbb{T}^d}F(t,x,m)m\,dxdt$$
$$+\int_t^T\int_{\mathbb{T}^d}m\,\left[H_p(x,Du)\cdot Du-H(x,Du)\right]dxdt$$

for a.e. $t\in(0,T)$.

Proof. Let us take t such that $m(t)\in L^1(\mathbb{T}^d)$ (this is true for a.e. $t\in(0,T)$). First we apply Lemma 15 in the time interval (t,T). Notice that, since $F(x,m)m\in L^1(Q_T)$, $G(x,m(T))m(T)\in L^1(\mathbb{T}^d)$, then the requirements of the Lemma hold with $\phi_1(s)=f(s)s$ and $\phi_2(s)=g(s)s$, where f,g are given by (1.115)-(1.116). We obtain that

$u(t)m(t) \in L^1(\mathbb{T}^d)$ (where $u(t)$ is the right-continuous Borel representative of u) and

$$\begin{aligned}\int_{\mathbb{T}^d} m(t)\, u(t)\, dx \le \int_{\mathbb{T}^d} G(x, m(T))\, m(T)\, dx + \int_t^T \int_{\mathbb{T}^d} F(x,m) m\, dx dt \\ + \int_t^T \int_{\mathbb{T}^d} m\, [H_p(x, Du) \cdot Du - H(x, Du)]\, dx dt \\ = \int_{\mathbb{T}^d} u(0) m_0 - \int_0^t \int_{\mathbb{T}^d} F(x,m) m\, dx dt - \int_0^t \int_{\mathbb{T}^d} m\, [H_p(x, Du) \cdot Du - H(x, Du)]\, dx dt\end{aligned} \tag{1.139}$$

where we used (1.123) in the last equality. Now we wish to apply once more Lemma 15 in the interval $(0,t)$; but this needs to be done in two steps. First of all, we replace u with $u_k = \min(u,k)$; u_k is itself a sub solution and satisfies (see e.g. [158, Lemma 5.3])

$$-\partial_t u_k + H(x, Du_k) \le F(x,m)\, 1_{\{u<k\}} + c\, 1_{\{u>k\}}$$

for some constant $c > 0$. Since $u_k(t) \in L^\infty(\mathbb{T}^d)$ and $m(t) \in L^1(\mathbb{T}^d)$, we can apply Lemma 15 in $(0,t)$ to get

$$\int_{\mathbb{T}^d} u(0) m_0 \le \int_{\mathbb{T}^d} u_k(t)\, m(t) + \int_0^t \int_{\mathbb{T}^d} [F(x,m)\, 1_{\{u<k\}} + c\, 1_{\{u>k\}}] m + \int_0^t \int_{\mathbb{T}^d} m\, [H_p(x, Du) \cdot Du_k - H(x, Du_k)]\ .$$

Letting $k \to \infty$ is allowed since $u(t)m(t) \in L^1(\mathbb{T}^d)$, and we deduce that

$$\int_{\mathbb{T}^d} u(0) m_0 \le \int_{\mathbb{T}^d} u(t)\, m(t) + \int_0^t \int_{\mathbb{T}^d} F(x,m) m\ + \int_0^t \int_{\mathbb{T}^d} m\, [H_p(x, Du) \cdot Du - H(x, Du)]\ .$$

Using this information in (1.139) we conclude the proof of the desired equality. □

We are ready for the uniqueness result, where we further invoke the following lemma. This is a particular case of what proved in [158, Lemma 5.3] for solutions in the whole space; the proof follows the same steps in the setting of $x \in \mathbb{T}^d$. A similar statement is also contained in [58, Thm 6.2].

Lemma 17 *[[158]] Let u_1, u_2 be two weak sub solutions of (1.117). Then $v := \max(u_1, u_2)$ is also a sub solution of the same problem.*

We have all ingredients for the uniqueness result.

Theorem 16 *Assume that F, G satisfy (1.115)-(1.116), and that $H(x,p)$ satisfies (1.66)-(1.68). Let $m_0 \in L^\infty(\mathbb{T}^d)$, and let (u,m), $(\tilde u, \tilde m)$ be two weak solutions of (1.121)–(1.122), in the sense of Definition 8. Then we have $F(x,m) = F(x,\tilde m)$ and, if $F(x,\cdot)$ is an increasing function, then $m = \tilde m$ and $u = \tilde u$ $m-$almost everywhere.*

Proof. After condition (1.115), there is no loss of generality in assuming that $F(x,s) \le f(s)$ (which is the case up to addition of a (same) constant to both H ad F). Therefore, using the monotonicty of $F(x,\cdot)$, we have

$$F(x,m)s \le F(x,m)m + F(x,s)s \le F(x,m)m + f(s)s$$

Hence, if we denote by $\phi_1(s) = f(s)s$, we have that $\phi_1^*(F(x,m)) \in L^1(Q_T)$, while $\phi_1(m) \in L^1(Q_T)$. This is of course true for $\tilde m$ as well. Similarly we reason for $G(x, m(T))$ with $\phi_2(s) = g(s)s$ given by (1.116). Therefore, we can apply Lemma 15 to u and to $\tilde m$ as well as to $\tilde u$ and m. We obtain

$$\begin{aligned}\int_{\mathbb{T}^d} u(0) m_0 \le \int_0^T \int_{\mathbb{T}^d} F(x,m)\tilde m + \int_{\mathbb{T}^d} G(x, m(T)) \tilde m(T) + \int_0^T \int_{\mathbb{T}^d} \tilde m [H_p(x, D\tilde u) Du - H(x, Du)]\,, \\ \int_{\mathbb{T}^d} \tilde u(0) m_0 \le \int_0^T \int_{\mathbb{T}^d} F(x,\tilde m) m + \int_{\mathbb{T}^d} G(x, \tilde m(T)) m(T) + \int_0^T \int_{\mathbb{T}^d} m [H_p(x, Du) D\tilde u - H(x, D\tilde u)]\,.\end{aligned}$$

We use (1.123) in the first inequality, and similarly we use (1.123) written for $(\tilde u, \tilde m)$ in the second one. When we add the two contributions we deduce the usual inequality

$$\int_0^T\int_{\mathbb{T}^d}(F(x,m)-F(x,\tilde m))(m-\tilde m)+\int_{\mathbb{T}^d}[G(x,m(T))-G(x,\tilde m(T))][m(T)-\tilde m(T)]$$
$$\int_0^T\int_{\mathbb{T}^d} m\,[H(x,D\tilde u)-H(x,Du)-H_p(x,Du)(D\tilde u-Du)]$$
$$+\int_0^T\int_{\mathbb{T}^d}\tilde m\,[H(x,Du)-H(x,D\tilde u)-H_p(x,D\tilde u)(Du-D\tilde u)]\le 0\,.$$

This implies that $F(x,m)=F(x,\tilde m)$ and $G(x,m(T))=G(x,\tilde m(T))$, and we have

$$\begin{aligned} &H(x,D\tilde u)-H(x,Du)=H_p(x,Du)(D\tilde u-Du) &&\text{in } \{(t,x):\ m(t,x)>0\}\\ &H(x,Du)-H(x,D\tilde u)=H_p(x,D\tilde u)(Du-D\tilde u) &&\text{in } \{(t,x):\ \tilde m(t,x)>0\}.\end{aligned} \tag{1.140}$$

Of course, if $F(x,\cdot)$ is increasing, we deduce that $m=\tilde m$ almost everywhere.

Now we use Lemma 17, which says that $z:=\max(u,\tilde u)$ is a sub solution of the HJ equation. Then we can apply Lemma 15 and we obtain, for a.e. $t\in(0,T)$:

$$\int_{\mathbb{T}^d} m(t)\,z(t)\ \le \int_{\mathbb{T}^d} G(x,m(T))\,m(T)+\int_t^T\int_{\mathbb{T}^d} F(x,m)\,m\ +\int_t^T\int_{\mathbb{T}^d} m\ [H_p(x,Du)\cdot Dz-H(x,Dz)]\ . \tag{1.141}$$

Now we have

$$\begin{aligned}\int_t^T\int_{\mathbb{T}^d} m\ [H_p(x,Du)\cdot Dz-H(x,Dz)] &=\int_t^T\int_{\mathbb{T}^d} m\ [H_p(x,Du)\cdot D\tilde u-H(x,D\tilde u)]\,1_{\{u\le\tilde u\}}\\ &\quad+\int_t^T\int_{\mathbb{T}^d} m\ [H_p(x,Du)\cdot Du-H(x,Du)]\,1_{\{u>\tilde u\}}\\ &=\int_t^T\int_{\mathbb{T}^d} m\ [H_p(x,Du)\cdot Du-H(x,Du)]\end{aligned}$$

thanks to (1.140); thus we deduce from (1.141)

$$\begin{aligned}\int_{\mathbb{T}^d} m(t)\,z(t)\ &\le \int_{\mathbb{T}^d} G(x,m(T))\,m(T)+\int_t^T\int_{\mathbb{T}^d} F(x,m)\,m\ +\int_t^T\int_{\mathbb{T}^d} m\ [H_p(x,Du)\cdot Du-H(x,Du)]\\ &=\int_{\mathbb{T}^d} m(t)\,u(t)\end{aligned}$$

where we used Lemma 16. We conclude that

$$\int_{\mathbb{T}^d} m(t)\,[z(t)-u(t)]\le 0$$

which implies that $u(t)=z(t)$ (i.e. $u(t)\ge\tilde u(t)$) m-almost everywhere. Reversing the roles of the two functions we conclude that $u=\tilde u$ m-almost everywhere. □

Remark 16 There are other approaches to study the first order MFG system (1.121)–(1.122), especially if model cases are considered. One possible strategy, introduced in [145], consists in transforming the system into a second order elliptic equation for u in time space. More precisely, using that F is one-to-one and replacing m in the continuity equation by

$$m(t,x)=F^{-1}(x,-\partial_t u+H(x,Du)),$$

one finds an elliptic equation in (t,x) for u. This elliptic equation is fully nonlinear and degenerate (at least on the points (t,x) where $m(t,x)=0$). This strategy is the starting point of regularity results proved by P.-L. Lions in [149] (Lessons 6-27/11 2009), which in particular lead to uniform bounds for the density m.

Other regularity results, including L^∞- bounds or Sobolev regularity for the density, were obtained by F. Santambrogio in [147, 148, 175] using completely different techniques inspired by optimal transport theory. Those results are just one by-product of the Lagrangian approach developed by F. Santambrogio, for which we refer to his presentation in this same volume.

1.3.7.2 Variational approach and optimality conditions

Following [144] the MFG system (1.62) can be viewed as an optimality condition for two optimal control problems: the first one is an optimal control of Hamilton-Jacobi equations and the second one an optimal control of the Fokker-Planck equation.

In order to be more precise, let us first introduce some assumptions and notations: without loss of generality, we suppose that $F(x,0)=0$ (indeed we can always subtract $F(x,0)$ to both sides of (1.62)). Then we set

$$\Phi(x,m)=\begin{cases}\displaystyle\int_0^m F(x,\rho)d\rho \text{ if } m\geq 0\\ 0 \qquad\qquad\qquad \text{otherwise}\,.\end{cases}$$

As F is nondecreasing with respect to the second variable, $\Phi(x,m)$ is convex with respect to m, so we denote by $\Phi^* = \Phi^*(x,\alpha)=\sup_{m\in\mathbb{R}}\ (\alpha m-\Phi(x,m))$ its convex conjugate. Note that Φ^* is convex and nondecreasing with respect to the second variable. We also recall the convex conjugate $H^*(x,q)=\sup_{p\in\mathbb{R}^d}(q\cdot p-H(x,p))$ already used before. For simplicity, we neglect here the coupling at $t=T$ and we let $G=G(x)$.

The first optimal control problem is the following: the distributed control is $\alpha:\mathbb{T}^d\times[0,T]\to\mathbb{R}$ and the state function is u. The goal is to minimize the functional

$$\mathcal{J}^{HJ}(\alpha)=\int_0^T\int_{\mathbb{T}^d}\Phi^*\left(x,\alpha(t,x)\right)\,dxdt-\int_{\mathbb{T}^d}u(0,x)dm_0(x)$$

over Lipschitz continuous maps $\alpha:(0,T)\times\mathbb{T}^d\to\mathbb{R}^d$, where u is the unique classical solution to the backward Hamilton-Jacobi equation

$$\begin{cases}-\partial_t u-\varepsilon\Delta u+H(x,Du)=\alpha(t,x) \qquad \text{in } (0,T)\times\mathbb{T}^d\\ u(T,x)=G(x) \qquad \text{in } \mathbb{T}^d\,.\end{cases} \tag{1.142}$$

The second is an optimal control problem where the state is the solution m of the Fokker-Planck equation: the (distributed and vector valued) control is now the drift term $v:[0,T]\times\mathbb{T}^d\to\mathbb{R}^d$. The goal here is to minimize the functional

$$\mathcal{J}^{FP}(v)=\int_0^T\int_{\mathbb{T}^d}\left[m\,H^*\left(x,-v\right)+F(x,m)\right]dxdt+\int_{\mathbb{T}^d}G(x)m(T)dx,$$

where the pair (m,v) solves the Fokker-Planck equation

$$\partial_t m-\varepsilon\Delta m+\mathrm{div}(mv)=0 \text{ in } (0,T)\times\mathbb{T}^d, \qquad m(0)=m_0. \tag{1.143}$$

Assuming that F^* and H^* are smooth enough, the equivalence between the MFG system and the optimality conditions of the previous two problems can be checked with a direct verification.

Theorem 17 *[[149]] Assume that $(\bar m,\bar u)$ is of class $C^2((0,T)\times\mathbb{T}^d)$, with $\bar m(0)=m_0$ and $\bar u(T,x)=G(x)$. Suppose furthermore that $\bar m(t,x)>0$ for any $(t,x)\in(0,T)\times\mathbb{T}^d$. Then the following statements are equivalent:*

(i) $(\bar u,\bar m)$ is a solution of the MFG system (1.62).

(ii) The control $\bar\alpha(t,x) := F(x,\bar m(t,x))$ *is optimal for* $\mathcal{J}^{HJ}$ *and the solution to* (1.142) *is given by* $\bar u$.
(iii) The control $\bar v(t,x) := -H_p(x, D\bar u(t,x))$ *is optimal for* $\mathcal{J}^{FP}$, $\bar m$ *being the solution of* (1.143).

Let us stress that the above equivalence holds even for $\varepsilon = 0$, say for the deterministic problem. But of course a formal equivalence for smooth solutions is of very little help. However, it is possible to exploit the convexity of the optimal control problems in order to export the equivalence principle to suitably relaxed optimization problems. To this purpose, observe that the optimal control problem of Hamilton-Jacobi equation can be rewritten as

$$\inf_u \int_0^T \int_{\mathbb{T}^d} F^*\left(x, -\partial_t u(x,t) - \varepsilon\Delta u(x,t) + H(x, Du(x,t))\right)\, dxdt - \int_{\mathbb{T}^d} u(0,x)dm_0(x)$$

under the constraint that u is sufficiently smooth, with $u(\cdot,T) = G(\cdot)$. Remembering that H is convex with respect to the last variable and that F is convex and increasing with respect to the last variable, it is clear that the above problem is convex.

The optimal control problem of the Fokker-Planck equation is also a convex problem, up to a change of variables which appears frequently in optimal transportation theory since the pioneering paper [28]. In fact, if we set $w = mv$, then the problem can be rewritten as

$$\inf_{(m,w)} \int_0^T \int_{\mathbb{T}^d} [m(t,x)H^*\left(x, -\frac{w(t,x)}{m(t,x)}\right) + F(x, m(t,x))]\, dxdt + \int_{\mathbb{T}^d} G(x)m(T,x)dx,$$

where the pair (m,w) solves the Fokker-Planck equation

$$\partial_t m - \varepsilon\Delta m + \operatorname{div}(w) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \qquad m(0) = m_0. \tag{1.144}$$

This problem is convex because the constraint (1.144) is linear and the map $(m,w) \to mH^*\left(x, -\frac{w}{m}\right)$ is convex.

It turns out that the two optimal control problems just defined are conjugate in the Fenchel-Rockafellar sense (see, for instance, [100]) and they share the same optimality condition. Minimizers of such problems are expected to provide with weak solutions for (1.62). This approach has been extensively used for first order problems since [53], [57], leading to weak solutions in the sense of Definition 8. A similar analysis was later extended to second order degenerate MFG problems in [58], as well as to problems with density constraints, in which case one enforces the constraint that the density $m = m(t,x)$ is below a certain threshold. In this case, a penalization term appears in the HJ equation as an extra price to go through the zones where the density saturates the constraint (see [64, 124, 155, 173]).

A similar variational approach was also specially developed for the planning problem in connection with optimal transportation ([125], [158]).

We do not comment more on the optimal control approach because this is also extensively recalled in the contributions by Y. Achdou & M. Lauriere, and in the one by F. Santambrogio, in this volume.

1.3.8 Further comments, related topics and references

Boundary conditions, exit time problems, state constraints, planning problem.

The existence and uniqueness results presented here for second order problems remain valid, with no additional difficulty, for the case of Neumann boundary conditions, i.e. when the controlled process lives in a bounded domain $\Omega \subset \mathbb{R}^d$ with reflection on the boundary. Results in this setting can be found e.g. in [165], or in [86]. A similar situation occurs when the domain happens to be invariant for the controlled process, and the trajectory cannot reach the boundary because of the degeneracy of the diffusion or due to the direction of the controlled drift. The study of the MFG system in this situation appears in [168].

By contrast, in many models, players can leave the game before the terminal time and the population is not constant: this is for instance the case of MFG with exit time, which lead to Dirichlet boundary conditions for the two unknowns

(u, m) in the system. An interesting problem arises when the agents also control the time they stay in the game. The optimal control problem then becomes

$$u(t,x) = \inf_{\tau,\alpha} \mathbb{E}\left[\int_0^\tau f(X_t, \alpha_t, m(t))dt + g(X_T, m(T))\right]$$

where (X_t) is driven by the usual controlled diffusion process. Here the controls are α and the stopping time τ. The measure $m(t)$ can be (depending on the model) either the law of X_t given $\{\tau \geq t\}$ (in which case the mass of $m(t)$ is constant, but the equation for m is no longer a simple Fokker-Planck equation) or simply the measure $m(t)$ defined by

$$\int_{\mathbb{R}^d} \phi(x)m(t,dx) = \mathbb{E}\left[\phi(X_t)\mathbf{1}_{\tau\geq t}\right] \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d).$$

In this case the mass of $m(t)$ is non increasing in time.

Such models have been studied in the framework of bank run in [70] or in exhaustible commodities trade [84, 126]. In [32], the author provides a general PDE framework to study the problem and shows that the players might be led to use random strategies (see also [38]). An early work on the topic is [112] while surprising phenomena in the mean field analysis are discussed in [46, 112, 156, 157]. Minimal exit time problems for first order MFG problems are also studied in [154].

In many applications, the MFG system also involves state constraints. Namely, the optimal control problem for a small player takes the form

$$u(t,x) = \inf_{\gamma} \int_t^T L(\gamma(s), \alpha(s), m(s)ds + G(\gamma(T), m(T)),$$

where the infimum is taken over solution of

$$\begin{cases} \dot{\gamma}(s) = b(\gamma(s), m(s))ds, & \gamma(s) \in \overline{\Omega} \qquad \forall s \in [t,T], \\ \gamma(t) = x \end{cases}$$

where Ω is an open subset of $\mathbb{R}^d$ (in general with a smooth boundary). This is the case of the Aiyagari problem [13] in economy for instance (see also [3, 7]). The natural set-up of the HJ problem is the so-called viscosity solution with state-constraints and is well understood. However, the analysis led in this section no longer applies: the measure m develops singularities (not only on the boundary) and one cannot expect uniqueness of the flow m given the vector field $-H_p(x, Du, m)$. To overcome this issue one can device a Lagrangian approach (see [47]). The PDE analysis of this problem is only partially understood (see [48, 49]).

The initial-terminal conditions may also be changed, in what is called the *planning problem*. In that model one wishes to prescribe both the initial and the final distribution, while no terminal condition is assumed on u. This variant of MFG problem fits into the models of optimal transportation theory, since the goal is to transport the density from $m(0) = m_0$ to $m(T) = m_1$ in a way which is optimal for the agents' control. Early results for this problem were given by P.-L. Lions in [149] (Lessons 4-11/12 2009); the second order case was later studied in [163], [164], [165], and the deterministic case in [125], [158]. Very recently, the planning problem has been also addressed for the master equation with finite states, see [35].

Numerical methods.

The topic will be developed in detail in the contribution of Achdou and Lauriere (see the references therein). Let us just remark here that the computation of the solution of the MFG system is difficult because it involves a forward equation and a backward equation. Let us just quote on this point the pioneering work [5], where a finite difference numerical scheme was proposed in a way that the discretized equations preserve the structure of the MFG system. In some cases one can also take advantage of the fact that the MFG system has a variational structure [29].

MFG systems with several populations.

MFG models may very naturally involve several populations (say, to fix the ideas, I populations). In this case the system takes the form

$$\begin{cases} (i)\ -\partial_t u_i - \Delta u_i + H_i(x, Du_i, m(t)) = 0 \qquad \text{in } (0,T)\times\mathbb{T}^d \\ (ii)\ \partial_t m_i - \Delta m_i - \operatorname{div}(m_i\ D_pH_i(x, Du_i(t,x), m(t))) = 0 \qquad \text{in } (0,T)\times\mathbb{T}^d \\ (iii)\ m_i(0) = m_{i,0}\ ,\ u_i(T,x) = G_i(x, m(T)) \qquad \text{in } \mathbb{T}^d \end{cases}$$

where $i = 1,\dots,I$, u_i denotes the value function of each player in population i and $m = (m_1,\dots,m_I)$ denotes the collection of densities m_i of the population i. The coupling functions H_i and G_i depend on all the densities. Existence of solutions can be proved by fixed point arguments as in Theorem 5. Uniqueness, however, is a difficult issue.

The MFG models with several populations were introduced in the early paper by Caines, Huang and MalhamÃ© [132] and revisited by Cirant [86] (for Neumann boundary conditions, see also [24, 155]) and by Kolokoltsov, Li and Yang [137] (for very general diffusions, possibly with jumps). Analysis of segregation phenomena is pursued in [2], [93].

MFG of control.

In most application in economics, the coupling between the agents is not only through the distribution of their positions but also of their instantaneous controls. This kind of problem is more subtle than the classical MFG system since it requires, in its general version, a new constraint explaining how to derive the distribution of controls. For instance this new system can take the form (for problems with a constant diffusion):

$$\begin{cases} (i)\quad -\partial_t u(t,x) - \Delta u(t,x)) + H(t,x,Du(t,x),\mu(t)) = 0 \text{ in } (0,T)\times\mathbb{R}^d \\ (ii)\quad \partial_t m(t,x) - \Delta m(t,x) - \operatorname{div}(m(t,x)H_p(t,x,Du(t,x),\mu(t))) = 0 \text{ in } (0,T)\times\mathbb{R}^d \\ (iii)\ \mu(t) = (id, -H_p(t,x,Du,\mu))\sharp m(t) \text{ in } (0,T) \\ (iv)\ \ m(0,x) = m_0(x),\ u(T,x) = g(x,m(T)) \qquad \text{in } \mathbb{R}^d \end{cases}$$

Here $\mu(t)$ is the distribution of states and controls of the players. Its first marginal $m(t)$ is the usual distribution of players. The new relation (iii) explains how to compute $\mu(t)$ from the distribution of the states $m(t)$ and the optimal feedback $-H_p(t,x,Du,\mu)$. Note that (iii) is itself a fixed point problem. In many applications, the players only interact through some moments of the distribution of controls, which simplifies the system. Existence of a solution for MFG of controls can be achieved under rather general conditions (some structure condition, ensuring (iii) to have a solution, is however required). Uniqueness is largely open.

Analysis of such problems can be found, among many other references, in [8, 34, 62, 72, 115, 122].

MFG with common noise and with a major player.

Throughout this section we have discussed models in which the agents are subject to individual noises (“idiosyncratic noise") which are independent. However, it is also important to be able to deal with problems in which some random perturbation affects all the players. This perturbation can be quite rough (a white noise) or simply the (random) position of a single player (who, since he/she cannot be considered as an infinitesimal player in this game, is often called a major player). In these setting, the MFG system becomes random. For instance, in the case of MFG with a Brownian common noise, it takes the form

$$
\begin{cases}
du(t,x) = \big[-2\Delta u(t,x)) + H(x, Du(t,x), m(t)) \\
\qquad\qquad\qquad -\sqrt{2}\mathrm{div}(v(t,x))\big]dt + v(t,x)\cdot dW_t \qquad\qquad \text{in } (0,T)\times\mathbb{R}^d, \\
dm(t,x) = \big[2\Delta m(t,x)) + \mathrm{div}\big(m(,x)D_pH(x, Du(t,x), m(t))\big)\big]dt \\
\qquad\qquad\qquad -\mathrm{div}(m(t,x)\sqrt{2}dW_t\big), \qquad\qquad \text{in } (0,T)\times\mathbb{R}^d, \\
u(T,x) = G(x, m(T)),\ m(0) = m_0, \qquad \text{in } \mathbb{R}^d
\end{cases}
$$

Here W is the common noise (a Brownian motion). The new variable v is an unknown function which ensures the solution u to the backward HJ to be adapted to the filtration generated by W. Another formulation of this problem involves the master equation and will be discussed below, at the end of Section 1.4. Let us just mention now that the analysis of MFG with common noise goes back to [149] (Lessons 12-26/11 2010), where in particular the structure of the master equation with common noise is described. The probabilistic approach of the MFG system is studied in [69] (see also [11, 141]) while the first results on the PDE system above and on the associated master equation are in [56].

MFG problems with a major player have been introduced by Huang in [130]. In a series of papers, Carmona and al. introduced a different notion of solution for the problem [77, 75, 76], mainly in a finite state space framework, and they showed that this notion actually corresponds to a Nash equilibrium for infinitely many players. This result is confirmed in [54] where the Nash equilibria for the $N-$player problem is shown to converge to the corresponding master equation. The master equation for the major problem is also studied in [146] (mostly in finite time horizon and in a finite state space) and in [55] (short-time existence in continuous space). Variants on the major player problem are discussed in [30] (MFG with a dominating player in a Stackelberg equilibrium), [101] (for a principal-agent problem) and in [36].

Miscellaneous.

Other MFG systems. Let us first mention other variations on the MFG system. Besides the standard continuous-time, continuous-spaces models, the most relevant class of MFG models is probably the MFG on finite state space: see, among main other works, [27, 34, 78, 114]. In these problems the state of a typical player jumps from one state to another. The coupling between the HJ and the FP equations takes a much simpler form of a “forward-backward” system of ordinary differential equations. Another class of MFG problems are MFG on networks [6, 45], in which the state space is a (one dimensional) network. Motivated by knowledge growth models [152], some authors considered MFGs in which the interaction between players leads to a Boltzmann type equation or a Fisher-KPP equation for the distribution function [42, 43, 160, 169]. MFGs involving jump processes, where the diffusion term becomes a fractional Laplacian, have been studied in [44, 82, 89, 103, 137], while MFGs involving dynamics with deterministic jumps have been investigated in [33].

MFGs vs Agent Based Models. In MFG theory, agents are assumed to be rational. On the contrary, in Agent Based Models, the agents are supposed to adopt an automatic behavior. The link between the two approach has been discussed in [23, 34, 99] where it is shown that MFG models degenerate into Agent Based Models as the agents become more and more myopic or more and more impatient.

Learning. A natural question in Mean Field Games, in view of the complexity of the notion of MFG equilibria, is how players can achieve in practice to play a MFG Nash equilibrium. This kind of problem, also related to the concept of adaptative control [136], has been discussed in particular in the following references: [59, 62, 73, 74, 102, 129].

Efficiency of MFGs. In game theory, a classical question is the (in)efficiency of Nash equilibria (the so-called “price of anarchy”): to what extent are Nash equilibria doing socially worse than a global planner? This question has also been addressed for Mean Field Games in [21, 66, 71].

1.4 The master equation and the convergence problem

In Section 1.3 we have explained in detail how the mean field game problem can often be reduced to a system of PDEs, the MFG system. If this MFG system is suitable for the analysis of problems in which players have only independent individual noises (the so-called idiosyncratic noises), it is no longer satisfactory to investigate more complex models (for instance models in which the players are subject to a common randomness, the so-called "MFG models with a common noise"). Nor does it allow to understand the link between $N-$player differential games and mean field games. To proceed, we need to introduce another equation: the master equation. The master equation is an infinite dimensional hyperbolic equation stated in the space of probability measures. As explained below, it is helpful for the following reasons:

- for standard MFG models, it allows to write the optimal control of a player in feedback form in function of the current time, the current position and *the current distribution of the other players*. This is meaningful since one can expect in practice that players adapt their behavior in function of these data;
- it provides a key tool to investigate the convergence of the $N-$player game to the MFG system;
- it allows to formalize and investigate more complex MFG models, as MFG with a common noise or MFG with a major player.

In order to discuss the master equation, we first need to have a closer look at the space of probability measures (Subsection 1.4.1) and then understand the notion of derivative in this space (Subsection 1.4.2). Then we present the master equation and state, almost without proof, the existence and the uniqueness of the solution (Subsection 1.4.3). We then discuss the convergence of $N-$player differential games by using the master equation (Subsection 1.4.4).

1.4.1 The space of probability measures (revisited)

We have already seen the key role of the space of probability measures in the mean field game theory. It is now time to investigate the basic properties of this space more thoroughly. The results are given mostly without proofs, which can be found, for instance, in the monographs [19, 174, 178, 179].

1.4.1.1 The Monge-Kantorovich distances

Let X be a Polish space (i.e., a complete separable metric space) and $\mathcal{P}(X)$ be the set of Borel probability measures on X. There are several ways to metricize the topology of narrow convergence, at least on some subsets of $\mathcal{P}(X)$. Let us denote by d the distance on X and, for $p \in [1, +\infty)$, by $\mathcal{P}_p(X)$ the set of probability measures m such that

$$\int_X d^p(x_0, x)dm(x) < +\infty \qquad \text{for some (and hence for all) point } x_0 \in X.$$

The Monge-Kantorowich distance on $\mathcal{P}_p(X)$ is given by

$$\mathbf{d}_p(m, m') = \inf_{\gamma \in \Pi(m,m')} \left[\int_{X^2} d(x,y)^p d\gamma(x,y) \right]^{1/p} \tag{1.145}$$

where $\Pi(m, m')$ is the set of Borel probability measures on $X \times X$ such that $\gamma(A \times X) = m(A)$ and $\gamma(X \times A) = m'(A)$ for any Borel set $A \subset X$. In other words, a Borel probability measure γ on $X \times X$ belongs to $\Pi(m, m')$ if and only if

$$\int_{X^2} \varphi(x)d\gamma(x,y) = \int_X \varphi(x)dm(x) \qquad \text{and} \qquad \int_{X^2} \varphi(y)d\gamma(x,y) = \int_X \varphi(y)dm'(y) \ ,$$

for any Borel and bounded measurable map $\varphi : X \to \mathbb{R}$. Note that $\Pi(m,m')$ is non-empty, because for instance $m \otimes m'$ always belongs to $\Pi(m,m')$. Moreover, by Hölder inequality, $\mathcal{P}_p(X) \subset \mathcal{P}_{p'}(X)$ for any $1 \leq p' \leq p$ and

$$\mathbf{d}_{p'}(m,m') \leq \mathbf{d}_p(m,m') \qquad \forall m,m' \in \mathcal{P}_p(X) \; .$$

We now explain that there exists at least an optimal measure in (1.145). This optimal measure is often referred to as *an optimal transport plan* from m to m'.

Lemma 18 (Existence of an optimal transport plan) *For any $m,m' \in \mathcal{P}_p(X)$, there is at least one measure $\bar{\gamma} \in \Pi(m,m')$ with*

$$\mathbf{d}_p(m,m') = \left[\int_{X^2} d(x,y)^p d\bar{\gamma}(x,y)\right]^{1/p} \; .$$

Proof. We first show that $\Pi(m,m')$ is tight and therefore relatively compact for the weak-* convergence. For any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that $m(K_\varepsilon) \geq 1 - \varepsilon/2$ and $m'(K_\varepsilon) \geq 1-\varepsilon/2$. Then, for any $\gamma \in \Pi(m,m')$, we have

$$\begin{array}{rl} \gamma(K_\varepsilon \times K_\varepsilon) & \geq \gamma(K_\varepsilon \times X) - \gamma(K_\varepsilon \times (X\backslash K_\varepsilon)) \\ & \geq m(K_\varepsilon) - \gamma(X \times (X\backslash K_\varepsilon)) \\ & \geq 1 - \varepsilon/2 - m'(X\backslash K_\varepsilon) \; \geq \; 1-\varepsilon \; . \end{array}$$

This means that $\Pi(m,m')$ is tight. It is also closed for the weak-* convergence. Since the map $\gamma \to \int_{X^2} |x-y|^p d\gamma(x,y)$ is lower semi-continuous for the weak-* convergence, it has a minimum on $\Pi(m,m')$. □

Let us now check that $\mathbf{d}_p$ is a distance.

Lemma 19 *For any $p \geq 1$, $\mathbf{d}_p$ is a distance on $\mathcal{P}_p$.*

The proof uses the notion of disintegration of a measure, see Theorem 10.

Proof. Only the triangle inequality presents some difficulty. Let $m, m', m'' \in \mathcal{P}_p$ and γ, γ' be optimal transport plans from m to m' and from m' to m'' respectively. We disintegrate the measures γ and γ' with respect to m': $d\gamma(x,y) = d\gamma_y(x)dm'(y)$ and $d\gamma'(y,z) = d\gamma'_y(z)dm'(y)$ and we define the measure π on $X \times X$ by

$$\int_{X\times X} \varphi(x,z)d\pi(x,z) = \int_{X\times X\times X} \varphi(x,z)d\gamma_y(x)d\gamma'_y(z)dm'(y) \qquad \forall \phi \in C^0_b(X\times X) \; .$$

Then one easily checks that $\pi \in \Pi(m,m'')$ and we have, by Hölder inequality,

$$\begin{array}{rl} \left[\int_{X\times X} d^p(x,z)d\pi(x,z)\right]^{1/p} & \leq \left[\int_{X\times X\times X} (d(x,y)+d(y,z))^p d\gamma_y(x)d\gamma'_y(z)dm'(y)\right]^{1/p} \\ & \leq \left[\int_{X\times X} d^p(x,y)d\gamma_y(x)dm'(y)\right]^{1/p} + \left[\int_{X\times X} d^p(y,z)d\gamma_y(z)dm'(y)\right]^{1/p} \\ & = \mathbf{d}_p(m,m') + \mathbf{d}_p(m',m'') \end{array}$$

So $\mathbf{d}_p(m,m'') \leq \mathbf{d}_p(m,m') + \mathbf{d}_p(m',m'')$. □

In these notes, we are mainly interested in two Monge-Kantorovich distances, $\mathbf{d}_1$ and $\mathbf{d}_2$. The distance $\mathbf{d}_2$, which is often called the Wasserstein distance, is particularly useful when X is a Euclidean or a Hilbert space. Its analysis will be the object of the next subsection.

As for the distance $\mathbf{d}_1$, which often takes the name of the Kantorovich-Rubinstein distance, we have already encountered it several times. Let us point out the following equivalent representation, which explains the link with the notion introduced in Subsection 1.2.2:

Theorem 18 (Kantorovich-Rubinstein Theorem) *For any $m,m' \in \mathcal{P}_1(X)$,*

$$\mathbf{d}_1(m,m') = \sup\left\{\int_X f(x)dm(x) - \int_X f(x)dm'(x)\right\}$$

where the supremum is taken over the set of all $1-$*Lipschitz continuous maps* $f : X \to \mathbb{R}$.

Remark 17 In fact the above "Kantorovich duality result" holds for much more general costs (i.e., it is not necessary to minimize the power of a distance). The typical assertion in this framework is, for any lower semicontinuous map $c : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$, the following equality holds:

$$\inf_{\gamma \in \Pi(m,m')} \int_{X\times X} c(x,y)d\gamma(x,y) = \sup_{f,g} \int_X f(x)dm(x) + \int_X g(y)dm'(y) \ ,$$

where the supremum is taken over the maps $f \in L^1_m(X)$, $g \in L^1_{m'}(X)$ such that

$$f(x) + g(y) \leq c(x,y) \qquad \text{for } m-\text{a.e. } x \text{ and } m'-\text{a.e } y.$$

The proof of this result exceeds the scope of these notes and can be found in several textbooks (see [178] for instance).

Let us finally underline the link between convergence for the $\mathbf{d}_p$ distance and narrow convergence:

Proposition 4 *A sequence of measures* (m_n) *of* $\mathcal{P}_p(X)$ *converges to* m *for* $\mathbf{d}_p$ *if and only if if* (m_n) *narrowly converges to* m *and*

$$\lim_{n\to+\infty} \int_X d^p(x,x_0)dm_n(x) = \int_X d^p(x,x_0)dm(x) \qquad \text{for some (and thus any) } x_0 \in X \ .$$

The proof for $p = 1$ is a simple consequence of Proposition 1 and Theorem 18. For the general case, see [178].

1.4.1.2 The Wasserstein space of probability measures on $\mathbb{R}^d$

From now on we work in $X = \mathbb{R}^d$. Let $\mathcal{P}_2 = \mathcal{P}_2(\mathbb{R}^d)$ be the set of Borel probability measures on $\mathbb{R}^d$ with a finite second order moment: m belongs to $\mathcal{P}_2$ if m is a Borel probability measure on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} |x|^2 m(dx) < +\infty$. The Wasserstein distance is just the Monge-Kankorovich distance when $p = 2$:

$$\mathbf{d}_2(\mu,\nu) = \inf_{\gamma\in\Pi(\mu,\nu)} \left[\int_{\mathbb{R}^{2d}} |x-y|^2 d\gamma(x,y)\right]^{1/2} \tag{1.146}$$

where $\Pi(\mu,\nu)$ is the set of Borel probability measures on $\mathbb{R}^{2d}$ such that $\gamma(A\times\mathbb{R}^d) = \mu(A)$ and $\gamma(\mathbb{R}^d \times A) = \nu(A)$ for any Borel set $A \subset \mathbb{R}^d$.

An important point, that we shall use sometimes, is the fact that the optimal transport plan can be realized as *an optimal transport map* whenever μ is absolutely continuous.

Theorem 19 (Existence of an optimal transport map) *If* $\mu \in \mathcal{P}_2$ *is absolutely continuous, then, for any* $\nu \in \mathcal{P}_2$, *there exists a convex map* $\Phi : \mathbb{R}^N \to \mathbb{R}$ *such that the measure* $(id_{\mathbb{R}^d}, D\Phi)\sharp\mu$ *is optimal for* $\mathbf{d}_2(\mu,\nu)$. *In particular* $\nu = D\Phi\sharp\mu$.

Conversely, if the convex map $\Phi : \mathbb{R}^N \to \mathbb{R}$ *satisfies* $\nu = D\Phi\sharp\mu$, *then the measure* $(id_{\mathbb{R}^d}, D\Phi)\sharp\mu$ *is optimal for* $\mathbf{d}_2(\mu,\nu)$.

The proof of this result, due to Y. Brenier [39], exceeds the scope of these notes. It can be found in various places, such as [178].

1.4.2 Derivatives in the space of measures

In this section, we discuss different notions of derivatives in the space of probability measures and explain how they are related. This part is, to a large extent, borrowed from [56, 68]. For simplicity, we work in the whole space $\mathbb{R}^d$ and set $\mathcal{P}_2 = \mathcal{P}_2(\mathbb{R}^d)$.

1.4.2.1 The flat derivative

Definition 9 Let $U : \mathcal{P}_2 \to \mathbb{R}$. We say that U is of class C^1 if there exists a jointly continuous and bounded map $\frac{\delta U}{\delta m} : \mathcal{P}_2 \times \mathbb{R}^d \to \mathbb{R}$ such that

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-h)m + hm', y)(m' - m)(dy)dh \qquad \forall m, m' \in \mathcal{P}_2.$$

Moreover we adopt the normalization convention

$$\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, y)m(dy) = 0 \qquad \forall m \in \mathcal{P}_2. \tag{1.147}$$

Remark 18 If $U : \mathcal{P}_2(\mathbb{T}^d) \to \mathbb{R}$, then the derivative is defined in the same way, with $\frac{\delta U}{\delta m} : \mathcal{P}_2(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-h)m + hm', y)(m' - m)(dy)dh \qquad \forall m, m' \in \mathcal{P}_2(\mathbb{T}^d).$$

If U is of class C^1, then the following equality holds for any $m \in \mathcal{P}_2$ and $y \in \mathbb{R}^d$

$$\frac{\delta U}{\delta m}(m, y) = \lim_{h \to 0^+} \frac{1}{h}\left(U((1-h)m + h\delta_y) - U(m)\right).$$

Here is a kind of converse.

Proposition 5 *Let $U : \mathcal{P}_2 \to \mathbb{R}$ and assume that the limit*

$$V(m, y) := \lim_{h \to 0^+} \frac{1}{h}\left(U((1-h)m + h\delta_y) - U(m)\right)$$

exists and is jointly continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d$. Then U is C^1 and $\frac{\delta U}{\delta m}(m, y) = V(m, y)$.

Proof. Although the result can be expected, the proof is a little involved and can be found in [55]. □

Let us recall that, if $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a Borel measurable map and m is a Borel probability measure on $\mathbb{R}^d$, the image of m by ϕ is the Borel probability measure $\phi\sharp m$ defined by

$$\int_{\mathbb{R}^d} f(x)\phi\sharp m(dx) = \int_{\mathbb{R}^d} f(\phi(y))m(dy) \qquad \forall f \in C_b^0(\mathbb{R}^d).$$

Proposition 6 *Let U be C^1 and be such that $D_y \frac{\delta U}{\delta m}$ exists and is jointly continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d$. Then, for any Borel measurable map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ with at most a linear growth, the map $s \to U((id_{\mathbb{R}^d} + s\phi)\sharp m)$ is differentiable at 0 and*

$$\frac{d}{ds} U((id_{\mathbb{R}^d} + s\phi)\sharp m)_{|s=0} = \int_{\mathbb{R}^d} D_y \frac{\delta U}{\delta m}(m, y) \cdot \phi(y)m(dy).$$

Proof. Indeed

$$U((id_{\mathbb{R}^d}+s\phi)\sharp m)-U(m)=\int_0^1\int_{\mathbb{R}^d}\frac{\delta U}{\delta m}(m_{h,s},y)((id_{\mathbb{R}^d}+s\phi)\sharp m)-m)(dy)dh$$
$$=\int_0^1\int_{\mathbb{R}^d}(\frac{\delta U}{\delta m}(m_{h,s},y+s\phi(y))-\frac{\delta U}{\delta m}(m_{h,s},y))m(dy)dh$$
$$=s\int_0^1\int_0^1\int_{\mathbb{R}^d}D_y\frac{\delta U}{\delta m}(m_{h,s},y+s\tau\phi(y))\cdot\phi(y)m(dy)dhd\tau,$$

where

$$m_{h,s}=(1-h)m+h(id_{\mathbb{R}^d}+s\phi)\sharp m.$$

Dividing by s and letting $s\to 0^+$ gives the desired result. □

Let us recall that, if $m,m'\in\mathcal{P}_2$, the set $\Pi^{opt}(m,m')$ denotes the set of optimal transport plans between m and m' (see Lemma 18).

Proposition 7 *Under the assumptions of the previous Proposition, let $m,m'\in\mathcal{P}_2$ and $\pi\in\Pi^{opt}(m,m')$. Then*

$$\left|U(m')-U(m)-\int_{\mathbb{R}^{2d}}D_y\frac{\delta U}{\delta m}(m,x)\cdot(y-x)\pi(dx,dy)\right|\le o(\mathbf{d}_2(m,m')).$$

Remark 19 The same proof shows that, if π is a transport plan between m and m' (not necessarily optimal), then

$$\left|U(m')-U(m)-\int_{\mathbb{R}^{2d}}D_y\frac{\delta U}{\delta m}(m,x)\cdot(y-x)\pi(dx,dy)\right|\le o\left(\left(\int_{\mathbb{R}^{2d}}|x-y|^2\pi(dx,dy)\right)^{1/2}\right).$$

Proof. Let $\phi_t(x,y)=(1-t)x+ty$ and $m_t=\phi_t\sharp\pi$. Then $m_0=m$ and $m_1=m'$ and, for any $t\in(0,1)$ and any s small we have

$$U(\phi_{t+s}\sharp\pi)-U(\phi_t\sharp\pi)=\int_0^1\int_{\mathbb{R}^d}\frac{\delta U}{\delta m}(m_{s,h},y)(\phi_{t+s}\sharp\pi-\phi_t\sharp\pi)(dy)dh$$
$$=\int_0^1\int_{\mathbb{R}^{2d}}\frac{\delta U}{\delta m}(m_{s,h},(1-t-s)x+(t+s)y)-\frac{\delta U}{\delta m}(m_{s,h},(1-t)x+ty)\;\pi(dx,dy)dh$$
$$=s\int_0^1\int_0^1\int_{\mathbb{R}^{2d}}D_y\frac{\delta U}{\delta m}(m_{s,h},(1-t-\tau s)x+(t+\tau s)y)\cdot(y-x)\;\pi(dx,dy)dhd\tau,$$

where $m_{s,h}=(1-h)\phi_{t+s}\sharp\pi+h\phi_t\sharp\pi$. So, dividing by s and letting $s\to 0$ we find:

$$\frac{d}{dt}U(\phi_t\sharp\pi)=\int_{\mathbb{R}^{2d}}D_y\frac{\delta U}{\delta m}(\phi_t\sharp\pi,(1-t)x+ty)\cdot(y-x)\;\pi(dx,dy).$$

As $D_y\frac{\delta U}{\delta m}$ is continuous and bounded by C, for any $\varepsilon,R>0$, there exists $r>0$ such that, if $\mathbf{d}_2(m,m')\le r$ and $|x|,|y|\le R$, then

$$|D_y\frac{\delta U}{\delta m}(\phi_t\sharp\pi,(1-t)x+ty)-D_y\frac{\delta U}{\delta m}(m,x)|\le\varepsilon+2C\mathbf{1}_{|y-x|\ge r}.$$

So

$$\left|\int_{\mathbb{R}^{2d}}D_y\frac{\delta U}{\delta m}(\phi_t\sharp\pi,(1-t)x+ty)\cdot(y-x)\;\pi(dx,dy)-\int_{\mathbb{R}^{2d}}D_y\frac{\delta U}{\delta m}(m,x)\cdot(y-x)\pi(dx,dy)\right|$$
$$\le\delta_R+\int_{(B_R)^2}(\varepsilon+2C\mathbf{1}_{|x-y|\ge r})|y-x|\pi(dx,dy)\le\delta_R+\varepsilon\mathbf{d}_2(m,m')+\frac{2C}{r}\mathbf{d}_2^2(m,m').$$

where

$$\delta_R := \int_{\mathbb{R}^{2d}\backslash(B_R)^2} |D_y \frac{\delta U}{\delta m}(\phi_t \sharp \pi, (1-t)x+ty)\cdot(y-x)| + |D_y \frac{\delta U}{\delta m}(m,x)\cdot(y-x)|\pi(dx,dy)$$
$$\le C\int_{\mathbb{R}^{2d}\backslash(B_R)^2} |y-x|\pi(dx,dy) \le C\mathbf{d}_2(m,m')\pi^{1/2}(\mathbb{R}^{2d}\backslash(B_R)^2) = \mathbf{d}_2(m,m')o_R(1).$$

This proves the result. □

1.4.2.2 *W*−differentiability

Next we turn to a more geometric definition of derivative in the space of measures. For this, let us introduce the notion of tangent space to $\mathcal{P}_2$.

Definition 10 (Tangent space) The tangent space $\mathrm{Tan}_m(\mathcal{P}_2)$ of $\mathcal{P}_2$ at $m \in \mathcal{P}_2$ is the closure in $L^2_m(\mathbb{R}^d)$ of $\{D\phi,\ \phi \in C^\infty_c(\mathbb{R}^d)\}$.

Following [18] we define the super and the subdifferential of a map defined on $\mathcal{P}_2$:

Definition 11 Let $U : \mathcal{P}_2 \to \mathbb{R}$, $m \in \mathcal{P}_2$ and $\xi \in L^2_m(\mathbb{R}^d, \mathbb{R}^d)$. We say that ξ belongs to the superdifferential $\partial^+ U(m)$ of U at m if, for any $m' \in \mathcal{P}_2$ and any transport plan π from m to m',

$$U(m') \le U(m) + \int_{\mathbb{R}^d\times\mathbb{R}^d} \xi(x)\cdot(y-x)\pi(dx,dy) + o\left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2\pi(dx,dy)\right)^{1/2}\right).$$

We say that ξ belongs to the subdifferential $\partial^- U(m)$ of U at m if $-\xi$ belongs to $D^+(-U)(m)$. Finally, we say that the map U is $W-$differentiable at m if $\partial^+U(m)\cap\partial^-U(m)$ is not empty.

One easily checks the following:

Proposition 8 *If U is $W-$differentiable at m, then $\partial^+U(m)$ and $\partial^-U(m)$ are equal and reduce to a singleton, denoted* $\{D_m U(m,\cdot)\}$.

Remark 20 On can actually check that $D_mU(m,\cdot)$ belongs to $\mathrm{Tan}_m(\mathcal{P}_2)$.

Proof. Let $\xi_1 \in D^+U(m)$ and $\xi_2 \in D^-U(m)$. We have, for any $m' \in \mathcal{P}_2$ and any transport plan π from m to m',

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \xi_2(x)\cdot(y-x)\pi(dx,dy) + o\left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2\pi(dx,dy)\right)^{1/2}\right)$$
$$\le U(m')-U(m) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} \xi_1(x)\cdot(y-x)\pi(dx,dy) + o\left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2\pi(dx,dy)\right)^{1/2}\right).$$

In particular, if we choose $m' = (1+h\phi)\sharp m$ and $\pi = (Id, Id+h\phi)\sharp m$ for some $\phi \in L^2_m(\mathbb{R}^d,\mathbb{R}^d)$ and $h>0$ small, we obtain

$$h\int_{\mathbb{R}^d} \xi_2(x)\cdot\phi(x)m(dx) + o\,(h) \le U(m')-U(m) \le h\int_{\mathbb{R}^d} \xi_1(x)\cdot\phi(x)m(dx) + o\,(h)\,,$$

from which we easily infer that $\xi_1 = \xi_2$ in $L^2_m(\mathbb{R}^d)$. □

Remark 19 implies that, if U is C^1 with $D_y\delta U/\delta m$ continuous and bounded, then U is W-differentiable. In this case it is obvious that $D_y\delta U/\delta m$ belongs to $\mathrm{Tan}_m(\mathcal{P}_2)$ by definition and that $D_mU(m,\cdot) = D_y\delta U/\delta m$. From now on we systematically use the notation $D_mU(m,\cdot) = D_y\delta U/\delta m$ in this case.

1.4.2.3 Link with the $L-$ derivative

Another possibility for the notion of derivative is to look at probability measures as the law of random variables with values in $\mathbb{R}^d$ and to use the fact that the set of random variables, under suitable moment conditions, is a Hilbert space.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ an atomless probability space (meaning that, for any $E \in \mathcal{F}$ with $\mathbb{P}[E] > 0$, there exists $E' \in \mathcal{F}$ with $E' \subset E$ and $0 < \mathbb{P}[E'] < \mathbb{P}[E]$). Given a map $U : \mathcal{P}_2 \to \mathbb{R}$, we consider its extension $\tilde{U}$ to the set of random variables $L^2(\Omega, \mathbb{R}^d)$:

$$\tilde{U}(X) = U(\mathcal{L}(X)) \qquad \forall X \in L^2(\Omega, \mathbb{R}^d).$$

(recall that $\mathcal{L}(X)$ is the law of X, i.e., $\mathcal{L}(X) := X\sharp\mathbb{P}$. Note that $\mathcal{L}(X)$ belongs to $\mathcal{P}_2$ because $X \in L^2(\Omega)$). The main point is that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Hilbert space, in which the notion of Frechet differentiability makes sense.

For instance, if U is a map of the form

$$U(m) = \int_{\mathbb{R}^d} \phi(x)m(dx) \qquad \forall m \in \mathcal{P}_2, \tag{1.148}$$

where $\phi \in C_c^0(\mathbb{R}^d)$ is given, then

$$\tilde{U}(X) = \mathbb{E}[\phi(X)] \qquad \forall X \in L^2(\Omega, \mathbb{R}^d).$$

Definition 12 The map $U : \mathcal{P}_2 \to \mathbb{R}$ is $L-$differentiable at $m \in \mathcal{P}_2$ if there exists $X \in L^2(\Omega, \mathbb{R}^d)$ such that $\mathcal{L}(X) = m$ and the extension $\tilde{U}$ of U is Frechet differentiable at X.

The following result says that the notion of $L-$differentiability coincides with that of $W-$differentiability and is independent of the probability space and of the representative X. The first statement in that direction goes back to Lions [149] (Lesson 31/10 2008), the version given here can be found in [109] (see also [15], from which the sketch of proof of Lemma 21 is largely inspired).

Theorem 20 *The map U is $W-$differentiable at $m \in \mathcal{P}_2$ if and only if $\tilde{U}$ is Frechet differentiable at some (or thus any) $X \in L^2(\Omega, \mathbb{R}^d)$ with $\mathcal{L}(X) = m$. In this case*

$$\nabla\tilde{U}(X) = D_m U(m, X).$$

The result can be considered as a structure theorem for the L-derivative.

For instance, if U is as in (1.148) for some map $\phi \in C_c^1(\mathbb{R}^d)$, then it is almost obvious that

$$\nabla\tilde{U}(X) = D\phi(X)$$

and thus

$$D_m U(m, x) = D\phi(x).$$

The proof of Theorem 20 is difficult and we only sketch it briefly. Complete proofs can be found in [109] or [15]. The first step is the fact that, if X and X' have the same law, then so do $\nabla\tilde{U}(X)$ and $\nabla\tilde{U}(X')$:

Lemma 20 *Let $U : \mathcal{P}_2 \to \mathbb{R}$ and $\tilde{U}$ be its extension. Let X, X' be two random variables in $L^2(\Omega, \mathbb{R}^d)$ with $\mathcal{L}(X) = \mathcal{L}(X')$. If $\tilde{U}$ is Frechet differentiable at X, then $\tilde{U}$ is differentiable at X' and $(X, \nabla\tilde{U}(X))$ has the same law as $(X', \nabla\tilde{U}(X'))$.*

(Sketch of) proof. The idea behind this fact is that, if X and X' have the same law, then one can “almost" find a bi-measurable and measure-preserving transformation $\tau : \Omega \to \Omega$ such that $X = X' \circ \tau$. Admitting this statement for a while, we have, for any $H' \in L^2$ small,

$$\begin{aligned}\tilde{U}(X' + H') &= \tilde{U}((X' + H') \circ \tau) = \tilde{U}(X + H' \circ \tau) = \tilde{U}(X) + \mathbb{E}\left[\nabla\tilde{U}(X) \cdot H' \circ \tau\right] + o(\|H' \circ \tau\|_2) \\ &= \tilde{U}(X') + \mathbb{E}\left[\nabla\tilde{U}(X) \circ \tau^{-1} \cdot H'\right] + o(\|H'\|_2).\end{aligned}$$

This shows that $\tilde{U}$ is differentiable at X' with differential given by $\nabla\tilde{U}(X)\circ\tau^{-1}$. Thus $(X', \nabla\tilde{U}(X')) = (X, \nabla\tilde{U}(X))\circ\tau^{-1}$, which shows that $(X, \nabla\tilde{U}(X))$ and $(X', \nabla\tilde{U}(X'))$ have the same law.

In fact the existence of τ does not hold in general. However, one can show that, for any $\varepsilon > 0$, there exists $\tau : \Omega \to \Omega$ bi-measurable and measure preserving and such that $\|X' - X\circ\tau\|_\infty \leq \varepsilon$. A (slightly technical) adaptation of the proof above then gives the result (see [51] or [68] for the details). □

Next we show that $\nabla\tilde{U}(X)$ is a function of X:

Lemma 21 *Assume that $\tilde{U}$ is differentiable at $X \in L^2(\Omega, \mathbb{R}^d)$. Then there exists a Borel measurable map $g : \mathbb{R}^d \to \mathbb{R}^d$ such that $\nabla\tilde{U}(X) = g(X)$ a.s..*

(Sketch of) proof. To prove the claim, we just need to check that $\nabla\tilde{U}(X)$ is $\sigma(X)-$measurable (see Theorem 20.1 in [37]), which can be recasted into the fact that $\nabla\tilde{U}(X) = \mathbb{E}\left[\nabla\tilde{U}(X)|X\right]$. Let $\mu = \mathcal{L}(X, \nabla\tilde{U}(X))$ and let $\mu(dx, dy) = (\delta_x \otimes \nu_x(dy))\mathbb{P}_X(dx)$ be its disintegration with respect to its first marginal $\mathbb{P}_X$. Let λ be the restriction of the Lebesgue measure to $Q_1 := [0,1]^d$. Then, as λ has an L^1 density, the optimal transport from λ to ν_x is unique and given by the gradient of a convex map $\psi_x(\cdot)$ (Brenier's Theorem, see [179]). So we can find[2] a measurable map $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that, for $\mathbb{P}_X-$a.e. $x \in \mathbb{R}^d$, $\psi_x(\cdot)\sharp\lambda = \nu_x$. Let Z be a random variable with law λ and independent of $(X, \nabla\tilde{U}(X))$.

Note that $\mu = \mathcal{L}(X, \nabla\tilde{U}(X)) = \mathcal{L}(X, \psi_X(Z))$ because, for any $f \in C_b^0(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\begin{aligned}\mathbb{E}\left[f(X, \psi_X(Z))\right] &= \int_{\mathbb{R}^d}\int_{Q_1} f(x, \psi_x(z))\lambda(dz)\mathbb{P}_X(dx) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y)(\psi_x\sharp\lambda)(dy)\mathbb{P}_X(dx)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y))\nu_x(dy)\mathbb{P}_X(dx) = \int_{\mathbb{R}^{2d}} f(x,y)\mu(dx,dy).\end{aligned}$$

So, for any ε,

$$\tilde{U}(X + \varepsilon\nabla\tilde{U}(X)) = \tilde{U}(X + \varepsilon\psi_X(Z)),$$

from which we infer, taking the derivative with respect to ε at $\varepsilon = 0$:

$$\mathbb{E}\left[\left|\nabla\tilde{U}(X)\right|^2\right] = \mathbb{E}\left[\nabla\tilde{U}(X)\cdot\psi_X(Z)\right].$$

Note that, as Z is independent of $(X, \nabla\tilde{U}(X))$, we have

$$\mathbb{E}\left[\nabla\tilde{U}(X)\cdot\psi_X(Z)\right] = \mathbb{E}\left[\nabla\tilde{U}(X)\cdot\mathbb{E}\left[\psi_x(Z)\right]_{x=X}\right],$$

where, for $\mathbb{P}_X-$a.e. x,

$$\mathbb{E}\left[\psi_x(Z)\right] = \int_{Q_1}\psi_x(z)\lambda(dz) = \int_{Q_1} y\ (\psi_x\sharp\lambda)(dy) = \int_{\mathbb{R}^d} y\ \nu_x(dy) = \mathbb{E}\left[\nabla\tilde{U}(X)|X = x\right].$$

So, by the tower property of the conditional expectation, we have

$$\mathbb{E}\left[\left|\nabla\tilde{U}(X)\right|^2\right] = \mathbb{E}\left[\nabla\tilde{U}(X)\cdot\mathbb{E}\left[\nabla\tilde{U}(X)|X\right]\right] = \mathbb{E}\left[\left|\mathbb{E}\left[\nabla\tilde{U}(X)|X\right]\right|^2\right].$$

Using again standard properties of the conditional expectation we infer the equality $\nabla\tilde{U}(X) = \mathbb{E}\left[\nabla\tilde{U}(X)|X\right]$, which shows the result. □

Proof of Theorem 20. Let us first assume that U is $W-$differentiable at some $m \in \mathcal{P}_2$. Then there exists $\xi := D_mU(m,\cdot) \in L^2_m(\mathbb{R}^d)$ such that, for any $m' \in \mathcal{P}_2$ and any transport plan π between m and m' we have

[2] Warning: here the proof is sloppy and the possibility of a measurable selection should be justified.

$$\left|U(m') - U(m) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \xi(x)\cdot(y-x)\pi(dx,dy)\right| \leq o\left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2\pi(dx,dy)\right)^{1/2}\right).$$

Therefore, for any $X \in L^2$ such that $\mathcal{L}(X) = m$, for any $H \in L^2$, if we denote by m' the law of $X + H$ and by π the law of $(X, X+H)$, we have

$$\begin{aligned}\left|\ \tilde{U}(X+H) - \tilde{U}(X) - \mathbb{E}\left[\xi(X)\cdot H\right]\right| &= \left|\ U(m') - \tilde{U}(m) - \int_{\mathbb{R}^{2d}} \xi(x)\cdot(y-x)\pi(x,y)\right| \\ &\leq o\left(\left(\int_{\mathbb{R}^{2d}} |x-y|^2\pi(dx,dy)\right)^{1/2}\right)) \\ &= o\left(\mathbb{E}\left[|X-Y|^2\right]^{1/2}\right).\end{aligned}$$

This shows that U is L-differentiable.

Conversely, let us assume that U is L-differentiable at m. We know from Lemma 21 that, for any $X \in L^2$ such that $\mathcal{L}(X) = m$, $\tilde{U}$ is differentiable at X and $\nabla\tilde{U}(X) = \xi(X)$ for some Borel measurable map $\xi : \mathbb{R}^d \to \mathbb{R}^d$. In view of Lemma 20, the map ξ does not depend on the choice of X. So, for any $\varepsilon > 0$, there exists $r > 0$ such that, for any X with $\mathcal{L}(X) = m$ and any $H \in L^2$ with $\|H\| \leq r$, one has

$$\left|\ \tilde{U}(X+H) - \tilde{U}(X) - \mathbb{E}\left[\xi(X)\cdot H\right]\right| \leq \varepsilon.$$

Let now $m' \in \mathcal{P}_2$ and π be a transport plan between m and m' such that $\int_{\mathbb{R}^{2d}} |x-y|^2\pi(dx,dy) \leq r^2$. Let (X,Y) with law π. We set $H = Y - X$ and note that $\|H\|_2 \leq r$. So we have

$$\left|\ U(m') - \tilde{U}(m) - \int_{\mathbb{R}^{2d}} \xi(x)\cdot(y-x)\pi(x,y)\right| = \left|\ \tilde{U}(X+H) - \tilde{U}(X) - \mathbb{E}\left[\xi(X)\cdot H\right]\right| \leq \varepsilon.$$

This proves the W-differentiability of U. □

1.4.2.4 Higher order derivatives

We say that U is partially C^2 if U is C^1 and if $D_y\delta U/\delta m$ and $D^2_{yy}\delta U/\delta m$ exist and are continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d$.

We say that U is C^2 if $\frac{\delta U}{\delta m}$ is C^1 in m with a continuous and bounded derivative: namely $\frac{\delta^2 U}{\delta m^2} = \frac{\delta}{\delta m}(\frac{\delta U}{\delta m}) : \mathcal{P}_2 \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in all variables and bounded. We say that U is twice $L-$differentiable if the map D_mU is $L-$differentiable with respect to m with a second order derivative $D^2_{mm}U = D^2_{mm}U(m,y,y')$ which is continuous and bounded on $\mathcal{P}_2 \times \mathbb{R}^d \times \mathbb{R}^d$ with values in $\mathbb{R}^{d\times d}$. One can check that this second order derivative enjoys standard properties of derivatives, such as the symmetry:

$$D^2_{mm}U(m,y,y') = D^2_{mm}U(m,y',y).$$

See [56, 68].

1.4.2.5 Comments

For a general description of the notion of derivatives and the historical background, we refer to [68], Chap. V. The notion of flat derivative is very natural and has been introduced in several contexts and under various assumptions. We follow here [56]. Let us note however that these notions of derivatives can be traced back to [14], while the construction of Proposition 5 has already a counterpart in [159].

The initial definition of sub and super differential in the space $\mathcal{P}_2$, introduced in [19], is the following: ξ belongs to $\partial^+ U(m)$ if $\xi \in \mathrm{Tan}_m(\mathcal{P}_2)$ and

$$U(m') \leq U(m) + \inf_{\pi \in \Pi^{opt}(m,m')} \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y-x) \pi(dx,dy) + o(\mathbf{d}_2(m,m')).$$

It is proved in [109] that this definition coincides with the one introduced in Definition 11.

The notion of L-derivative and the structure of this derivative has been first discussed by Lions in [149] (see also [51] for a proof of Theorem 20 in which the function is supposed to be continuously differentiable). The proof of Theorem 20, without the extra continuity condition, is due to Gangbo and Tudorascu [109] (see also [15], revisited here in a loose way).

1.4.3 The Master equation

In this section we investigate the partial differential equation:

$$\begin{cases} -\partial_t U - \Delta_x U + H(x, D_x U) - \displaystyle\int_{\mathbb{T}^d} \mathrm{div}_y D_m U(t,x,m,y)\, dm(y) \\ \qquad + \displaystyle\int_{\mathbb{T}^d} D_m U(t,x,m,y) \cdot H_p(y, D_x U)\, dm(y) = F(x,m) \\ \qquad\qquad\qquad \text{in } (0,T) \times \mathbb{T}^d \times \mathcal{P}_2 \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}_2 \end{cases} \tag{1.149}$$

In this equation, $U = U(t,x,m)$ is the unknown. As explained below, $U(t,x,m)$ can be interpreted as the minimal cost, in the mean field problem, for a small player at time t in position x, if the distribution of the other players is m. Equation (1.149) is often called the *first order master equation* since it only involves first order derivatives with respect to the measure. This is in contrast with what happens for MFG problems with a common noise, for which the corresponding master equation also involves second order derivatives (see Subsection 1.4.3.3). After explaining the existence of the uniqueness of a solution for (1.149) (Subsection 1.4.3.1), we discuss other frameworks for the master equation: the case of finite state space (Subsection 1.4.3.2) and the MFG problem with a common noise (Subsection 1.4.3.3).

Throughout this part, we work in the torus $\mathbb{T}^d$ and in the space $\mathcal{P}_2 = \mathcal{P}_2(\mathbb{T}^d)$ of Borel probability measures on $\mathbb{T}^d$ endowed with the Wasserstein distance $\mathbf{d}_2$. The notion of derivative is the one discussed in the previous part (with the minor difference explained in Remark 18).

1.4.3.1 Existence and uniqueness of a solution for the master equation

Definition 13 We say that a map $V : [0,T] \times \mathbb{T}^d \times \mathcal{P}_2 \to \mathbb{R}$ is a classical solution to the Master equation (1.149) if

- V is continuous in all its arguments (for the $\mathbf{d}_1$ distance on $\mathcal{P}_2$), is of class C^2 in x and C^1 in time,
- V is of class C^1 with respect to m with a derivative $\frac{\delta V}{\delta m} = \frac{\delta V}{\delta m}(t,x,m,y)$ having globally continuous first and second order derivatives with respect to the space variables.
- The following relation holds for any $(t,x,m) \in (0,T) \times \mathbb{T}^d \times \mathcal{P}_2$:

$$\begin{cases} -\partial_t V(t,x,m) - \Delta_x V(t,x,m) + H(x, D_x V(t,x,m)) - \displaystyle\int_{\mathbb{T}^d} \mathrm{div}_y D_m V(t,x,m,y)\, dm(y) \\ \qquad + \displaystyle\int_{\mathbb{T}^d} D_m V(t,x,m,y) \cdot H_p(y, D_x V(t,y,m))\, dm(y) = F(x,m) \end{cases}$$

and $V(T,x,m) = G(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}_2$.

Throughout the section, $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth, globally Lipschitz continuous and satisfies the coercivity condition:

$$C^{-1}\frac{I_d}{1+|p|} \leq H_{pp}(x,p) \leq CI_d \qquad \text{for } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{1.150}$$

We also always assume that the maps $F, G : \mathbb{T}^d \times \mathcal{P}_1 \to \mathbb{R}$ are globally Lipschitz continuous and monotone:

$$F \text{ and } G \text{ are monotone.} \tag{1.151}$$

Note that assumption (1.151) implies that $\frac{\delta F}{\delta m}$ and $\frac{\delta G}{\delta m}$ satisfy the following monotonicity property (explained for F):

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\delta F}{\delta m}(x,m,y)\mu(x)\mu(y)dxdy \geq 0$$

for any smooth map $\mu : \mathbb{T}^d \to \mathbb{R}$ with $\int_{\mathbb{T}^d} \mu = 0$.

Let us fix $n \in \mathbb{N}^*$ and $\alpha \in (0,1/2)$. We set

$$\mathrm{Lip}_n(\frac{\delta F}{\delta m}) := \sup_{m_1 \neq m_2} (\mathbf{d}_1(m_1,m_2))^{-1} \left\| \frac{\delta F}{\delta m}(\cdot, m_1, \cdot) - \frac{\delta F}{\delta m}(\cdot, m_2, \cdot) \right\|_{C^{n+2\alpha}\times C^{n-1+2\alpha}}$$

and use the symmetric notation for G. We call **(HF(n))** the following regularity conditions on F:

$$(\mathbf{HF}(\mathbf{n})) \qquad \sup_{m\in\mathcal{P}_1} \left(\|F(\cdot,m)\|_{C^{n+2\alpha}} + \left\| \frac{\delta F(\cdot,m,\cdot)}{\delta m} \right\|_{C^{n+2\alpha}\times C^{n+2\alpha}} \right) + \mathrm{Lip}_n(\frac{\delta F}{\delta m}) \; < \; \infty.$$

and **(HG(n))** the symmetric condition on G:

$$(\mathbf{HG}(\mathbf{n})) \qquad \sup_{m\in\mathcal{P}_1} \left(\|G(\cdot,m)\|_{C^{n+2\alpha}} + \left\| \frac{\delta G(\cdot,m,\cdot)}{\delta m} \right\|_{C^{n+2\alpha}\times C^{n+2\alpha}} \right) + \mathrm{Lip}_n(\frac{\delta G}{\delta m}) \; < \; \infty.$$

In order to explain the existence of a solution to the master equation, we need to introduce the solution of the MFG system: for any $(t_0, m_0) \in [0,T) \times \mathcal{P}_2$, let (u,m) be the solution to:

$$\begin{cases} -\partial_t u - \Delta u + H(x,Du) = F(x,m(t)) \\ \partial_t m - \Delta m - \mathrm{div}(mH_p(x,Du)) = 0 \\ u(T,x) = G(x,m(T)), \; m(t_0,\cdot) = m_0 \end{cases} \tag{1.152}$$

Thanks to the monotonicity condition (1.151), we know that the system admits a unique solution, see Theorem 4. Then we set

$$U(t_0,x,m_0) := u(t_0,x) \tag{1.153}$$

Theorem 21 *Assume that* **(HF(n))** *and* **(HG(n))** *hold for some* $n \geq 4$. *Then the map* U *defined by* (1.153) *is the unique classical solution to the master equation* (1.149).
Moreover, U *is globally Lipschitz continuous in the sense that*

$$\|U(t_0,\cdot,m_0) - U(t_0,\cdot,m_1)\|_{C^{n+\alpha}} \leq C_n \mathbf{d}_1(m_0,m_1) \tag{1.154}$$

with Lipschitz continuous derivatives:

$$\|D_m U(t_0,\cdot,m_0,\cdot) - D_m U(t_0,\cdot,m_1,\cdot)\|_{C^{n+\alpha}\times C^{n+\alpha}} \leq C_n \mathbf{d}_1(m_0,m_1) \tag{1.155}$$

for any $t_0 \in [0,T]$, $m_0, m_1 \in \mathcal{P}_1$.

Relation (1.153) says that the solutions of the MFG system (1.152) can be considered as characteristics of the master equation (1.149). As it will be transparent in the analysis of the MFG problem on a finite state space (Subsection 1.4.3.2 below), this means that the master equation is a kind of transport in the space of measures. The difficulty is that it is nonlinear, nonlocal (because of the integral terms) and without a comparison principle.

The proof of Theorem 21, although not very difficult in its principle, is quite technical and will be mostly omitted here. The main issue is to check that the map U defined by (1.153) satisfies (1.154), (1.155). This exceeds the scope of these notes and we refer the reader to [56] for a proof. Once we know that U is quite smooth, the conclusion follows easily:

Sketch of proof of Theorem 21 (existence). Let $m_0 \in \mathcal{P}(\mathbb{T}^d)$ with a C^1, positive density. Let $t_0 > 0$, (u, m) be the solution of the MFG system (1.152) starting from m_0 at time t_0. Then

$$\frac{U(t_0+h,x,m_0)-U(t_0,x,m_0)}{h} = \frac{U(t_0+h,x,m_0)-U(t_0+h,x,m(t_0+h))}{h} + \frac{U(t_0+h,x,m(t_0+h))-U(t_0,x,m_0)}{h}.$$

As

$$\partial_t m - \mathrm{div}[m(D(\ln(m)) + H_p(x, Du))] = 0,$$

Lemma 22 below says that

$$\mathbf{d}_1(m(t_0+h), (id - h\Phi)\sharp m_0) = o(h)$$

where

$$\Phi(x) := D(\ln(m_0(x))) + H_p(x, Du(t_0, x))$$

and $o(h)/h \to 0$ as $h \to 0$. So, by Lipschitz continuity of U and then differentiability of U,

$$\begin{aligned} U(t_0+h,x,m(t_0+h)) &= U(t_0+h,x,(id-h\Phi)\sharp m_0) + o(h) \\ &= U(t_0+h,x,m_0) - h\int_{\mathbb{T}^d} D_mU(t_0+h,x,m_0,y)\cdot\Phi(y)\, m_0(y)dy + o(h), \end{aligned}$$

and therefore, by continuity of U and D_mU,

$$\lim_{h\to 0} \frac{U(t_0+h,x,m(t_0+h))-U(t_0+h,x,m_0)}{h} = -\int_{\mathbb{T}^d} (D_mU(t_0,x,m_0,y)\cdot[D(\ln(m_0)) + H_p(y, Du(t_0))])\, m_0(y)dy.$$

On the other hand, for $h > 0$,

$$U(t_0+h,x,m(t_0+h)) - U(t_0,x,m_0) = u(t_0+h,x) - u(t_0,x) = h\partial_t u(t_0,x) + o(h),$$

so that

$$\lim_{h\to 0^+} \frac{U(t_0+h,x,m(t_0+h))-U(t_0,x,m_0)}{h} = \partial_t u(t_0,x).$$

Therefore $\partial_t U(t_0, x, m_0)$ exists and is equal to

$$\begin{aligned}\partial_t U(t_0,x,m_0) &= \int_{\mathbb{T}^d}(D_mU(t_0,x,m_0,y)\cdot[D(\ln(m_0))+H_p(y,Du(t_0))])m_0(y)dy+\partial_t u(t_0,x)\\ &= -\int_{\mathbb{T}^d}\mathrm{div}_y D_mU(t_0,x,m_0,y)m_0(y)dy\\ &\qquad+\int_{\mathbb{T}^d}D_mU(t_0,x,m_0,y)\cdot H_p(y,Du(t_0))\,m_0(y)dy\\ &\qquad-\Delta u(t_0,x)+H(x,Du(t_0,x))-F(x,m_0)\\ &= -\int_{\mathbb{T}^d}\mathrm{div}_y D_mU(t_0,x,m_0,y)m_0(y)dy\\ &\qquad+\int_{\mathbb{T}^d}D_mU(t_0,x,m_0,y)\cdot H_p(y,D_xU(t_0,y,m_0))\,m_0(y)dy\\ &\qquad-\Delta_{xx}U(t_0,x,m_0)+H(x,D_xU(t_0,x,m_0))-F(x,m_0)\end{aligned}$$

This means that U satisfies (1.149) at (t_0,x,m_0). By continuity, U satisfies the equation everywhere. □

Lemma 22 *Let $V=V(t,x)$ be a C^1 vector field, $m_0\in\mathcal{P}_2$ and m be the weak solution to*

$$\begin{cases}\partial_t m+\mathrm{div}(mV)=0\\ m(0)=m_0\,.\end{cases}$$

Then

$$\lim_{h\to 0^+}\mathbf{d}_1(m(h),(id+hV(0,\cdot))\sharp m_0)/h=0.$$

Proof. Recall that $m(h)=X^\cdot(h)\sharp m_0$, where $X^x(h)$ is the solution to the ODE

$$\begin{cases}\frac{d}{dt}X^x(t)=V(t,X^x(t))\\ X^x(0)=x\,.\end{cases}$$

Let ϕ be a Lipschitz test function. Then

$$\begin{aligned}\int_{\mathbb{T}^d}\phi(x)(m(h)-(id+hV(0,\cdot))\sharp m_0)(dx) &= \int_{\mathbb{T}^d}(\phi(X^x(h))-\phi(x+hV(0,x)))m_0(dx)\\ &\le \|D\phi\|_\infty\int_{\mathbb{T}^d}|X^x(h)-x-hV(0,x)|m_0(dx)=\|D\phi\|_\infty o(h),\end{aligned}$$

which proves that $\mathbf{d}_1(m(h),(id+hV(0,\cdot))\sharp m_0)=o(h)$. □

Proof of Theorem 21 (uniqueness). We use a technique introduced in [149] (Lesson 5/12/2008), consisting at looking at the MFG system (1.152) as a system of characteristics for the master equation (1.149). We reproduce here this argument for the sake of completeness. Let V be another solution to the master equation. The main point is that, by definition of solution $D^2_{xy}\frac{\delta V}{\delta m}$ is bounded, and therefore D_xV is Lipschitz continuous with respect to the measure variable.

Let us fix (t_0,m_0). In view of the Lipschitz continuity of D_xV, one can easily uniquely solve the PDE (by standard fixed point argument):

$$\begin{cases}\partial_t\tilde m-\Delta\tilde m-\mathrm{div}(\tilde m H_p(x,D_xV(t,x,\tilde m))=0\\ \tilde m(t_0)=m_0\end{cases}$$

Then let us set $\tilde u(t,x)=V(t,x,\tilde m(t))$. By the regularity properties of V, $\tilde u$ is at least of class $C^{2,1}$ with

$$
\begin{aligned}
\partial_t \tilde u(t,x) &= \partial_t V(t,x,\tilde m(t)) + \langle \frac{\delta V}{\delta m}(t,x,\tilde m(t),\cdot), \partial_t \tilde m(t)\rangle_{C^2,(C^2)'} \\
&= \partial_t V(t,x,\tilde m(t)) + \langle \frac{\delta V}{\delta m}(t,x,\tilde m(t),\cdot), \Delta \tilde m + \mathrm{div}(\tilde m H_p(\cdot, D_x V(t,\cdot,\tilde m)))\rangle_{C^2,(C^2)'} \\
&= \partial_t V(t,x,\tilde m(t)) + \int_{\mathbb{T}^d} \mathrm{div}_y D_m V(t,x,\tilde m(t),y)\, d\tilde m(t)(y) \\
&\qquad - \int_{\mathbb{T}^d} D_m V(t,x,\tilde m(t),y)\cdot H_p(y, D_x V(t,y,\tilde m))\, d\tilde m(t)(y)
\end{aligned}
$$

Recalling that V satisfies the master equation we get

$$
\begin{aligned}
\partial_t \tilde u(t,x) &= -\Delta_x V(t,x,\tilde m(t)) + H(x, D_x V(t,x,\tilde m(t))) - F(x,\tilde m(t)) \\
&= -\Delta \tilde u(t,x) + H(x, D\tilde u(t,x)) - F(x,\tilde m(t))
\end{aligned}
$$

with terminal condition $\tilde u(T,x) = V(T,x,\tilde m(T)) = G(x,\tilde m(T))$. Therefore the pair $(\tilde u, \tilde m)$ is a solution of the MFG system (1.152). As the solution of this system is unique, we get that $V(t_0,x,m_0) = U(t_0,x,m_0)$ is uniquely defined. □

1.4.3.2 The master equation for MFG problems on a finite state space

We consider here a MFG problem on a finite state space: let $I \in \mathbb{N}$, $I \geq 2$ be the number of states. Players control their jump rate from one state to another; their cost depends on the jump rate they choose and on the distribution of the other players on the states. In this finite state setting, this distribution is simply an element of the simplex $\mathcal{S}_{I-1}$ with

$$
\mathcal{S}_{I-1} := \left\{ m \in \mathbb{R}^I,\ m = (m_i)_{i=1,\dots,I},\ m_i \geq 0,\ \forall i,\ \sum_i m_i = 1 \right\}.
$$

Given $m = (m_i) \in \mathcal{S}_{I-1}$, m_i is the proportion of players in state i.

The MFG system. In this setting the MFG system takes the form of a coupled system of ODEs: for $i = 1, \dots, I$,

$$
\begin{cases}
-\dfrac{d}{dt} u^i(t) + H^i((u^j(t) - u^i(t))_{j\neq i}, m(t)) = 0 \qquad \text{in } (0,T) \\
\dfrac{d}{dt} m_i(t) - \displaystyle\sum_{j\neq i} m_j(t) \frac{\partial H^j}{\partial p_i}((u^k(t) - u^j(t))_{k\neq j}, m(t)) \\
\qquad\qquad + m_i(t) \displaystyle\sum_{j\neq i} \frac{\partial H^i}{\partial p_j}((u^k(t) - u^i(t))_{k\neq i}, m(t)) = 0 \qquad \text{in } (0,T) \\
m_i(t_0) = m_{i,0},\ u^i(T) = g^i(m(T)).
\end{cases}
\tag{1.156}
$$

In the above system, the unknown is $(u,m) = (u^i(t), m^i(t))$, where $u^i(t)$ is the value function of a player at time t and in position i while $m(t)$ is the distribution of players at time t, with $m(t) \in \mathcal{S}_{I-1}$ for any t. The map $H^i : \mathbb{R}^{I-1} \times \mathcal{S}_{I-1} \to \mathbb{R}$ is the Hamiltonian of the problem in state i while $m_0 = (m_{i,0}) \in \mathcal{S}_{I-1}$ is the initial distribution at time $t_0 \in [0,T)$ and $g^i : \mathcal{S}_{I-1} \to \mathbb{R}$ is the terminal cost in state i. As usual, this is a forward-backward system.

The structure for uniqueness. As for standard MFG systems, the existence of a solution is relatively easy; the uniqueness relies on a specific structure of the coupling and on a monotonicity condition which become here:

$$
H^i(z,m) = h^i(z) - f^i(m) \tag{1.157}
$$

where h^i is strictly convex in z and

$$\sum_{i=1}^I (f^i(m)-f^i(m'))(m^i-(m')^i)\geq 0, \ \sum_{i=1}^I (g^i(m)-g^i(m'))(m^i-(m')^i)\geq 0, \qquad \forall m,m'\in\mathcal{S}_{I-1}. \quad (1.158)$$

The master equation. To find a solution of this MFG problem in feedback form (i.e., such that the control of a players depends on the state of this player and on the distribution of the other players), one can proceed as in the continuous space case and set $U^i(t,m_0)=u^i(t_0)$, where m_0 is the initial distribution of the players at time t_0 and (u,m) is the solution to (1.156). Then U solves the following hyperbolic system, for $i=1,\dots,I$,

$$\begin{cases} -\partial_t U^i(t,m)+H^i((U^j(t,m)-U^i(t,m))_{j\neq i},m)-\displaystyle\sum_{j=1}^I \frac{\partial U^i}{\partial p_j}(t,m)\Big(\sum_{k\neq j} m_k\frac{\partial H^k}{\partial p_j}((U^l(t,m)-U^k(t,m))_{l\neq k},m) \\ \qquad\qquad -m_j\displaystyle\sum_{k\neq j}\frac{\partial H^j}{\partial p_k}((U^l(t,m)-U^j(t,m))_{l\neq j},m)\Big)=0 \text{ in } (0,T)\times\mathcal{S}_{I-1} \\ U^i(T,m)=g^i(m) \qquad \text{in } \mathcal{S}_{I-1} \end{cases}$$

This is the master equation in the framework of the finite state space problem. It can be rewritten in a more compact way in the form

$$\partial_t U+(F(m,U)\cdot D)U=G(m,U) \tag{1.159}$$

where $F,G:\mathcal{S}_{I-1}\times\mathbb{R}^I\to\mathbb{R}^I$ are defined by

$$F(m,U)=\Big(\sum_{k\neq j} m_k\frac{\partial H^k}{\partial p_j}((U^l-U^k)_{l\neq k},m)-m_j\sum_{k\neq j}\frac{\partial H^j}{\partial p_k}((U^l-U^j)_{l\neq j},m)\Big)_j$$

and $G(m,U)=-(H^j(U,m))_j$. Equation (1.159) has to be understood as follows: for any $i\in\{1,\dots,I\}$,

$$\partial_t U^i+(F(m,U)\cdot D)U^i=-H^i(U,m).$$

Link between two notions of monotonicity. The monotonicity condition stated in (1.158) is equivalent with the fact that the pair (G,F) is monotone (in the classical sense) from $\mathbb{R}^{2d}$ into itself. Indeed, recalling the structure condition (1.157), we have

$$\begin{aligned}
&\langle (G,F)(m,U)-(G,F)(m',U'),(m,U)-(m',U')\rangle \\
&=\sum_j (h^j((U^k-U^j)_{k\neq j})-h^j((U'^k-U'^j)_{k\neq j}))(m_j-m'_j)-\sum_j (f^j(m)-f^j(m'))(m_j-m'_j) \\
&\quad+\sum_{j\neq k}\Big(m_k\frac{\partial h^k}{\partial p_j}((U^l-U^k)_{l\neq k})-m'_k\frac{\partial h^k}{\partial p_j}((U'^l-U'^k)_{l\neq k})\Big)(U^j-U'^j) \\
&\quad-\sum_{j=1}^I\Big(m_j\sum_{k\neq j}\frac{\partial h^j}{\partial p_k}((U^l-U^j)_{l\neq j})-m'_j\sum_{k\neq j}\frac{\partial h^j}{\partial p_k}((U'^l-U'^j)_{l\neq j})\Big)(U^j-U'^j) \\
&=-\sum_j (f^j(m)-f^j(m'))(m_j-m'_j) \\
&\quad-\sum_{j=1}^I m_j\Big(h^j((U'^k-U'^j)_{k\neq j})-h^j((U^k-U^j)_{k\neq j})-\sum_{k\neq j}\frac{\partial h^j}{\partial p_k}((U^l-U^j)_{l\neq j})(U'^k-U^k-U'^j-U^j)\Big) \\
&\quad-\sum_{j=1}^I m'_j\Big(h^j((U^k-U^j)_{k\neq j})-h^j((U'^k-U'^j)_{k\neq j})-\sum_{k\neq j}\frac{\partial h^j}{\partial p_k}((U'^l-U'^j)_{l\neq j})(U^k-U'^k-U^j-U'^j)\Big),
\end{aligned}$$

which is nonnegative since (1.158) holds and h is convex.

The finite state space is very convenient in the analysis of MFGs: it makes complete sense in terms of modeling and, in addition, it simplifies a lot the analysis of the master equation. First of all, this is a finite dimensional problem. Secondly, under the monotonicity condition, the solution of the master equation is also monotone and it is known that monotone maps are BV in open sets in which they are finite: so some regularity is easily available.

1.4.3.3 The MFG problem with a common noise

The aim of this part is to say a few words about the MFGs in which all agents are subject to a common source of randomness. This kind of models are often met in macro-economy, after the pioneering work of Krusell-Smith [138]. We start with a toy example, in which the agents are subject to a single shock. Then we describe the more delicate model where the shock is a Brownian motion.

An illustrative example.

We consider here a problem in which the agents face a common noise which, in this elementary example, is a random variable Z on which the coupling costs F and G depend: $F = F(x, m, Z)$ and $G = G(x, m, Z)$. The game is played in finite horizon T and the exact value of Z is revealed to the agents at time $T/2$ (to fix the ideas).

To fix the ideas, we assume that the agents directly control their drift:

$$dX_t = \alpha_t dt + \sqrt{2} dB_t$$

(where (α_t) is the control with values in $\mathbb{R}^d$ and B a Brownian motion). In contrast with the previous discussions, the control α_t is now adapted to the filtration generated by B *and to the noise* Z when $t \geq T/2$. The cost is now of the form

$$J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t|^2 + F(X_t, m(t), Z)\, dt + G(X_T, m(T), Z)\right],$$

where F and G depend on the position of the player, on the distribution of the agents and on the common noise Z. As all the agents will choose their optimal control in function of the realization of Z (of course after time $T/2$), one expect the distribution of players to be random after $T/2$ and to depend on the noise Z.

On the time interval $[T/2, T]$, the agents have to solve a classical control problem (which depends on Z and on $(m(t))$):

$$u(t, x) := \inf_\alpha \mathbb{E}\left[\int_t^T \frac{1}{2}|\alpha_t|^2 + F(X_t, m(t), Z)\, dt + G(X_T, m(T)) \mid Z\right]$$

which depends on the realization of Z and solves the HJ equation (with random coefficients):

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x, m(t), Z) \text{ in } (T/2, T) \times \mathbb{R}^d \\ u(T, x, Z) = G(x, m(T), Z) \text{ in } \mathbb{R}^d. \end{cases} \tag{1.160}$$

On the other hand, on the time interval $[0, T/2)$, the agent has no information on Z and, by dynamic programming, one expects to have

$$u(t, x) := \inf_\alpha \mathbb{E}\left[\int_t^{T/2} \frac{1}{2}|\alpha_t|^2 + \bar{F}(X_t, m(t))\, dt + u(T/2^+, X_{T/2})\right],$$

where $\bar{F}(x, m) = \mathbb{E}\left[F(x, m, Z)\right]$ (recall that $m(t)$ is deterministic on $[0, T/2]$). Thus, on the time interval $[0, T/2]$, u solves

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = \bar{F}(x, m(t)) \text{ in } (0, T/2) \times \mathbb{R}^d \\ u(T/2^-, x) = \mathbb{E}\left[u(T/2^+, x)\right] \text{ in } \mathbb{R}^d \end{cases} \tag{1.161}$$

As for the associated Kolmogorov equation, on the time interval $[0, T/2]$ (where the optimal feedback $-Du$ is purely deterministic) we have as usual:

$$\partial_t m - \Delta m - \mathrm{div}\,(m\, Du(t,x)) = 0 \text{ in } (0, T/2) \times \mathbb{R}^d, \qquad m(0) = m_0. \tag{1.162}$$

while on the time interval $[T/2, T]$, m becomes random (as the control $-Du$) and solves

$$\partial_t m - \Delta m - \mathrm{div}\,(m\, Du(t,x,Z)) = 0 \text{ in } (T/2) \times \mathbb{R}^d, \qquad m(T/2^-) = m(T/2^+). \tag{1.163}$$

Note the relation: $m(T/2^-) = m(T/2^+)$, which means that the dynamics of the crowd is continuous in time.

Let us point out some remarkable features of the problem. Firstly, the pairs (u, m) are no longer deterministic, and are adapted to the filtration generated by the common noise (here this filtration is trivial up to time $T/2$ and is the $\sigma-$algebra generated by Z after $T/2$). Secondly, the map u is discontinuous: this is due to the shock of information at time $T/2$.

The existence of a solution to the MFG system (1.160)-(1.163) can be obtained in two steps. First one solves the MFG system on $[T/2, T]$: given any measure $m_0 \in \mathcal{P}_1(\mathbb{R}^d)$, let (u, m) be the solution to

$$\begin{cases} -\partial_t u(t,x,Z) - \Delta u(t,x,Z) + \dfrac{1}{2}|Du(t,x,Z)|^2 = F(x, m(t), Z) \text{ in } (T/2, T) \times \mathbb{R}^d \\ \partial_t m(t,x,Z) - \Delta m(t,x,Z) - \mathrm{div}(m(t,x,Z) Du(t,x,Z)) = 0 \text{ in } (T/2, T) \times \mathbb{R}^d \\ m(T/2, dx, Z) = m_0(dx), \qquad u(T,x,Z) = G(x, m(T,x,Z), Z) \text{ in } \mathbb{R}^d \end{cases}$$

Note that u and m depend of course on m_0. If we require F and G to be monotone, then this solution is unique and we can set $U(x, m_0, Z) = u(T/2, x, Z)$ (with the notation of Section 1.4.3.1, it should be $U(T/2^+, x, m_0, Z)$, but we omit the $T/2$ for simplicity). It is not difficult to check that, if the couplings F, G are smoothing, then U is continuous in m (uniformly in (x, Z)), measurable in Z and C^2 in x uniformly in (m, Z). In addition, it is a simple exercise to prove that U is monotone as well. Therefore, if we set $\bar{U}(x, m) = \mathbb{E}[U(x, m, Z)]$, then $\bar{U}$ is also continuous in m and C^2 in x and monotone. So the system

$$\begin{cases} -\partial_t u(t,x) - \Delta u(t,x) + \dfrac{1}{2}|Du(t,x)|^2 = \bar{F}(x, m(t)) \text{ in } (0, T/2) \times \mathbb{R}^d \\ \partial_t m(t,x) - \Delta m(t,x) - \mathrm{div}(m(t,x) Du(t,x)) = 0 \text{ in } (0, T/2) \times \mathbb{R}^d \\ m(0, dx) = m_0(dx), \qquad u(T,x) = \bar{U}(x, m(T/2)) \text{ in } \mathbb{R}^d \end{cases}$$

has a unique solution. Note that u is a discontinuous function of time, but the discontinuity

$$u(T/2^+, x) - u(T/2^-, x) = u(T/2^+, x) - \mathbb{E}\left[u(T/2^+, x)\right]$$

has zero mean, that is, it is a "one-step martingale".

Common noise of Brownian type.

In general, MFG with a common noise involve much more complex randomness than a single shock that occurs at a given time. We discuss here very briefly a case in which the common noise is a Brownian motion. As before, we just consider an elementary model in order to fix the ideas.

The game is played in finite horizon T. The agents control directly their drift: their state solves therefore the SDE

$$dX_t = \alpha_t dt + \sqrt{2} dB_t + \sqrt{2\beta} dW_t,$$

where (α_t) is the control with values in $\mathbb{R}^d$, B the idiosyncratic noise (a Brownian motion, independent for each player) and W is the common noise (a Brownian motion, the same for each player), $\beta \geq 0$ denoting the intensity of this noise. The control α_t is now adapted to the filtration generated by B and W. The cost is of the (standard) form

$$J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t|^2 + F(X_t, m(t))\, dt + G(X_T, m(T))\right],$$

where F and G depend on the the position of the player and on the distribution of the agents.

The main difference with the classical case is that now the flow of measures m is random and adapted to the filtration generated by W. To understand why it should be so, let us come back to the setting with finitely many agents (in which one sees better the difference between B and W). If there are N agents, controlling their state with a feedback control $\alpha = \alpha_t(x)$ (possibly random), then the state of player i, for $i \in \{1, \dots, N\}$, solves

$$dX_t^i = \alpha_t(X_t^i)dt + \sqrt{2}dB_t^i + \sqrt{2\beta}dW_t.$$

Note that the B^i are independent (idiosyncratic noise) and independent of the common noise W. Let m_t^N be the empirical measure associated to the X^i:

$$m_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

Let us assume that m^N converges to some m (formally) and let us try to guess the equation for m. We have, for any smooth test function $\phi = \phi(t,x)$ with a compact support,

$$\int_{\mathbb{R}^d} \phi(t,x)m_t(dx) = \lim_N \int_{\mathbb{R}^d} \phi(t,x)m_t^N(dx),$$

where, by ItÃ´'s formula,

$$\begin{aligned}
\int_{\mathbb{R}^d} \phi(t,x)m_t^N(dx) &= \frac{1}{N}\sum_{i=1}^N \phi(t,X_t^i) \\
&= \frac{1}{N}\sum_{i=1}^N \phi(t,X_0^i) + \frac{1}{N}\sum_{i=1}^N \int_0^t (\partial_t\phi(s,X_s^i) + D\phi(s,X_s^i)\cdot\alpha_t(X_s^i) + (1+\beta)\Delta\phi(s,X_s^i))ds \\
&\quad + \frac{1}{N}\sum_{i=1}^N \int_0^t D\phi(s,X_s^i)\cdot(dB_s^i + dW_s) \\
&= \int_{\mathbb{R}^d} \phi(t,x)m^N(0,dx) + \int_0^t \int_{\mathbb{R}^d} (\partial_t\phi(s,x) + D\phi(s,x)\cdot\alpha_t(x) + (1+\beta)\Delta\phi(s,x))m_s^N(dx)ds \\
&\quad + \beta\int_0^t (\int_{\mathbb{R}^d} D\phi(s,x)m_s^N(dx))\cdot dW_s + \frac{1}{N}\sum_{i=1}^N \int_0^t D\phi(s,X_s^i)\cdot dB_s^i.
\end{aligned}$$

As $N \to +\infty$, the last term vanishes because, by ItÃ´'s isometry,

$$\lim_{N\to+\infty} \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \int_0^t D\phi(s,X_s^i)\cdot dB_s^i\right|^2\right] = \lim_{N\to+\infty} \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}\left[\int_0^t |D\phi(s,X_s^i)|^2 ds\right] = 0.$$

So we find

$$\begin{aligned}
\int_{\mathbb{R}^d} \phi(t,x)m_t(dx) = \int_{\mathbb{R}^d} \phi(t,x)m(0,dx) + \int_0^t \int_{\mathbb{R}^d} (\partial_t\phi(s,x) + D\phi(s,x)\cdot\alpha_t(x) + (1+\beta)\Delta\phi(s,x))m_s(dx)ds \\
+ \beta\int_0^t (\int_{\mathbb{R}^d} D\phi(s,x)m_s(dx))\cdot dW_s.
\end{aligned}$$

This means that m solves in the sense of distributions the stochastic Kolmogorov equation:

$$dm_t = [(1+\beta)\Delta m_t - \mathrm{div}(m_t\alpha)]\,dt - \sqrt{2\beta}\mathrm{div}(m_t dW_t).$$

As the flow m is stochastic and adapted to the filtration generated by W, the value function u is stochastic as well and is adapted to the filtration generated by W. It turns out that u solves a backward Hamilton-Jacobi equation. The precise form of this equation is delicate because, as it is random and backward, it has to involve an extra unknown vector field $v = v_t(x)$ which ensures the solution u to be adapted to the filtration generated by W (see, on that subject, the pioneering work by Peng [162] and the discussion in [56] (Chapter 4) or in [68] (Part II, Section 1.4.2)). The stochastic MFG system associated with the problem becomes (if the initial distribution of the players is $\bar{m}_0$):

$$\begin{cases} du_t = \left[-(1+\beta)\Delta u_t + \frac{1}{2}|Du_t|^2 - F(x,m_t) - \sqrt{2\beta}\mathrm{div}(v_t)\right]dt - \sqrt{2\beta}v_t \cdot dW_t \\ dm_t = [(1+\beta)\Delta m_t + \mathrm{div}(m_t Du_t)]\,dt - \sqrt{2\beta}\mathrm{div}(m_t dW_t) \\ m_0 = \bar{m}_0, \ u_T = G(\cdot, m_T) \end{cases}$$

Finally, one can associate with the problem a master equation, which plays the same role as without common noise. It takes the form of a second order (in measure) equation on the space of measures:

$$\begin{cases} -\partial_t U - (1+\beta)\Delta_x U + \frac{1}{2}|D_x U|^2 - (1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y D_m U(t,x,m,y)\,m(dy) \\ \qquad + \int_{\mathbb{R}^d} D_m U(t,x,m,y)\cdot D_x U(t,y,m)\,m(dy) - 2\beta\int_{\mathbb{R}^d} \mathrm{div}_x D_m U(t,x,m,y)\,m(dy) \\ \qquad -\beta\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{Tr}(D^2_{mm}U(t,x,m,y,y'))m(dy)m(dy') = F(x,m) \\ \qquad\qquad\qquad \text{in } (0,T)\times\mathbb{R}^d\times\mathcal{P}_2 \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}_2 \end{cases}$$

where the unknown is $U = U(t,x,m)$.

1.4.3.4 Comments

Most formal properties of the Master equation have been introduced and discussed by Lions in [149] (Lesson 5/12/2008 and the Course 2010-'11), including of course the representation formula (1.153). The actual proof of the existence of a solution of the master equation is a tedious verification that (1.153) actually gives a solution. This has required several steps: the first paper in this direction is [41], where a master equation is studied for linear Hamiltonian and without coupling terms ($F = G = 0$); [108] analyzes the master equation in short time and without the diffusion term; [85] obtains the existence and uniqueness for the master equation (1.149); [56] establishes the existence and uniqueness of solutions for the master equation with common noise under the Lasry-Lions monotonicity condition (see also [68]). There has been few works since then on the subject outside the above references and the analysis on finite state space in [34, 26]: see [11, 12, 55]. Another approach, not discussed in these notes, is the so-called "Hilbertian approach" developed by Lions in [149] (see e.g. Lesson 31/10 2008, and later the seminar 08/11/2013): the idea is to write the master equation (or, more precisely, its space derivative) in the Hilbert space of square integrable random variables and use this Hilbert structure to obtain existence and uniqueness results.

The reader may notice that we have worked here under the monotonicity assumption. We could have also considered the problem in short time, or with a "small coupling". All these settings correspond to situation in which the MFG system has a unique solution for any initial measure. When this does not hold, the solution of the master equation is expected to be discontinuous. One knows almost nothing on the definition of the master equation outside of the smooth set-up: this remains one of the major issues of the topic. To overcome this difficulty, an idea would be to add a common noise to smoothen the solution. Although this approach is not understood in the whole space, there are now a few results in this direction in the finite state space: we discuss this point now.

The MFG problem on finite state space has been first described by Lions [149] (Lesson 14/1 2011 and the Course 2011-'12). The probabilistic interpretation is carefully explained in [78], while the well-posedness of the master equation (and its use for the convergence of the Nash system) is discussed in this setting in [27] and [79]. The addition of a common noise to the master equation in finite state space is described in [34] and [26]. In particular, [26] provides the existence of smooth solutions even without the monotonicity assumption (see also [146], on problems with a major player). Finally, for the master equation on finite state space we definitively refer to the contribution by F. Delarue in the present volume.

1.4.4 Convergence of the Nash system

In this section, we study the convergence of Nash equilibria in differential games with a finite number of players, as the number of players tends to infinity. We would like to know if the limit is a MFG model. Let us recall that, in Subsection 1.3.3 we explained how to use the MFG system to find an $\varepsilon-$Nash equilibrium in a $N-$player game. So here we consider the converse problem. As we will see, this question is much more subtle and, in fact, not completely understood.

On one hand, this problem depends on the structure of information one allows to the players in the finite player game. If in this game players observe only their own position (but they are aware of the controls played by the other players and hence their average distribution), then the limit problem is (almost always) a MFG game (see the notes below). On the other hand, if players observe each other closely and remember all the past actions, the convergence cannot be expected because a deviating player can always be punished in the game with finitely many players (this is the so-called Folk Theorem), while it is not the case in Mean Field Games. This kind of strategy, however, is not always convincing because a player is often led to punish him/herself in order to punish a deviation. So the most interesting case is when players play in closed loop strategies (in function of the current position of the other players): indeed, this kind of strategy is time consistent (and is associated with a PDE, the Nash system). However, the answer to the convergence problem is then much more complicated and we only have a partial picture.

We consider here a very smooth case, in which the Nash equilibrium in the $N-$player game satisfies a time-consistency condition. More precisely, we assume that the Nash equilibrium is given through the solution $(v^{N,i})$ of the so-called Nash system:

$$\begin{cases} -\partial_t v^{N,i} - \sum_j \Delta_{x_j} v^{N,i} + H(x_i, D_{x_i} v^{N,i}) \\ \qquad\qquad + \sum_{j\neq i} H_p(x_j, D_{x_j} v^{N,j}) \cdot D_{x_j} v^{N,i} = F(x_i, m_X^{N,i}) \qquad \text{in } (0,T)\times\mathbb{T}^{Nd} \\ v^{N,i}(T,x) = G(x_i, m_X^{N,i}) \qquad \text{in } \mathbb{T}^{Nd} \end{cases} \tag{1.164}$$

where we set, for $X=(x_1,\dots,x_N)\in(\mathbb{T}^d)^N$, $m_X^{N,i} = \dfrac{1}{N-1}\sum_{j\neq i}\delta_{x_j}$. We explain below how this system is associated with a Nash equilibrium.

Assuming that the coupling functions F and G are monotone, our aim is to show that the solution $(v^{N,i})$ converges, in a suitable sense, to the solution of the master equation without a common noise.

Throughout this part we denote by $U=U(t,x,m)$ the solution of the master equation built in Theorem 21 which satisfies (1.154) and (1.155). It solves

$$\begin{cases} -\partial_t U - \Delta_x U + H(x, D_x U) - \int_{\mathbb{T}^d} \mathrm{div}_y D_m U\, dm(y) \\ \quad + \int_{\mathbb{T}^d} D_m U(t,x,m,y)\cdot H_p(y, D_x U(t,y,m))\, dm(y) = F(x,m) \\ \qquad\qquad \text{in } (0,T)\times\mathbb{T}^d\times\mathcal{P}_2 \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{T}^d\times\mathcal{P}_2 \end{cases} \tag{1.165}$$

Throughout the section, we suppose that the assumptions of the previous section are in force.

1.4.4.1 The Nash system

Let us first explain the classical interpretation of the Nash system (1.164):

The game consists, for each player $i = 1, \dots, N$ and for any initial position $x_0 = (x_0^1, \dots, x_0^N)$, in minimizing

$$J_i(t_0, x_0, (\alpha^j)) = \mathbb{E}\left[\int_{t_0}^T L(X_t^i, \alpha_t^i) + F(X_t^i, m_{X_t}^{N,i})\, dt + G(X_t^i, m_{X_t}^{N,i})\right]$$

where, for each $i = 1, \dots, N$,

$$dX_t^i = \alpha_t^i dt + \sqrt{2} dB_t^i, \qquad X_{t_0}^i = x_0^i$$

We have set $X_t = (X_t^1, \dots, X_t^N)$. The Brownian motions (B_t^i) are independent, but the controls (α^i) are supposed to depend on the filtration $\mathcal{F}$ generated by all the Brownian motions.

Proposition 9 (Verification Theorem) *Let $(v^{N,i})$ be a classical solution to the above system. Then the $N-$uple of maps $(\alpha^{i,*})_{i=1,\dots,d} := (-H_p(x_i, D_{x_i} v^{N,i}))_{i=1,\dots,d}$ is a Nash equilibrium in feedback form of the game: for any $i = 1, \dots, d$, for any initial condition $(t_0, x_0) \in [0,T] \times \mathbb{T}^{Nd}$, for any control α^i adapted to the whole filtration $\mathcal{F}$, one has*

$$J_i(t_0, x_0, (\alpha^{j,*})) \leq J_i(t_0, x_0, \alpha^i, (\alpha^{j,*})_{j \neq i})$$

Proof. The proof relies on a standard verification argument and is left to the reader. □

1.4.4.2 Finite dimensional projections of *U*

Let U be the solution to the master equation (1.165). For $N \geq 2$ and $i \in \{1, \dots, N\}$ we set

$$u^{N,i}(t, X) = U(t, x_i, m_X^{N,i}) \quad \text{where } X = (x_1, \dots, x_N) \in (\mathbb{T}^d)^N, \ m_X^{N,i} = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}. \tag{1.166}$$

Note that the $u^{N,i}$ are at least C^2 with respect to the x_i variable because so is U. Moreover, $\partial_t u^{N,i}$ exists and is continuous because of the equation satisfied by U. The next statement says that $u^{N,i}$ is actually globally $C^{1,1}$ in the space variables:

Proposition 10 *For any $N \geq 2$, $i \in \{1, \dots, N\}$, $u^{N,i}$ is of class $C^{1,1}$ in the space variables, with*

$$D_{x_j} u^{N,i}(t, X) = \frac{1}{N-1} D_m U(t, x_i, m_X^N, x_j) \qquad (j \neq i)$$

and

$$\left\| D_{x_k, x_j} u^{N,i}(t, \cdot) \right\|_\infty \leq \frac{C}{N} \qquad (k \neq i, \ j \neq i).$$

Proof. Let $X = (x_j) \in (\mathbb{T}^d)^N$ be such that $x_j \neq x_k$ for any $j \neq k$. Let $\varepsilon := \min_{j \neq k} |x_j - x_k|$. For $V = (v_j) \in (\mathbb{R}^d)^N$ with $v_i = 0$, we consider a smooth vector field $\phi : \mathbb{T}^d \to \mathbb{R}^d$ such that

$$\phi(x) = v_j \qquad \text{if } x \in B(x_j, \varepsilon/4).$$

Then, as U satisfies (1.154), (1.155), we can apply Proposition 6 which says that, (omitting the dependence with respect to t for simplicity)

$$\begin{aligned}
u^{N,i}(X+V) - u^{N,i}(X) &= U((id + \phi)\sharp m_X^{N,i}) - U(m_X^{N,i}) \\
&= \int_{\mathbb{T}^d} D_m U(m_X^{N,i}, y) \cdot \phi(y)\, dm_X^{N,i}(y) + O(\|\phi\|^2_{L^2(m_X^{N,i})}) \\
&= \frac{1}{N-1} \sum_{j \neq i} D_m U(m_X^{N,i}, x_j) \cdot v_j + O(\sum_{j \neq i} |v_j|^2)
\end{aligned}$$

This shows that $u^{N,i}$ has a first order expansion at X with respect to the variables $(x_j)_{j\neq i}$ and that

$$D_{x_j}u^{N,i}(t,X) = \frac{1}{N-1} D_m U(t, x_i, m_X^N, x_j) \qquad (j \neq i).$$

As $D_m U$ is continuous with respect to all its variables, $u^{N,i}$ is C^1 with respect to the space variables in $[0,T] \times \mathbb{T}^{Nd}$.

The second order regularity of the $u^{N,i}$ can be established in the same way. □

We now show that $(u^{N,i})$ is "almost" a solution to the Nash system (1.164). More precisely, next Proposition states that the $(u^{N,i})$ solve the Nash system (1.164) up to an error of size $1/N$.

Proposition 11 *One has, for any $i \in \{1, \dots, N\}$,*

$$\left\{\begin{array}{l} -\partial_t u^{N,i} - \sum_j \Delta_{x_j} u^{N,i} + H(x_i, D_{x_i} u^{N,i}) \\ \qquad + \sum_{j \neq i} D_{x_j} u^{N,i}(t,X) \cdot H_p(x_j, D_{x_j} u^{N,j}(t,X)) = F(x_i, m_X^{N,i}) + r^{N,i}(t,X) \\ \qquad\qquad\qquad\qquad \text{in } (0,T) \times \mathbb{T}^{Nd} \\ u^{N,i}(T,X) = G(x_i, m_X^{N,i}) \qquad \text{in } \mathbb{T}^{Nd} \end{array}\right. \tag{1.167}$$

where $r^{N,i} \in L^\infty((0,T) \times \mathbb{T}^{dN})$ with

$$\|r^{N,i}\|_\infty \leq \frac{C}{N}.$$

Proof. As U solves (1.165), one has at a point $(t, x_i, m_X^{N,i})$:

$$-\partial_t U - \Delta_x U + H(x_i, D_x U) - \int_{\mathbb{T}^d} \operatorname{div}_y D_m U(t, x_i, m_X^{N,i}, y)\, dm_X^{N,i}(y) \\ + \int_{\mathbb{T}^d} D_m U(t, x_i, m_X^{N,i}, y) \cdot H_p(y, D_x U(t, y, m_X^{N,i}))\, dm_X^{N,i}(y) = F(x_i, m_X^{N,i})$$

So $u^{N,i}$ satisfies:

$$-\partial_t u^{N,i} - \Delta_{x_i} u^{N,i} + H(x_i, D_{x_i} u^{N,i}) - \frac{1}{N-1} \sum_{j \neq i} \operatorname{div}_y D_m U(t, x_i, m_X^{N,i}, y_j) \\ + \frac{1}{N-1} \sum_{j \neq i} D_{x_j} u^{N,i}(t,X) \cdot H_p(x_j, D_x U(t, x_j, m_X^{N,i})) = F(x_i, m_X^{N,i})$$

By the Lipschitz continuity of $D_x U$ with respect to m, we have

$$\left| D_x U(t, x_j, m_X^{N,i}) - D_x U(t, x_j, m_X^{N,j}) \right| \leq C \mathbf{d}_1(m_X^{N,i}, m_X^{N,j}) \leq \frac{C}{N-1},$$

so that, by Proposition 10,

$$\left| \frac{1}{N-1} D_x U(t, x_j, m_X^{N,i}) - D_{x_j} u^{N,j}(t,X) \right| \leq \frac{C}{N^2}$$

and

$$\frac{1}{N-1} \sum_{j \neq i} D_{x_j} u^{N,i}(t,X) \cdot H_p(x_j, D_x U(t, x_j, m_X^{N,i})) \\ = \sum_{j \neq i} D_{x_j} u^{N,i}(t,X) \cdot H_p(x_j, D_{x_j} u^{N,j}(t,X)) + O(1/N).$$

On the other hand,

$$\sum_j \Delta_{x_j} u^{N,i} = \Delta_{x_i} u^{N,i} + \sum_{j \neq i} \Delta_{x_j} u^{N,i}$$

where, using Proposition 10 and the Lipschitz continuity of D_mU with respect to m,

$$\sum_{j\neq i} \Delta_{x_j} u^{N,i} = \int_{\mathbb{T}^d} \mathrm{div}_y D_m U(t, x_i, m_X^{N,i}, y) dm_X^{N,i}(y) + O(1/N) \qquad \text{a.e.}$$

Therefore

$$\begin{aligned} -\partial_t u^{N,i} - \sum_j \Delta_{x_j} u^{N,i} + H(x_i, D_{x_i} u^{N,i}) \\ + \sum_{j\neq i} D_{x_j} u^{N,i}(t,X) \cdot H_p(x_j, D_{x_j} u^{N,j}(t,X)) + O(1/N) = F(x_i, m_X^{N,i}). \end{aligned}$$

□

1.4.4.3 Convergence

We are now ready to state the main convergence results of [56]: the convergence of the value function and the convergence of the optimal trajectories. Let us strongly underline that we have to work here under the restrictive assumption that there exists a classical solution to the master equation. This solution is known to exist only on short time intervals or under the Lasry-Lions monotonicity assumption. Outside this framework, a recent (and beautiful) result of Lacker [140]Â states that the limit problem is a weak solution of a MFG model (i.e., involving some extra randomness), provided the idiosyncratic noise is non degenerate.

Let us start with the convergence of the value function:

Theorem 22 *Let* $(v^{N,i})$ *be the solution to* (1.164) *and* U *be the classical solution to the master equation* (1.165). *Fix* $N \geq 1$ *and* $(t_0, m_0) \in [0,T] \times \mathcal{P}_1$.

(i) For any $\mathbf{x} \in (\mathbb{T}^d)^N$, *let* $m_{\mathbf{x}}^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. *Then*

$$\sup_{i=1,\cdots,N} \left| v^{N,i}(t_0,\mathbf{x}) - U(t_0, x_i, m_{\mathbf{x}}^N) \right| \leq CN^{-1}.$$

(ii)For any $i \in \{1,\dots,N\}$ *and* $x_i \in \mathbb{T}^d$, *let us set*

$$w^{N,i}(t_0, x_i, m_0) := \int_{\mathbb{T}^d} \dots \int_{\mathbb{T}^d} v^{N,i}(t_0, \mathbf{x}) \prod_{j\neq i} m_0(dx_j),$$

where $\mathbf{x} = (x_1, \dots, x_N)$. *Then,*

$$\left\| w^{N,i}(t_0,\cdot,m_0) - U(t_0,\cdot,m_0) \right\|_{L^1(m_0)} \leq \begin{cases} CN^{-1/d} & \text{if } d \geq 3 \\ CN^{-1/2}\log(N) & \text{if } d = 2 \\ CN^{-1/2} & \text{if } d = 1 \end{cases}.$$

In (i) and (ii), the constant C *does not depend on* t_0, m_0, i *nor* N.

Theorem 22 says, in two different ways, that the $(v^{N,i})_{i\in\{1,\cdots,N\}}$ are close to U. In the first statement, one compares $v^{N,i}(t,\mathbf{x})$ with the solution of the master equation evaluated at the empirical measure $m_{\mathbf{x}}^N$ while, in the second statement, the averaged quantity $w^{N,i}$ can directly be compared with the solution of the MFG system (1.152) thanks to the representation formula (1.153) for the solution U of the master equation.

The proof of Theorem 22 consists in comparing the "optimal trajectories" for $v^{N,i}$ and for $u^{N,i}$, for any $i \in \{1,\dots,N\}$. For this, let us fix $t_0 \in [0,T)$, $m_0 \in \mathcal{P}_2$ and let $(Z_i)_{i\in\{1,\dots,N\}}$ be an i.i.d family of N random variables of law m_0. We set $\mathbf{Z} = (Z_i)_{i\in\{1,\dots,N\}}$. Let also $((B_t^i)_{t\in[0,T]})_{i\in\{1,\dots,N\}}$ be a family of N independent d-dimensional Brownian motions which is also independent of $(Z_i)_{i\in\{1,\dots,N\}}$. We consider the systems of SDEs with variables $(\mathbf{X}_t = (X_{i,t})_{i\in\{1,\dots,N\}})_{t\in[0,T]}$ and $(\mathbf{Y}_t = (Y_{i,t})_{i\in\{1,\dots,N\}})_{t\in[0,T]}$(the SDEs being set on $\mathbb{R}^d$ with periodic coefficients):

$$\begin{cases} dX_{i,t} = -H_p\big(X_{i,t}, D_{x_i}u^{N,i}(t,\mathbf{X}_t)\big)dt \\ \qquad\qquad\qquad\qquad\qquad +\sqrt{2}dB^i_t, \quad t\in[t_0,T], \\ X_{i,t_0} = Z_i, \end{cases} \tag{1.168}$$

and

$$\begin{cases} dY_{i,t} = -H_p\big(Y_{i,t}, D_{x_i}v^{N,i}(t,\mathbf{Y}_t)\big)dt \\ \qquad\qquad\qquad\qquad\qquad +\sqrt{2}dB^i_t, \quad t\in[t_0,T], \\ Y_{i,t_0} = Z_i. \end{cases} \tag{1.169}$$

Note that the (Y_i) are the optimal solutions for the Nash system, while, by the mean field theory, the (X_i) are close to the optimal solutions in the mean field limit.

Since the $(u^{N,i})_{i\in\{1,\dots,N\}}$ are symmetrical, the processes $((X_{i,t})_{t\in[t_0,T]})_{i\in\{1,\dots,N\}}$ are exchangeable. The same holds for the $((Y_{i,t})_{t\in[t_0,T]})_{i\in\{1,\dots,N\}}$ and, actually, the N $\mathbb{R}^{2d}$-valued processes $((X_{i,t},Y_{i,t})_{t\in[t_0,T]})_{i\in\{1,\dots,N\}}$ are also exchangeable.

Theorem 23 *We have, for any $i\in\{1,\dots,N\}$,*

$$\mathbb{E}\big[\sup_{t\in[t_0,T]}|Y_{i,t}-X_{i,t}|\big] \le \frac{C}{N}, \qquad \forall t\in[t_0,T], \tag{1.170}$$

$$\mathbb{E}\left[\int_{t_0}^T |D_{x_i}v^{N,i}(t,\mathbf{Y}_t) - D_{x_i}u^{N,i}(t,\mathbf{Y}_t)|^2 dt\right] \le CN^{-2}, \tag{1.171}$$

and, $\mathbb{P}-$almost surely, for all $i=1,\dots,N$,

$$|u^{N,i}(t_0,\mathbf{Z}) - v^{N,i}(t_0,\mathbf{Z})| \le CN^{-1}, \tag{1.172}$$

where C is a (deterministic) constant that does not depend on t_0, m_0 and N.

The main step of the proof of Theorem 22 and Theorem 23 consists in comparing the maps $v^{N,i}$ and $u^{N,i}$ along the optimal trajectory Y_i. Using the presence of the idiosyncratic noises B^i and Proposition 11 gives (1.171), from which one derives that the X_i and the Y_i solve almost the same SDE, whence (1.170). We refer to [56] for details.

1.4.4.4 Comments

The question of the convergence of $N-$player games to the MFG system has been and is still one of the most puzzling questions of the MFG theory (together with the notion of discontinuous solution for the master equation). In their pioneering works [143, 144, 145] Lasry and Lions first discussed the convergence for open-loop problems in a Markovian setting, because in this case the Nash equilibrium system reduces to a coupled system of N equations in $\mathbb{R}^d$ (instead of N equations in $\mathbb{R}^{Nd}$), and in short time, where the estimates on the derivatives of the $v^{N,i}$ propagate from the initial condition.

The convergence of open-loop Nash equilibria (in a general setting) is now completely understood thanks to the works of Fischer [105] and Lacker [139], who identified completely the possible limits: these limits are always MFG equilibria. If these results are technically subtle, they are not completely surprising because at the limit players actually play open-loop controls: so there is not a qualitative difference between the game with finitely many players and the mean field game.

The question of convergence of closed-loop equilibria is more subtle. As shows a counter-example in [68, I.7.2.5], this convergence does not hold in full generality: however, the conditions under which it holds are still not clear. We have presented above what happens in MFG problems for monotone coupling and nondegenerate idiosyncratic noise. The result also holds for MFG problems with a common noise: see [56]. The convergence is quite strong, and there is a convergence rate. In that same setting, [95] and [96] study the central limit theorem and the large deviation. Lacker's result [140], on the other hand, allows to prove the convergence towards (weak) solutions of MFG equilibria without using the master equation, under the assumption of nondegeneracy of the idiosyncratic noise only. The result relies on the fact that, in some average sense, the deviation of a player barely affects the distribution of the players when N is large. Heuristically, this is due to the presence of the noise, which prevents the players to guess if another player has deviated or not. One of the drawbacks of Lacker's paper is that there might be a lot of (weak) MFG equilibria, outside of the monotone case where it is unique. It is possible that actually only one of these equilibria is selected at the limit: this is what happens in the examples discussed in [80, 97].

1.5 Appendix: P.-L. Lions' courses on Mean Field Games at the Collège de France

Mean Field Game theory has been largely developed from Lions's ideas on the topic as presented in his courses at the Collège de France during the period 2007-2012. These courses have been recorded and can be found at the address:

`http://www.college-de-france.fr/site/pierre-louis-lions/_course.htm`

To help the reader to navigate between the different years, we collect here some informal notes on the organization of the courses. We will use brackets to link some of the topics below to the content of the previous Sections.

1.5.1 Organization 2007-2008

(Symmetric functions of many variables; differentiability on the Wasserstein space)

- 09/11/2007
  Behavior as $N \to \infty$ of symmetric functions of N variables. Distances on spaces of measures. Eikonal equation in the space of measures (by Lax-Oleinik formula). Monomial on the space of measures. Hewitt-Savage theorem.
- 16/11/2007
  A proof of Hewitt-Savage theorem by the use of monomials on the space of measures.
- 23/11/2007
  1st hour: A remark on quantum mechanics (antisymmetric functions of N variables).
  2nd hour: extensions on the result about the behavior as $N \to \infty$ of symmetric functions of N variables.
  - other moduli of continuity ($|u^N(X) - u^N(Y)| \leq C \inf_\sigma \max_i |x_i - y_{\sigma(i)}|$).
  - relaxation of the symmetry assumption: symmetry by blocs.
  - distances with weights (replacing $1/N$ by weights (λ_i)).
  Discussion on the differential calculus on $\mathcal{P}_2$: functions C^1 over $\mathcal{P}_2$ defined through conditions on their restriction to measures with finite support.
- 07/12/2007
  1st hour: Back to the differential calculus on $\mathcal{P}_2$; application to linear transport equation, to 1st order HJ equations (discussion on scaling $(1/N)\sum_i H(ND_{x_i}u^N)$ - discussion on the restriction to subquadratic hamiltonians).
  2nd hour: second order equations. Heat equations (independent noise, common noise); case of diffusions depending on the measure.
- 14/12/2007
  Discussion about differentiability, C^1, $C^{1,1}$ on the Wasserstein space [cfr. Section 1.4.2]. Wasserstein distance computed by random variables.

1.5.2 Organization 2008-2009

(Hamilton-Jacobi equation in the Wasserstein space - Derivation and analysis of the MFG system)

- 24/10/08
  Nash equilibria in one shot symmetric games as the number of players tends to infinity (example of the towel on the beach).
  Characterization of the limit of Nash equilibria.
  Existence - Discussion on the uniqueness through an example.
  Nash equilibria (in the game with infinitely many players) as optima of a functional (efficiency principle).
- 31/10/2008
  Differentiability on $\mathcal{P}_2$ through the representation as a function of random variables. Definition of C^1, link with the differentiability of functions of many variables. Structure of the derivative: law independent of the choice of the representative, derivative as a function of the random variable [cfr. Section 1.4.2].
  First order Hamilton-Jacobi equations in the space of measures. Definition with test functions in $L^2(\Omega)$. Lax-Oleinik formula. Uniqueness of the solution.
- 07/11/2008
  First order Hamilton-Jacobi equations in the space of measures: comparison. Limit of HJ with many variables: Eikonal equation, extension to general Hamiltonians, weak coupling.
  Discussion about the choice of the test function: is it possible to take test functions on $L^2(\Omega)$ which depend on the law only?
- 14/11/2008
  1st hour: 2nd order equations in probability spaces. Back to the limit of equations (A) $\partial_t u^N - \Delta u^N = 0$ and (B) $\partial_t u^N - \sum_{i,j} \frac{\partial^2 u^N}{\partial x_i \partial x_j} = 0$: different expressions for the limit.
  2nd hour: strategies for the proof of uniqueness for the limit equation (A): (1) by verification—restricted to linear eq, (2) in $L^2(\mathbb{R}^d)$—requires coercivity conditions which are missing here, (3) Feng-Katsoulakis technique—works mostly for the heat equation and relies on the contracting properties of the heat eq in the Wasserstein space.
- 21/11/2008
  (Digression: Back to the family of polynomials: restriction to $U(m) = \Pi_k \int_{\mathbb{R}^d} \phi_k(x) m(x)$.)
  Analysis of the “limit heat equation" in the Wasserstein space (case (A)): explanation of the fact that it is a first order equation - interpretation as a geometric equation.
  Back to uniqueness: use of HJ in Hilbert spaces (cf. Lions, Swiech). Key point: diffusion almost in finite dimension. Proof of uniqueness by using formulation in $L^2(\Omega)$.
  Nonlinear equations of the form
  $$(*) \qquad \partial_t u^N - \frac{1}{N} \sum_i F(N\, D^2 u_i^N) = 0.$$
  Heuristics for the limit by polynomials.
  Limit equation of (*): $\partial_t U - \mathbb{E}_1[F(\mathbb{E}_2[U''(G,G)])] = 0$. Uniqueness: as before.
  Beginning of the analysis of the case of complete correlation.
- 28/11/2008
  Analysis of “limit heat equation" in the Wasserstein space (case (B)). Discussion on the well-posedness.
  Remark on the dual equation.
- 05/12/2008
  Derivation of the MFG system from the N-player game [cfr Section 1.4.4].
  Back to the system of N equations and link with Nash equilibria. Ref. Bensoussan-Frehse. Uniqueness of smooth solutions; existence: more difficult, requires conditions in x of the Hamiltonian (growth of $\frac{\partial H}{\partial x}$).
  Problem: understand what happens as $N \to +\infty$.
  Key point: one needs to have $|\frac{\partial u_j^N}{\partial x_j}| \leq C$ and $|\frac{\partial u_j^N}{\partial x_i}| \leq C/N$. Known for T small or special structure of H. Open in general.
  One then expects that $u_i^N \to U(x_i, m, t)$. Derivation of the Master equation for U (without common noise, [cfr. Section 1.4.3]).
  Discussion on the Master equation; uniqueness. No maximum principle.
  Derivation of the MFG system from the Master equation.

Direct derivation of the MFG system from the Nash system: evolution of the density of the players in the $\mathbb{R}^{Nd}$ system for the Nash equilibrium with N players when starting from an initial density m_0; cost of a player with respect to the averaged position of the other players. Propagation of chaos under the assumption $|\frac{\partial^2 u_j^N}{\partial x_j \partial x_k}| \leq C/N^2$.

- 19/12/2008
  Analysis of the MFG system for time dependent problems: second order [cfr. Thm 4 and Thm 11].
  Existence: H Lipschitz or regularizing coupling.
  Discussion on the coupling: local or nonlocal, regularizing.
  Case H Lipschitz + coupling of the form $g = g(m, \nabla m)$ with a polynomial growth in ∇m. A priori estimates for (m, u) and its derivatives.
  Case of a regularizing coupling $F = F(m)$ without condition on H (here $H = H(\nabla u)$): a priori estimates by Bernstein method.
- 09/01/2009
  Existence of solutions for the MFG system: by strategy of fixed point and approximation.
  Starting point: H Lipschitz and regularizing coupling.
  Other cases by approximation.
  Description of “la ola".
  Discussion on the uniqueness for the system MFG. Two regimes: monotone coupling versus small time horizon.
- Â 16/01/2009
  1st hour: Interpretation of the MFG system (with a local coupling and planning problem setting) as an optimal control problem of the Fokker-Planck equation [cfr. Thm 17].
  Comment on the existence of a minimum, on the uniqueness (counter-example to uniqueness when the monotonicity is lost).
  Loss of uniqueness by analysis of the linearized system (when existence of a trivial solution): the linearized problem is well-posed only if the horizon is small.
  2nd hour: Use of the Hopf-Cole transform for quadratic Hamiltonians [cfr. Remark 13].
  Back on the existence of the solution to the MFG system [cfr. Remark 12]:

  $$\begin{cases} -\partial_t u - \Delta u + H(p) = f(m) \\ \partial_t m - \Delta m - \mathrm{div}(m H_p(Du)) = 0 \end{cases}$$

  - if f is bounded and H is subquadratic, existence of smooth solutions (e.g., $H(p) = p^\alpha$, $\alpha \leq 2$). (works also for $f(m) = c_0 m^p$ for p small).
  - if H is superquadratic and f is nonincreasing: open problem.
  - if $f(m) = cm^\beta$ with $c > 0$, $H(p) = c_0|p|^\gamma$ with $\gamma > 1$. First a priori estimate on $\int\int m^{1+\beta} + m|Du|^\gamma \leq C$. Second a priori estimate obtained by multiplying by Δm the equation for u, and by Δu the equation of m and adding the resulting quantities (computation for $\gamma = 2$): one gets $\frac{d}{dt}\int DuDm = \int |D^2u|^2 m + f'(m)|Dm|^2$.

1.5.3 Organization 2009-2010

(Analysis of the MFG system: the local coupling - Variational approach to MFGs)

- 06/11/2009
  Presentation of the MFG system.
  1st hour: Maximum principle in the deterministic case for smooth solutions: if $u_0 \leq v_0$, then $u \leq v$.
  Proof by reduction to a time-space elliptic equation with boundary conditions Dirichlet and nonlinear Neumann (+ discussion on the link with Euler equation). Proof that this is an elliptic equation.
  2nd hour: generalization to the case where the initial condition on u is a function of m. Discussion of the maximum principle when the running cost f grows: not true in general.
  Discussion of the maximum principle when the continuity equation has a right-hand side.

- 13/11/2009
  Comparison principle in the second order setting with a quadratic hamiltonian.
  Quadratic Hamiltonian: change of variable (Hopf-Cole transform, [cfr. Remark 13]) and algorithm to build solutions.
  Conjecture: no comparison principle for more general Hamiltonians.
- 20/11/2009
  Comparison principle: second order setting with a quadratic Hamiltonian and stationary MFG systems.
  Comments on the convergence of the MFG system as $T \to +\infty$ [cfr. Section 1.3.6]: convergence of $m^T(t)$, $u^T(t) - < u^T(t) >$, and $< u^T(t) > /T$. Claim that $u^T(t) - \bar{\lambda}(T-t)$ converges.
  Ergodic problem: comparison in the deterministic setting: if $f_1 \leq f_2$, then $\bar{\lambda}_1 \leq \bar{\lambda}_2$. When $H(x,\xi) \geq H(x,0)$ for all ξ, then $m = [f^{-1}(x,\lambda)]_+$ where λ is such that $\int m = 1$. Then $u = constant$ in $\{m > 0\}$; solve $H(x, Du) = \lambda$ in $\{m = 0\}$ with boundary conditions. Justification by $\nu \to 0^+$ for instance.
  Comparison in the second order setting: quadratic H.
  Planification problems. Approach by penalization. Link with Wasserstein.
- 27/11/2009
  Link between MFG with optimal control of (backward) Fokker-Plank equation:

  $$\partial_t m + \Delta m + div(m\alpha) = 0, \qquad m(T,x) = m_1(x)$$

  where $\alpha = \alpha(x,t)$ and the cost is of the form

  $$\int_0^T \int_Q mL(x,\alpha)dxdt + \Psi(m) + \int_Q \Phi(x, m(0,x))dx$$

  Planing pb: $\Phi = \frac{1}{2\varepsilon}\|m - m_0\|_2^2$.
  Derivation of the optimality conditions. Generalization to the case $L(x,\alpha,m)$ which is a functional of m. Approach by optimal control to the planning problem. Leads to controllability issues. Discussion of the polynomial case.
  2nd hour: First order planning problem: existence of a smooth solution.
  Step 1: link with quasilinear elliptic equations with nonlinear boundary conditions [cfr. Remark 16].
  Step 2: L^∞ estimates on $w := \partial_t u + H(x, Du)$ (i.e., estimate on m): extension of Bernstein method by looking at the equation satisfied by w.
  Step 3: L^∞ estimate on u. Indeed u is smooth and solves $\partial_t u + H(Du) = f(m)$ where $f(m)$ is bounded. So it is a forward and backward solution which gives the result.
- 04/12/2009
  Planning problem (without diffusion): link with quasilinear elliptic equation (in time-space) with nonlinear boundary conditions. Lipschitz estimates on u: Bernstein method again. Difficulties: constants are subsolutions and boundary conditions.
- 11/12/2009
  First hour: Back to the first order planning problem.
  Dual problem, i.e., optimal control of HJ equation [cfr. Section 1.3.7.2]. Namely

  $$\inf_u \int\int G(\frac{\partial u}{\partial t} + H(Du)) - \int (m_1 u(T) - m_0 u(0)).$$

  Computation of the first variation, and link with the MFG system. Comment on the fact that $f = f(m)$ has to be strictly increasing. Generalization to second order problems.
  Counter-examples:

  (i) (reminder when H at most linear (first or second order): existence of solutions). In this case there is no existence of solution for the dual problem (at least for small time).
  (ii)Regularity? Normalization: $H(0) = 0$, $H'(0) = 0$, $f(1) = 0$, $A = H''(0) > 0$, $f'(1) = a > 0$. Then $m = 1$, $u = 0$ is the unique solution for $m_0 = m_T = 0$. One linearizes to get $\partial_t v - \nu\Delta v = an$, $\partial_t n + \nu\Delta n + div(ADv) = 0$ with $n(0) = n_0$ and $n(T) = n_T$ where $\int n_0 = \int n_1$. Stability requires that $A > 0$. Proposition: the linearized

(periodic) problem is well-posed iif $A > 0$, $a \geq 0$, $\nu \geq 0$. Proof for first order, straightforward; for second order, Fourier.

Second hour: end of the proof.
Second order planning problem. Approach by optimization (optimal control of Fokker-Planck equation) yields the existence and uniqueness of very weak solutions. Main issue: regularity. Understood when $H = \frac{1}{2}|p|^2$. Theorem: when $H = \frac{1}{2}|p|^2$, and f non decreasing with polynomial growth, then there is a unique smooth solution. Generalization to the case $|H''(p) - I| \leq \frac{C}{\sqrt{1+|p|^2}}$ (conj. could be generalized to the case $cI \leq H'' \leq CI$). Proof by the Hopf-Cole transformation.

- 18/12/2009
MFG problems with congestion terms [cfr. Example 1]: minimize $\mathbb{E}\left[\int_t^T q^{-1}|\alpha_s|^q(m(s,X_s))^a ds + u_0(X_T)\right]$ with $dX_s = \sigma dW_s - \alpha_s ds$ where $q > 1$ and $a > 0$. Leads to the MFG system of the form
$$\begin{cases} \partial_t u - \nu\Delta u + \frac{1}{p}\frac{|Du|^p}{m^b} = 0 \\ -\partial_t m - \nu\Delta m + \operatorname{div}(\frac{|Du|^{p-2}Du}{m^b}m) = 0 \\ m(T) = m_T,\ u(0) = u_0 \end{cases} \tag{1.173}$$
Discussion of the (lack of) link with the optimal control of the Fokker-Planck equation. Uniqueness condition for the MFG system (for $p = 2$ and $0 < b \leq 2$).
- 08/01/2010
Back to the congestion problem. Uniqueness of the solution of (1.173) in the case (1) where the Hamiltonian is of the form $|Du|^2/(2f(m))$ (and the term in the divergence by $mDu/f(m)$ and (2) $p > 1$ and $0 < b \leq 4/p'$).
Discussion on the existence of a solution for $\nu = 0$ by using the fact that the equation of u is an elliptic equation in time-space: bounds on u, m and on Du. Regularity issue if m vanishes.
Analysis of the case $\nu > 0$, $p = 2$, $0 < b(\leq 2)$: a priori estimates and notion of solution.
- 15/01/2010
Back to to the congestion problem (1.173) when $p = 2$, $b = 1$. A priori estimates continued (bounds on u, on $\int\int |Du|^2(1 + m^{-1})$, on $\int\int |D^2u|^2$ and on $\int\int |Du|^2|Dm|^2/m^2$). Existence of a solution by approximation (replacing $|Du|^2/m$ by $|Du|^2/(\delta + m)$ for $\delta > 0$).

1.5.4 Organization 2010-2011

(the master equation in infinite and finite dimension)

- 05/11/2010
Uniqueness for the MFG system when $H = H(Du, m)$ [cfr. Thm 13]. Different approaches: monotonicity, continuation, reduction to an elliptic equation.
- 12/11/2010
Uniqueness for the MFG system when $H = H(Du, m)$ (continued): linearization, problems with actualization rate.
On the Master equation (MFGf)[3]:
    1. Heuristics: Master equation as a limit system of Nash equilibria with N players as $N \to +\infty$
    2. The Master Equation contains the MFG equation (when $\beta = 0$)
    3. Back to the uniqueness proof: U is monotone
    4. Back to $N \to +\infty$: MFGf contains the Nash system without individual noise.
- 19/11/2010

[3] Warning: missing term in the MFGf.

1st hour: Back to the Master equation[4]. Check that when $\nu \neq 0$ the equation does not match with Nash eq for N players. Link with optimal control problems in the case of separate variables (discussion of the case of non separate variables).
2nd hour: Hamilton-Jacobi equation associated with an optimal control of Fokker-Planck equation. Derivation of the master equation by taking the derivative of the Hamilton-Jacobi equation.

- 26/11/2010
  Erratum on the master equation. Interpretation of the Master Equation as a limit as $N \to +\infty$: explanation of the second order terms [cfr. Section 1.4.3.3].
  1) Interpretation in terms of optimal control problem ($\beta = 0$)
  2) Uniqueness related to the convexity of F and Φ
  3) General principle for the link between optimal control and the Master Equation in infinite dimension.
- 03/12/2010
  System derived from Hamilton-Jacobi: propagation of monotonicity.
- 10/12/2010
  System derived from Hamilton-Jacobi:
  - Propagation of monotonicity for second order systems.
  - Propagation of smoothness, method of characteristics.
- 17/12/2010
  Propagation of monotonicity for $\frac{\partial U}{\partial t} + (H'(DU)D)U = f(x) + \sum a_{\alpha,\beta} \frac{\partial^2 U}{\partial x_\alpha x_\beta}$.
- 07/01/2011
  Existence and uniqueness of a monotone solution for $\frac{\partial U}{\partial t} + (H'(DU)D)U = f(x)$.
  Remarks on semi-concavity for HJ equations.
- 14/01/2011
  1st hour: Structure of the master equation in the discrete setting (without diffusion):

$$\partial_t U_i + (\sum_j x_j H_j'(x, \nabla U)\nabla)U_i + H_i(x, \nabla U) = 0.$$

  Propagation of monotonicity.
  2nd hour: Propagation of monotonicity for independent noises (in the infinite dimensional setting). Finite dimensional setting, in which the noise yields a term of the form $\sum_{k,l} a_{kl} x_l \partial_k U_i + \sum_k a_{ki} U_k$.
  Monotonicity for the common noise (in the infinite dimensional setting; the finite dimensional setting being open).

1.5.5 Organization 2011-2012

(Analysis of the master equation for MFG in the finite state space, [cfr Section 1.4.3.2])

- 28/10/2011
  Analysis of equation: $\frac{\partial U}{\partial t} + (U.\nabla)U = 0$ (where $U : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}^n$).
  - case $U_0 = \nabla\phi_0$: then $U = \nabla\phi$ with ϕ sol of HJ equation.
  - case U_0 monotone, bounded and Lipschitz continuous: existence and uniqueness of a monotone, bounded and Lipschitz continuous sol, which is smooth if U_0 is smooth.
  Generalization to $\frac{\partial V}{\partial t} + (F(V).\nabla)V = 0$, provided F and V_0 monotone (since $U = F(V)$ the initial equation)
  Explicit formula: linear case, method of characteristics: solution is given by $U = (U_0^{-1} + tI_d)^{-1}$ as long as there is no shock. Quid in general?
  Propagation of the condition $\frac{\partial U_i}{\partial x_j} \leq 0$, $j \neq i$.
- 04/11/2011
  Back to the system $\frac{\partial U}{\partial t} + (U.\nabla)U = 0$.

[4] Warning: missing term in the MFGf.

Propagation of the condition $\frac{\partial U_i}{\partial x_j} \leq 0$, $j \neq i$. Consequence: $\frac{\partial U_i}{\partial x_i}$ is a bounded measure.
A striking identity: if U is a classical solution of $\frac{\partial U}{\partial t} + (F(U).\nabla)U = 0$, then $\frac{\partial}{\partial t}\det(\nabla U) + \operatorname{div}(F(U)\det(\nabla U)) = 0$.

- 25/11/2011
  Application to non-convex HJ equations: examples of smooth solutions.
- 09/12/2011
  Propagation of monotonicity with second order terms.
- 16/12/2011
  Analysis of $\frac{\partial U}{\partial t} + (F(U).\nabla)U = 0$.
  Following Krylov idea: introduce $W(x, \eta, t) = U(x, t) \cdot \eta$.
- 06/01/2012
  Analysis of $\frac{\partial U}{\partial t} + (F(U).\nabla)U = f(x)$: existence of a smooth global solution under monotonicity assumptions.
  A priori estimates when U_0 satisfies $U_0'(z)\xi \cdot \xi \geq \alpha |U_0'(z)\xi|^2$ for some $\alpha > 0$ and any z, ξ.
- 13/01/2012
  Analysis of $\frac{\partial U}{\partial t} + (F(U).\nabla)U = 0$ with U_0 and F monotone (continued). A priori estimates on ∇U under the assumption that there exists $\alpha > 0$ such that $F'(z)\xi \cdot \xi \geq \alpha |F'(z)\xi|^2$ for any z, ξ.
  Generalization to the case with a right-hand side of the form $a_{kl}\partial_{kl}U^i + b^i_{kl}\partial_l U^k$ where $a_{\alpha\beta}$ symmetric ≥ 0.

1.5.6 Additional notes

- 08/11/2013
  Seminar: on the differentiability in Wasserstein space, point of view of the random variables. MFGs in the finite state case: the master equation as a first order hyperbolic system. Back to the infinite dimensional case, the Hilbertian approach: if $U(t, x, X)$ is the solution of the classical master equation, one sets $V(t, X) = U(t, X, \mathcal{L}(X))$. Discussion of the monotonicity in the Hilbertian framework.

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