LECTURE 15, April 27, 23

GAME THEORY, \( \Phi \in C(A \times B) \), \( A, B \) compact metric spaces.

MARGINAL FNS: \( \Phi^\max (b) = \max_a \Phi(a,b) \), \( \Phi^\min (a) = \min_b \Phi(a,b) \)

Best Response: \( R^A(b) = \arg \max_a \Phi(a,b) \), \( R^B(a) = \arg \min_b \Phi(a,b) \)

UPPER VALUE: \( v^+ = \min_b \max_a \Phi(a,b) = \min_b \max_a \Phi(a,b) \)

LOWER VALUE: \( v^- = \max_a \min_b \Phi(a,b) = \max_a \min_b \Phi(a,b) \)

Note: \( v^- \leq v^+ \). If \( v^- = v^+ \) game has a VALUE.

Examples of \( v^- \neq v^+ \).

EX. MATRIX GAMES \( \Phi(i,j) = \Phi_{ij} \)

\[
\begin{pmatrix}
\phi_{11} & \cdots & \phi_{1n} \\
\vdots & \ddots & \vdots \\
\phi_{m1} & \cdots & \phi_{mn}
\end{pmatrix}
\]

\[ \min_j \max_i \phi_{ij} = \Phi^\max (1) \]

\[ \max_i \min_j \phi_{ij} = \Phi^\min (m) \]

\[ v^- = \min_j \max_i \phi_{ij} \]

\[ v^+ = \max_i \min_j \phi_{ij} \]
Ex. 2. "Cake"

\[
\begin{pmatrix}
\frac{1}{2} + 3 & \frac{1}{2} - 3 \\
\frac{3}{5} & \frac{1}{5}
\end{pmatrix}
\]

\[
\text{max} \begin{pmatrix}
\frac{2}{5} & \frac{1}{2} - 3 \\
\frac{2}{5} & \frac{1}{2} - 3
\end{pmatrix}
\]

\[
v = \frac{1}{2} - 3 \geq v^+
\]

\[
\Rightarrow v = \frac{1}{2} - 3 \text{ is the value of the game.}
\]

Ex 3. "Head & Tail"

\[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\]

\[
\text{max} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} = -1
\]

\[
v^- = -1 < v^+ = 1
\]

The value does not exist!

\[
\text{Def. A saddle point of the game is } (a^*, b^*) \in A \times B.
\]

\[
\forall a \in A, \quad \overline{\Phi}(a, b^*) \leq \Phi(a^*, b^*) \leq \underline{\Phi}(a^*, b) \quad \forall b \in B
\]

Ex. \( A = B = [-1, 1] \) \( \Phi(a, b) = b^2 - a^2 \)

\[
\text{If } b^* = a^* = 0 \text{ is a saddle}
\]
Suppose \( R^A \) and \( R^B \) are functions (single-valued) \( R^A : B \rightarrow A, \) \( R^B : A \rightarrow B, \) \((a^*, b^*)\) is a saddle point =

\( a^* \) is a fixed point of \( R^A \circ R^B : A \rightarrow A \) because
\( R^A \circ R^B (a^*) = R^A (b^*) = a^*. \)

\( b^* \) is a fixed point of \( R^B \circ R^A : B \rightarrow B. \)

\[ \text{Def. SECURITY STRATEGIES.} \quad v^+ = \min_b \Phi^+(b) \]

is \( b^* : \quad v^+ = \Phi^+(b^*), \quad \text{i.e., } b^* \text{ conjectures } v^+ \]
for 2nd player.

\( a^+ \) is S.S. for 1st player \( \iff \quad v = \max_a \Phi^+(a^+) \)
\( \text{i.e., } a^+ \text{ can guess } \max_a \Phi^+(a^+) \)

Thus: The game has a value \( \iff \) a saddle point.

\[ \text{If } v^+ \leq v^- \quad \text{Ass. } (a^*, b^*) \text{ saddle pt. Goal: } v^+ \leq v^- \]

\[ v^- = \max_a \min_b \Phi(a, b) \geq \min_b \Phi(a^*, b) = \max_a \Phi(a, b^+) \]
\[ \geq \max_a \min_b \Phi(a, b) = v^+ \]
Run $a^+$ is a SEC. STR. for 1st play. \( b^+ \) is S.S. for 2nd.

Ass \( v^+ = v^- \). Take \( a^+ \) a SEC. STR. For \( A \).

\[
V^- = \Phi_{\text{min}}(a^+) = \min_b \Phi(a^+, b).
\]

Take \( b^+ \) SEC. STR. For \( B \) : \( v^+ = \Phi_{\text{max}}(b^+) = \max_a \Phi(a, b^+) \)

\[
\forall a \in A \quad \Phi(a, b^+) \leq v^+ = \Phi_{\text{max}}(b^+) = \max_a \Phi(a, b^+) \leq \Phi(a^+, b) \forall b.
\]

For \( a = a^+, b = b^+ \implies \Phi(a^+, b^+) = v^+ = v^- \leq \Phi(a^+, b) \)

\( \implies \) sec. STR. \( a^+, b^+ \) are SADDLE PRT.

Corollary: If game has a value = \( v \)

(i) \( (a^+, b^+) \) is a saddle = \( a^+ \) is SEC. STR. for 1st

(ii) \( v \) is SADDLE. \( v \) is a SADDLE.

(iii) (EXCHANGEABILITY) : If \( (\overline{a}, \overline{b}) \) is also a SADDLE.

\( \implies (\overline{a}^+, \overline{b}^+) \) are SADDLES.

\( \Box \) (ii) Sec. PRT. of Thm. (iii) from (i).

\[ \text{THE MINIMAX THEOREM of Von Neumann (1928)} \]

Thm. \( A, B \subset \text{vector spaces}, \text{compact} \& \text{convex}, \Phi \in \text{EC}(A \times B) \)

\( \Phi \) CONCAVE-CONVEX, i.e.,
\[ \forall a, b, \quad a \mapsto \Phi(a, b) \text{ is concave} \]
\[ \forall a, \quad b \mapsto \Phi(a, b) \text{ is convex} \]
\[ \Rightarrow V^+ = V^-, \text{ i.e., } (A, B, \Phi) \text{ has a value \( \Phi \) at least one saddle point.} \]

**Proof:** will be done if \( A \subseteq \mathbb{R}^m, B \subseteq \mathbb{R}^n \).

**Remark:** Supp. \( \Phi(a, \ast) \) strictly convex \( \Rightarrow \) unique
\[ r(a) \in B : \quad \Phi(a, r(a)) = \min_{b \in B} \Phi(a, b) \]
\[ \Rightarrow R^B_a = \{ r(a) \} \]

**Lemma:** If \( \Phi(a, \ast) \) is strictly convex \( \forall a \in A, \Phi \in C \), \( A, B \) convex \( \Rightarrow \) \( r : A \rightarrow B \) is continuous.

**Rem.:** Not true without convexity.

\[ \begin{array}{c}
\Phi(a) \\
\Phi(b)
\end{array} \]

Here \( r \) jumps to right at \( \Phi^B_r \)

**Lemma:** \( \forall a \in A : a_n \rightarrow a \text{ conc: } r(a_n) \rightarrow r(a) \)

**Extract:** \( \forall a_n : r(a_n) \rightarrow b \in B \text{ (} B \text{ convex, conc.)} \)
\[ \Rightarrow \Phi(a_n, r(a_n)) \rightarrow \Phi(a, b) \quad n \rightarrow \infty \]
\[ \forall b \left( l \right) \leq \Phi(a_n, b) \rightarrow \Phi(a, b) \]
By the arbitrariness of \( a_n \Rightarrow r(a_n) \Rightarrow r(\bar{a}) \).

Proof of V.N. Theorem

Step 1. Suppose \( \Phi(x,b) \) is strictly concave. Then \( \Phi(x,b) \) is

\[ \Phi(x_1,b) = \min_{\Phi(x_1,b)} \Phi(x,b) \]

Step 2. Take \( a^+ \) s.t. \( \Phi(x,b) \) is s.t.

\[ \Phi(x_1,b) = \min_{\Phi(x_1,b)} \Phi(x,b) \]

Note:

\[ \Phi(x_1,b^*) = \min_{\Phi(x_1,b)} \Phi(x_1,b) \]

Remains the goal.

\[ \Phi(x^*,b^*) \geq \Phi(x,b^*) \] for all \( x \).

Step 3. Idea: approximate \( a^* \) with \( a = a^* + (1-d) a^* \)

Fix \( a \in A \), \( d \in [0,1] \), \( \mu = 1 - d \)

\[ \Phi(x^*,b^*) \geq \min_{\Phi(x^*,b^*)} \Phi(x,b^*) \]

\[ \Phi(x,b^*) \geq \Phi(x_1,b) \]

Conc. in \( a \).
Step 5. Remove the strict convexity of $b \mapsto \Phi(a, b)$.

For simplicity here $B \subseteq \mathbb{R}^k$. Fix $\varepsilon > 0$.

$$
\Phi_\varepsilon(a, b) = \Phi(a, b) + \varepsilon |b|^2
\quad \text{is strictly convex}
\quad \forall a.
$$

By composition of $A, B$, consider the $\varepsilon_n \to 0^+$:

$$
\begin{align*}
\lim_{\varepsilon_n \to 0^+} \Phi_\varepsilon(a, b_{\varepsilon_n}) &= \lim_{\varepsilon_n \to 0^+} \left( \Phi_\varepsilon(a, b_{\varepsilon_n}) - \varepsilon_n |b_{\varepsilon_n}|^2 \right) \\
&\leq \lim_{\varepsilon_n \to 0^+} \left( \Phi(a_{\varepsilon_n}, b_{\varepsilon_n}) + \varepsilon_n |b_{\varepsilon_n}|^2 \right) \\
&\leq \Phi(a_{\varepsilon_n}, b) + \varepsilon_n |b|^2
\end{align*}
$$

Let $\varepsilon_n \to 0$

$$
\Phi_\varepsilon(a, b^+) \leq \Phi(a^+, b^+) \leq \Phi(a^+, b) \quad \forall b
$$

$$
\Rightarrow \quad \Phi(a^+, b^+) \text{ is a SADDLE.}
$$

Examples. 1: $\Phi(a, b) = \Phi_1(a) - \Phi_2(b)$

$\Phi_1, \Phi_2$ concave in $C_1, C_2$, then applies.
2. IMPORTANT: \( \Phi(a, b) = a^T M b \) \( M \in M_{m \times n} \).

(1) \( A \in \mathbb{R}^m \), \( B \in \mathbb{R}^n \).

\( \Phi \) is bilinear \( \implies \)

cont. \& convex \( \implies \)

comp. \& convex \( \implies \)

\( \Phi \) conc. \& convex \( \therefore \) is OK.

3. MATRIX GAMES

\( A = \{ a_1, \ldots, a_m \} \), \( B = \{ b_1, \ldots, b_n \} \).

are NOT convex, V.M.T. does not apply & in fact we know examples without value.

MIXED STRATEGIES.
Idea: I choose a stochastic instead of deterministic way.

Def. A mixed strategy of 1st player is a \( \mu \in P(A) \) := probability measure on \( A \), & the 2nd player is \( \nu \in P(B) \) := \( \{ \ldots \} \).

Ex: \( \delta = \) Dirac measure centered in \( \bar{a} \in A \).

A 5CA Borel

\( \delta_{\bar{a}}(S) = \)

if \( S \ni \bar{a} \)

if \( S \not\ni \bar{a} \)

\( P(A) \ni \) "copy of \( A \)".

\( \Phi \) is pure strategies.

Def. \( \Phi(a, \nu) : = \int_S \Phi(a, b) \mu(a) d\nu(b) \)

\( \Phi : \ P(A) \times P(B) \rightarrow \mathbb{R} \).
\[
\Phi(x, \delta) \neq \int_{A \times B} \Phi(a, b) \, d\delta_a(a) \, d\delta_b(b) = \Phi(x, \delta)
\]

\[= \int_A f(a) \, d\delta_a(a) = f(x) .\]

\(\sim \) "EXTENDS" \(\Phi\) from \(A \times B\) to \(P(A) \times P(B)\)

Def. If 3 value of the gene \((1P(A), 1P(B), \Phi)\) is a saddle pt., they are called value & saddle of \((A, B, \Phi)\) in mixed strategies.