3.3. INTRODUCTION TO HAMILTON–JACOBI EQUATIONS

Let \( h \to 0^+ \) to compute
\[
\frac{x-z}{t} \cdot Du(x, t) + u_t(x, t) \geq L \left( \frac{x-z}{t} \right).
\]
Consequently
\[
u_t(x, t) + H(Du(x, t)) = u_t(x, t) + \max_{q \in \mathbb{R}^n} \{ q \cdot Du(x, t) - L(q) \}
\geq u_t(x, t) + \frac{x-z}{t} \cdot Du(x, t) - L \left( \frac{x-z}{t} \right)
\geq 0.
\]
This inequality and (31) complete the proof. \( \Box \)

We summarize:

**THEOREM 6 (Hopf–Lax formula as solution).** The function \( u \) defined by the Hopf–Lax formula (21) is Lipschitz continuous, is differentiable a.e. in \( \mathbb{R}^n \times (0, \infty) \), and solves the initial-value problem
\[
\begin{cases}
  u_t + H(Du) = 0 & \text{a.e. in } \mathbb{R}^n \times (0, \infty) \\
  u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \}.
\end{cases}
\]

3.3.3. Weak solutions, uniqueness.

a. Semiconcavity.

In view of Theorem 6 above it may seem reasonable to define a weak solution of the initial-value problem (18) to be a Lipschitz function which agrees with \( g \) on \( \mathbb{R}^n \times \{ t = 0 \} \), and solves the PDE a.e. on \( \mathbb{R}^n \times (0, \infty) \). However, this turns out to be an inadequate definition, as such weak solutions would not in general be unique.

**Example.** Consider the initial-value problem
\[
\begin{cases}
  u_t + |u|^2 = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
  u = 0 & \text{on } \mathbb{R} \times \{ t = 0 \}.
\end{cases}
\]
One obvious solution is
\[
u_1(x, t) \equiv 0.
\]
However the function
\[
u_2(x, t) := \begin{cases}
  0 & \text{if } |x| \geq t \\
  x - t & \text{if } 0 \leq x \leq t \\
  -x - t & \text{if } -t \leq x \leq 0
\end{cases}
\]
is Lipschitz continuous and also solves the PDE a.e. (everywhere, in fact, except on the lines \( x = \pm t \)). It is easy to see that actually there are infinitely many Lipschitz functions satisfying (33).

This example shows we must presumably require more of a weak solution than merely that it satisfy the PDE a.e. We will look to the Hopf–Lax formula (21) for a further clue as to what is needed to ensure uniqueness. The following lemma demonstrates that \( u \) inherits a kind of "one-sided" second-derivative estimate from the initial function \( g \).

**LEMMA 3 (Semiconcavity).** Suppose there exists a constant \( C \) such that
\[
g(x+z) - 2g(x) + g(x-z) \leq C|x|^2
\]
for all \( x, z \in \mathbb{R}^n \). Define \( u \) by the Hopf–Lax formula (21). Then
\[
u(x+z, t) - 2u(x, t) + u(x-z, t) \leq C|z|^2
\]
for all \( x, z \in \mathbb{R}^n, t > 0 \).

**Remark.** We say \( g \) is semiconcave provided (34) holds. It is easy to check (34) is valid if \( g \in C^3 \) and \( \sup_{x \in \mathbb{R}^n} |D^2g| < \infty \). Note that \( g \) is semiconcave if and only if the mapping \( x \mapsto g(x) - \frac{1}{2}|x|^2 \) is concave for some constant \( C \).

**Proof.** Choose \( y \in \mathbb{R}^n \) so that \( u(x, t) = tL \left( \frac{x-y}{t} \right) + g(y) \). Then, putting \( y+z \) and \( y-z \) in the Hopf–Lax formulas for \( u(x+z, t) \) and \( u(x-z, t) \), we find
\[
u(x+z, t) - 2u(x, t) + u(x-z, t)
\leq \left[ tL \left( \frac{x-y}{t} \right) + g(y+z) \right] - 2 \left[ tL \left( \frac{x-y}{t} \right) + g(y) \right]
+ \left[ tL \left( \frac{x-y}{t} \right) + g(y-z) \right]
= g(y+z) - 2g(y) + g(y-z)
\leq C|z|^2, \text{ by (34)}.\]

As a semiconcavity condition for \( u \) will turn out to be important, we pause to identify some other circumstances under which it is valid. We will no longer assume \( g \) to be semiconcave, but will suppose the Hamiltonian \( H \) to be uniformly convex.
3.3. INTRODUCTION TO HAMILTON–JACobi EQUATIONS

**DEFINITION.** A $C^2$ convex function $H : \mathbb{R}^n \to \mathbb{R}$ is called uniformly convex (with constant $\theta > 0$) if

$$\sum_{i,j=1}^n H_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } p, \xi \in \mathbb{R}^n. \quad (35)$$

We now prove that even if $g$ is not semiconcave, the uniform convexity of $H$ forces $u$ to become semiconcave for times $t > 0$: this is a kind of mild regularizing effect for the Hopf–Lax solution of the initial-value problem (18).

**LEMMA 4 (Semiconcavity again).** Suppose that $H$ is uniformly convex (with constant $\theta$) and $u$ is defined by the Hopf–Lax formula (21). Then

$$u(x + z, t) - 2u(x, t) + u(x - z, t) \leq \frac{1}{\theta t} |z|^2$$

for all $x, z \in \mathbb{R}^n$, $t > 0$.

**Proof.** 1. We note first using Taylor's formula that (35) implies

$$H \left( \frac{p_1 + p_2}{2} \right) \leq \frac{1}{2} H(p_1) + \frac{1}{2} H(p_2) - \frac{\theta}{8} |p_1 - p_2|^2. \quad (36)$$

Next we claim that for the Lagrangian $L$ we have the estimate

$$\frac{1}{2} L(q_1) + \frac{1}{2} L(q_2) \leq L \left( \frac{q_1 + q_2}{2} \right) + \frac{1}{8 \theta} |q_1 - q_2|^2 \quad (37)$$

for all $q_1, q_2 \in \mathbb{R}^n$. Verification is left as an exercise.

2. Now choose $y$ so that $u(x, t) = tL \left( \frac{x - y}{t} \right) + g(y)$. Then using the same value of $g$ in the Hopf–Lax formulas for $u(x + z, t)$ and $u(x - z, t)$, we calculate

$$u(x + z, t) - 2u(x, t) + u(x - z, t)$$

$$\leq tL \left( \frac{x + z - y}{t} \right) + g(y) - 2tL \left( \frac{x - y}{t} \right) + g(y)$$

$$+ \frac{1}{t} L \left( \frac{x - z + y}{t} \right) + g(y)$$

$$= 2t \left[ \frac{1}{2} L \left( \frac{x + z - y}{t} \right) + \frac{1}{2} L \left( \frac{x - z + y}{t} \right) - L \left( \frac{x - y}{t} \right) \right]$$

$$\leq 2 \frac{1}{8 \theta} \frac{1}{t} |z|^2 \leq \frac{1}{8 \theta} |z|^2,$$

the next-to-last inequality following from (37). $\Box$

3. NONLINEAR FIRST-ORDER PDE

**b. Weak solutions, uniqueness.**

In this section we show that semiconcavity conditions of the sorts discovered for the Hopf–Lax solution $u$ in Lemmas 3 and 4 can be utilized as uniqueness criteria.

**DEFINITION.** We say that a Lipschitz continuous function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a weak solution of the initial-value problem:

$$\begin{cases}
  u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
  u = g & \text{on } \mathbb{R}^n \times \{ t = 0 \}
\end{cases} \quad (38)$$

provided

(a) $u(x, 0) = g(x)$ \quad $(x \in \mathbb{R}^n)$,

(b) $u_t(x, t) + H(Du(x, t)) = 0$ \quad for a.e. \quad $(x, t) \in \mathbb{R}^n \times (0, \infty)$, and

(c) $u(x + z, t) - 2u(x, t) + u(x - z, t) \leq C \left( 1 + \frac{1}{t} \right) |z|^2$

for some constant $C \geq 0$ and all $x, z \in \mathbb{R}^n$, $t > 0$.

Next we prove that a weak solution of (38) is unique, the key point being that this uniqueness assertion follows from the inequality condition (c).

**THEOREM 7 (Uniqueness of weak solutions).** Assume $H$ is $C^2$ and satisfies (19), and $g$ satisfies (20). Then there exists at most one weak solution of the initial-value problem (38).

**Proof.** 1. Suppose that $u$ and $\tilde{u}$ are two weak solutions of (38) and write $w = u - \tilde{u}$.

Observe now at any point $(y, s)$ where both $u$ and $\tilde{u}$ are differentiable and solve our PDE, we have

$$w_t(y, s) = u_t(y, s) - \tilde{u}_t(y, s)$$

$$= -H(Du(y, s)) + H(D\tilde{u}(y, s))$$

$$= - \int_0^1 \frac{d}{dr} H(rDu(y, s) + (1 - r)D\tilde{u}(y, s)) \, dr$$

$$= - \int_0^1 \frac{d}{dr} \left( rDu(y, s) + (1 - r)D\tilde{u}(y, s) \right) \cdot (Du(y, s) - D\tilde{u}(y, s)) \, dr$$

$$= - \frac{1}{2} \left( Du(y, s) - D\tilde{u}(y, s) \right) \cdot (Du(y, s) - D\tilde{u}(y, s))$$

Consequently

$$\frac{1}{2} w_t = Du(y, s) \cdot D\tilde{u}(y, s). \quad (39)$$

Conclude that

$$w_t + b \cdot Du = 0 \quad \text{a.e.}$$

*Omit on first reading.*
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2. Write \( v := \varphi(u) \geq 0 \), where \( \varphi : \mathbb{R} \to (0, \infty) \) is a smooth function to be selected later. We multiply (39) by \( \varphi'(u) \) to discover

\[ v_t + b \cdot Dv = 0 \quad \text{a.e.} \tag{40} \]

3. Now choose \( \varepsilon > 0 \) and define \( u^\varepsilon := \eta_\varepsilon * v, \bar{u}^\varepsilon := \eta_\varepsilon * \bar{u} \), where \( \eta_\varepsilon \) is the standard mollifier in the \( x \) and \( t \) variables. Then according to \S C.4

\[ |Du^\varepsilon| \leq \text{Lip}(u), \quad |D\bar{u}^\varepsilon| \leq \text{Lip}(\bar{u}), \tag{41} \]

and

\[ D(u^\varepsilon) \to Du, \quad D\bar{u}^\varepsilon \to D\bar{u} \quad \text{a.e., as} \ \varepsilon \to 0. \tag{42} \]

Furthermore inequality (c) in the definition of weak solution implies

\[ D^2u^\varepsilon, D^2\bar{u}^\varepsilon \leq C \left( 1 + \frac{1}{\varepsilon} \right) I \tag{43} \]

for an appropriate constant \( C \) and all \( \varepsilon > 0, \, y \in \mathbb{R}^n, \, \varepsilon > 2 \varepsilon \). Verification is left as an exercise.

4. Write

\[ b_\varepsilon(y, \varepsilon) := \int_0^1 DH(rDu^\varepsilon(y, \varepsilon) + (1 - r)D\bar{u}^\varepsilon(y, \varepsilon)) \, dr. \tag{44} \]

Then (40) becomes

\[ v_t + b_\varepsilon \cdot Dv = (b_\varepsilon - b) \cdot Dv \quad \text{a.e.}; \tag{45} \]

hence

\[ v_t + \text{div}(eb_\varepsilon) = (\text{div} b_\varepsilon)v + (b_\varepsilon - b) \cdot Dv \quad \text{a.e.} \]

5. Now

\[ \text{div} b_\varepsilon = \int_0^1 \sum_{k,l=1}^n H_{kl}(rDu^\varepsilon + (1 - r)D\bar{u}^\varepsilon)(ru^\varepsilon_{t_kx_l} + (1 - r)\bar{u}^\varepsilon_{t_kx_l}) \, dr \leq C \left( 1 + \frac{1}{\varepsilon} \right) \tag{46} \]

for some constant \( C \), in view of (41), (43). Here we note that \( H \) convex implies \( D^2H \geq 0 \).

3. NONLINEAR FIRST-ORDER PDE

6. Fix \( x_0 \in \mathbb{R}^n, \, t_0 > 0 \), and set

\[ R := \max\{|DH(p)| \mid |p| \leq \max(\text{Lip}(u), \text{Lip}(\bar{u}))\}. \tag{47} \]

Define also the cone

\[ C := \{(x, t) \mid 0 \leq t \leq t_0, |x - x_0| \leq R(t_0 - t)\}. \]

Next write

\[ \varepsilon(t) = \int_{B(x_0,R(t_0-t))} v(x, t) \, dx \]

and compute for a.e. \( t > 0 \):

\[ \dot{\varepsilon}(t) = \int_{B(x_0,R(t_0-t))} v_t \, dx - R \int_{B(x_0,R(t_0-t))} v \, dS \]

\[ = \int_{B(x_0,R(t_0-t))} - \text{div}(eb_\varepsilon) + (\text{div} b_\varepsilon)v + (b_\varepsilon - b) \cdot Dv \, dx \]

\[ - R \int_{B(x_0,R(t_0-t))} v \, dS \quad \text{by (45)} \]

\[ = \int_{B(x_0,R(t_0-t))} v(b_\varepsilon \cdot v + R) \, dS \]

\[ + \int_{B(x_0,R(t_0-t))} (\text{div} b_\varepsilon)v + (b_\varepsilon - b) \cdot Dv \, dx \]

\[ \leq C \int_{B(x_0,R(t_0-t))} (\text{div} b_\varepsilon)v + (b_\varepsilon - b) \cdot Dv \, dx \quad \text{by (41), (44)} \]

\[ \leq C \left( 1 + \frac{1}{\varepsilon} \right) \varepsilon(t) + \int_{B(x_0,R(t_0-t))} (b_\varepsilon - b) \cdot Dv \, dx \]

by (46). The last term on the right hand side goes to zero as \( \varepsilon \to 0 \), for a.e. \( t_0 > 0 \), according to (41), (42) and the Dominated Convergence Theorem. Thus

\[ \dot{\varepsilon}(t) \leq C \left( 1 + \frac{1}{\varepsilon} \right) \varepsilon(t) \quad \text{for a.e.} \ 0 < t < t_0. \tag{48} \]

7. Fix \( 0 < \varepsilon < r < t \) and choose the function \( \varphi(z) \) to equal zero if

\[ |z| \leq \varepsilon |\text{Lip}(u, \text{Lip}(\bar{u})| \tag{49} \]

and to be positive otherwise. Since \( u = \bar{u} \) on \( \mathbb{R}^n \times \{t = 0\} \),

\[ v = \phi(u) = \phi(u - \bar{u}) = 0 \quad \text{at} \ (t = 0). \tag{50} \]
3.3 INTRODUCTION TO HAMILTON-JACOBI EQUATIONS

Thus \( c(\varepsilon) = 0 \). Consequently Gronwall's inequality (see §B.2) and (48) imply

\[
e(\tau) \leq c(\varepsilon)e^{\int_0^\tau c(t) \, dt} = 0.
\]

Hence

\[
|u - \bar{u}| \leq c[\text{Lip}(u) + \text{Lip}(\bar{u})] \quad \text{on} \quad B(x_0, R(t_0 - \tau)).
\]

This inequality is valid for all \( \varepsilon > 0 \), and so \( u = \bar{u} \) in \( B(x_0, R(t_0 - \tau)) \). Therefore, in particular, \( u(x_0, t_0) = \bar{u}(x_0, t_0) \). \( \square \)

In light of Lemmas 3, 4, and Theorem 7, we have

**THEOREM 8** (Hopf-Lax formula as weak solution). Suppose \( H \) is \( C^2 \) and satisfies (19), and \( g \) satisfies (20). If either \( g \) is semiconcave or \( H \) is uniformly convex, then

\[
u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\}
\]

is the unique weak solution of the initial-value problem (38) for the Hamilton-Jacobi equation.

**Examples.** (i) Consider the initial-value problem:

\[
\begin{cases}
u_t + \frac{1}{2} |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
u = |x| & \text{on } \mathbb{R}^n \times \{t = 0\}.
\end{cases}
\]

Here \( H(p) = \frac{1}{2} |p|^2 \) and so \( L(q) = \frac{1}{2} |q|^2 \). The Hopf-Lax formula for the unique, weak solution of (49) is

\[
u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ |x - y|^2 \right\}.
\]

Assume \( |x| > t \). Then

\[
D_y \left( \frac{|x - y|^2}{2t} + |y| \right) = \frac{y - x}{t} + \frac{y}{|y|} \quad (y \neq 0);
\]

and this expression equals zero if \( x = y + \frac{t}{|y|} y \). Thus \( \nu(x, t) = |x| - \frac{t}{2} \) if \( |x| > t \). If \( |x| \leq t \), the minimum in (50) is attained at \( y = 0 \). Consequently

\[
u(x, t) = \begin{cases}
|x| - t/2 & \text{if } |x| \geq t \\
|y| & \text{if } |x| \leq t.
\end{cases}
\]

3.3 INTRODUCTION TO CONSERVATION LAWS

In this section we investigate the initial-value problem for scalar conservation laws in one space dimension:

\[
\begin{cases}
u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
u = g & \text{on } \mathbb{R} \times \{t = 0\}.
\end{cases}
\]

Here \( F : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are given and \( u : \mathbb{R} \times [0, \infty) \to \mathbb{R} \) is the unknown, \( u = u(x, t) \). As noted in §3.2, the method of characteristics demonstrates that there does not in general exist a smooth solution of (1), existing for all times \( t > 0 \). By analogy with the developments in §3.3.5, we therefore look for some sort of weak or generalized solution.