$Y \sim N(\mu, \sigma^2)$, $X = e^Y = e^{\mu + \sigma Z}$ $\quad Z \sim N(0,1)$

Compute 1) $\mathbb{E}[X]$, 2) $\mathbb{E}[X^2]$, prove 3) $m_X(t) = \mathbb{E}[e^{tX}] < +\infty$?

1) $$\mathbb{E}[X] = \mathbb{E}[e^Y] = \mathbb{E}[e^{\mu+\sigma Z}] = \int_{-\infty}^{\infty} e^{\mu+\sigma z} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2} + \mu + \sigma z\right) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2 - 2\sigma z - 2\mu}{2}\right) dz =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2 - 2\sigma z + \sigma^2 - 2\mu - \sigma^2}{2}\right) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-\sigma)^2}{2}\right) \exp\left(\mu + \frac{\sigma^2}{2}\right) dz =$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{(z-\sigma)^2}{2}\right) dz = \boxed{\exp\left(\mu + \frac{\sigma^2}{2}\right)} \quad (\star)$$

2) $$\mathbb{E}[X^2] = \mathbb{E}[e^{2Y}] = \mathbb{E}[e^W]$$

with $W = 2Y$ $\quad W \sim N(2\mu, 2\sigma^2)$

$$\Rightarrow \mathbb{E}[X^2] = \mathbb{E}[e^W] = \boxed{\exp(2\mu + 2\sigma^2)}$$

using formula $(\star)$

same reasoning $\Rightarrow$ $$\boxed{\mathbb{E}[X^t] = \exp\left(t\mu + \frac{t^2\sigma^2}{2}\right)}$$

(~~As~~ it has all finite moments)

3) $$\mathbb{E}[e^{tX}] = \mathbb{E}[e^{te^Y}] = \int_{-\infty}^{\infty} \exp(te^y) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy =$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\Bigg(\underbrace{te^y - \frac{(y-\mu)^2}{2\sigma^2}}_{\xrightarrow[y\to\infty]{} +\infty}\Bigg) dy$$

$\Rightarrow$ the integral does not converge for ~~...~~ $t \neq 0$