LECTURE 5, March 14, 2023

Thm. \( K \subseteq \mathbb{R}^n \) convex, \( f : K \to \mathbb{R} \) convex \( \Rightarrow \) \( \forall x_0 \in K \exists \alpha \in \mathbb{R}^n \) \[ f(x) \geq f(x_0) + \alpha \cdot (x - x_0) \quad \forall x \in K. \]

Proof. \( A = (\text{epi} f) \), \( B = \{ (x_0, f(x_0)) \} \) the convex \( A \cap B = \emptyset \), \( A \) open : Thm on separation of convex sets in \( \mathbb{R}^n \times \mathbb{R} \Rightarrow \exists U = (p, y) \subseteq \mathbb{R}^n \times \mathbb{R} \neq (0, 0) \) \( \forall p \in \mathbb{R}^n : \)

\[ p \cdot x + t f(x) \leq \alpha \leq p \cdot x_0 + t f(x_0). \]

\( \forall t > f(x) \forall x \in K \). 

Claim 1. \( y \leq 0 \Rightarrow \) if \( y > 0 \), let \( t \to \infty \)

\( \lim_{t \to \infty} x = \infty \quad \bigcirc \quad \text{out.} \)

Claim 2. \( y = 0 \); \( p \cdot x = 0 \) \( \quad \forall x \in K \quad \bigcirc \quad \text{with } x_0 \in K \)

\[ \frac{p \cdot x - t}{|t|} \leq \alpha \leq \frac{p \cdot x_0 - f(x_0)}{|t|} \quad \forall t > f(x) \]

let \( t \downarrow \) \( f(x) \) \( \Rightarrow \quad \frac{p}{|t|} = \frac{p}{|t|} \quad \Rightarrow \quad \frac{p}{|t|} \cdot x - f(x) \leq \alpha \leq \frac{p}{|t|} \cdot x_0 - f(x_0) \]

\( \Rightarrow \quad f(x) \geq f(x_0) + \alpha \cdot (x - x_0) \)

\( \therefore \quad \boxed{\text{\( f(x) \) convex}} \)
CONVEX CONJUGATION (Fenchel Transform).

"Laurentian" \( L(q, x) \) convex in \( q \), \( x \) fixed. \( L^* \) is \( x \) convex.

Hypothesis:
\[ (c) \quad L : \mathbb{R}^n \to \mathbb{R} \text{ convex} \]
\[ (s) \quad \lim_{|q| \to \infty} \frac{L(q)}{|q|} = +\infty \quad \text{superlinear.} \]

Def. The convex conjugate of \( L \) is
\[ L^*(p) := \sup_{q \in \mathbb{R}^n} \left\{ q \cdot p - L(q) \right\}. \]

Lemma. \( L^*(p) < +\infty \) \( \forall p \in \mathbb{R}^n \) \( \sup \) is a max.

pf. \[ \frac{q \cdot p - L(q)}{|q|} \to -\infty \text{ as } |q| \to \infty \quad \text{by (s)} \]
\[ \Rightarrow q \cdot p - L(q) \to -\infty \quad \Rightarrow q \cdot p - L(q) \text{ has a max in } \mathbb{R}^n. \]

Weierstrass Thm. \( \Rightarrow q \cdot p - L(q) \) has a max in \( \mathbb{R}^n. \]

Connection with Legendre Transform \& Calc. Var.:
\( L \in C^1 \), \( L_q \) is bijective, \( p = L_q(q) \) has a unique sol.
\( q = Q(p) \) \( (\& Q \in C^1) \). Conclude \( L^* \).

\[ D_q \{\ldots\} = p - L_q(q) = 0 \quad \Rightarrow \quad q = Q(p) \]
\[ L^*(p) = p \cdot Q(p) - L(Q(p)) = H(p) \quad \text{def. last week.} \]

Def. \( H \) sat. \((c'(s))\) \( H(p) = L^*(p) \).
Then (Convex Duality, Fenchel)

Ass. \( L \) sets \( C \) s \( \Rightarrow H = L^* \) sets:

\[ \begin{align*}
&* \quad p \mapsto H(p) \text{ is convex in } \mathbb{R}^n \\
&* \quad \lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty \quad H \text{ superlinear}.
\end{align*} \]

H.D. Conjugacy is involutive: \( (L^*)^* = L \).

Pf. \( \top \quad H = L^* \text{ convex}; \quad p, \hat{p} \in \mathbb{R}^n, \quad 0 \leq \hat{p} \leq p \).

\[ H(2p + (1-2)\hat{p}) = \max \left\{ 2p \cdot q + (1-2)(\hat{p} \cdot q - \frac{1}{2}(L(q))_{\hat{p}} - \frac{1}{2}(1-2)\hat{p} \cdot q - (1-2)L(q)_{\hat{p}} \right\}. \]

\( \Rightarrow \quad H(p) = H(2p) - (1-2)H(\hat{p}) \quad \text{ (1)} \)

\[ \Rightarrow \quad \frac{H(p)}{|p|} \to +\infty \text{ as } |p| \to \infty. \quad \text{Fix } \hat{d} > 0 \]

\[ p \mapsto q = \frac{1}{|p|} \frac{\hat{d} p}{|\hat{p}|} \quad H(p) \geq p \cdot \frac{\hat{d} p}{|\hat{p}|} - L \left( \frac{\hat{d} p}{|\hat{p}|} \right) \geq \]

\[ \geq \frac{2}{\hat{d} |p|} - \max_{|\hat{d} |p| = 1} L(q) \quad \text{ (2)} \]

\[ \frac{H(p)}{|p|} \geq 2 - \frac{c_2}{|p|} \Rightarrow \lim_{|p| \to \infty} \frac{H(p)}{|p|} \geq 2 \quad \forall \hat{d} > 0 \]

\[ \Rightarrow \lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty. \]
3. Goal 3.1. \( H^* \leq L \)

\[
H(p) + L(q) = \sup_{\xi, \eta} \left\{ \xi \cdot p - L(\xi) + L(\eta) \geq q \cdot p - L(q) + L(\eta) \right\}
\]

\[
\geq q \cdot p \quad \forall q, \eta \in \mathbb{R}^n.
\]

\[
\Rightarrow L(q) \geq q \cdot p - H(p) \quad \forall p \in \mathbb{R}^n
\]

\[
\geq \sup_{p \in \mathbb{R}^n} \left\{ q \cdot p - H(p) \right\} = H^*(q).
\]

Goal 3.2. \( H^* \geq L \).

\[
H^*(q) = \sup_{p} \left\{ p \cdot q - \sup_{\xi} \left\{ q \cdot p - L(\xi) \right\} \right\} = \sup_{p} \inf_{\xi} \left\{ p \cdot q - p \cdot \xi + L(\xi) \right\} = \sup_{p} \inf_{\xi} \left\{ p \cdot q - \sum_{\xi} \right\}
\]

Thus support hyperplane: \( \exists \xi \in \mathcal{L}(q) \) such that

\[
L(\xi) \geq L(q) + \xi \cdot (q - \xi) \quad \forall \xi \in \mathbb{R}^n
\]

In (+) \( p = 0 \):

\[
H^*(q) \geq \inf_{\xi} \left\{ q \cdot (q - \xi) + L(q) + \xi \cdot (q - \xi) \right\} = L(q)
\]

Ref. T. Rockafellar: Convex Analysis, p. 70.

**Examples.**

HW. \( L(q) = \frac{1}{2} q^2 \), \( p \geq 1 \) \( \Rightarrow \) \( H(p) = \frac{1}{2} p^2 \).
JENSEN INEQUALITY.

Notation: \( f: \mathbb{R}^n \to \mathbb{R} \) convex, \( u: \mathbb{R}^n \to \mathbb{R} \) integrable.

\[ f(\frac{\int_U u \ dx}{|U|}) \leq \frac{1}{|U|} \int_U f(u(x)) \ dx \]

Proof: \( P = \frac{1}{|U|} \int_U u \ dx + c \mathbb{R}^n \). Take \( z \in \partial f(P) \):

\[ f(z) \geq f(P) + 2 \cdot (z - P) \]

Take \( q = u(x) \):

\[ f(u(x)) \geq f(P) + 2 \cdot (u(x) - P) \]

\[ \int_U f(u(x)) \geq f(P) + 2 \cdot (\frac{1}{|U|} \int_U u(x) \ dx - P) = f(\frac{1}{|U|} \int_U u \ dx) \]

Connection between Calc. vars & H-J eqs. continued.

(CP) \[ \begin{cases} u_t + H(D_x u, x) = 0 & (x, t) \in \mathbb{R}^n \times J_0, T \cr u(x, 0) = u_0(x) \end{cases} \]

\( H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) cont., \( P \to H(P, x) \) convex \( x \) fixed

\[ \lim_{|P| \to \infty} \frac{H(P)}{|P|} = +\infty \]
**Def.** \[ L(\cdot, x) \triangleq H^*(\cdot, x) \quad \forall x \in \mathcal{X}, \]

**Duality Thm.** \[ H(p, x) = \max_{q \in \mathcal{Q}} \left\{ q \cdot p - L(q, x) \right\}, \]

Last week I proposed the candidate sol. of (CP):

\[ \nu(x, t) \triangleq \inf \left\{ I[w] + g(w(0)) : w \in C([0, t], \mathbb{R}^k), \ w(0) = x \right\}, \]

\[ I[w] = \int_0^t L(w(t), w(t)) \, dt, \]

\[ w(0) = x \]

**Rmk.** \[ \nu(0, x) = g(x) \]

**Notat.** \[ F : x \in \mathbb{R} \mapsto \arg \max F = \left\{ x \in \mathbb{R} : F(x) = \max_{x \in \mathbb{R}} F \right\} \]

**Prop. (Verification Thm.).** \( u \in C^t([0, T], \mathbb{R}^k) \cap C([0, T], \mathbb{R}^k) \) solf. of (CP), then

(i) \[ u(x, t) \leq \nu(x, t) \]

(ii) \[ u(x, t) = \nu(x, t) \iff \exists \bar{x} \in C([0, T], \mathbb{R}^k) : \]

\[ \dot{x}(t) = Du(x(t), t) - L(x(t), x(t)) = H(Du(x(t), x(t)), x(t)) \quad \forall t \in [0, T), \]

and in such a case

\[ \nu(x, t) = \min \left\{ I[\bar{x}(\cdot)] + g(\bar{x}(0)) : \bar{x} \in C([0, T], \mathbb{R}^k) \right\} \]

**Rmk.** \[ H(p, x) = \max \left\{ \frac{1}{q} q \cdot p - L(q, x) \right\}, \quad (E) \iff \]

\[ \dot{x}(t) \in \arg \max \frac{1}{q} q \cdot Du(x(t), t) - L(q, x(t)) \]

\[ x(t) = x(0) \]
"Differential inclusion", it becomes an ODE if
er the set $x$ is a singleton.

$\mathcal{N}_B.2$ $\mathcal{N}_B.2$ means $x$ solving $(E)$ with $\Phi$ is optimal
for the minimization of $\int_{\Omega} f(w(x)) + \Phi$.

For $x, 0 < t \leq T$ we denote admissible by

$$\psi(t) := u(w(t), t) + \int_{0}^{t} L(w(s), w(s)) ds.$$ 

$$\dot{\psi}(t) = u_{\ell} (w(0), t) + D u_{\ell}(w(t)) \cdot \dot{w}(t) - L(w(t), w(t))$$

$$= \left( D u_{\ell}(w(t)) \cdot \dot{w}(t) - L(w(t), w(t)) \right)$$

$$\leq \| D u_{\ell}(w(t)) \| \cdot \| \dot{w}(t) \| - L(\| \dot{w}(t) \|, w(t))$$

$$= H(0, u_{\ell}(w(t)), w(t)) \geq 0$$

by HJ eq. $\psi(0) \geq 0$

$$\psi(t) \leq \psi(0) \forall t \leq T \quad \forall w(0) \in \Theta,$$

$$\psi(t) = \psi(0) \text{ if } w = \Xi \text{ solv. } (E)$$

$$\psi(t) = u(w(t), t) = u(x, t) \leq \psi(0) = u(w(0), 0) + \int_{0}^{t} L(w(s), w(s)) ds$$

$$= g(w(0)) + \int \Phi(w)$$

$$w_{\|e\|=d} \geq u$$

$$u(\|e\|=d) = u(x, t)$$

$u$ holds, $\Rightarrow w = \Xi$ solv. (E) (i.e. $\Xi$).
**Corollary.** Under the ass. of Verif. Thm.,

\[(C_{pCV}) \quad \exists \alpha, \beta \in Q \cap (p, x), c' \ni \bar{p} = L_{q}(Q(p, x), x).\]

Then the sol. of

\[
\begin{cases}
\bar{x}(t) = H_{p}(Du(x(0), 0), x(0)) \\
t \in [0, T]
\end{cases}
\]

is optimal for \(\min \int_{0}^{T} [I(x) + \varphi(w(t))] dt\) and w.e.c.

\(w(t) = x.\)

*Proof.* We saw (Ip CV)\( q \cdot P - L(q, P) = \frac{1}{2} Q(p, x)\)

Then \((\star)\) becomes the ODE

\[
\begin{cases}
\bar{x}(t) = \bar{Q}(Du(x(0), 0), x(0)) \\
\bar{x}(0) = x
\end{cases}
\]

Remains to prove \(Q = H_{p}:

\[
H(p, x) = Q(p, x) \cdot P - L(Q(p, x), x)
\]

\[
H_{p} = \bar{Q} + \frac{\partial}{\partial p} \bar{Q} \cdot P - L_{q}(Q(p, x), x) = 0
\]

*Remark.* If \(H_{p} = 0\) is \(c^{2}\) and \(Du(x) \) is loc. Lip., then \(Q(Du(x))\) is loc. Lip. \(\Rightarrow\) the trajectory of \((\star)\) exists at least in some interval \([0, \tau]\), \(\tau \leq \tau\) small enough.