Knowledge Representation and Learning

Modelling in Propositional Logic

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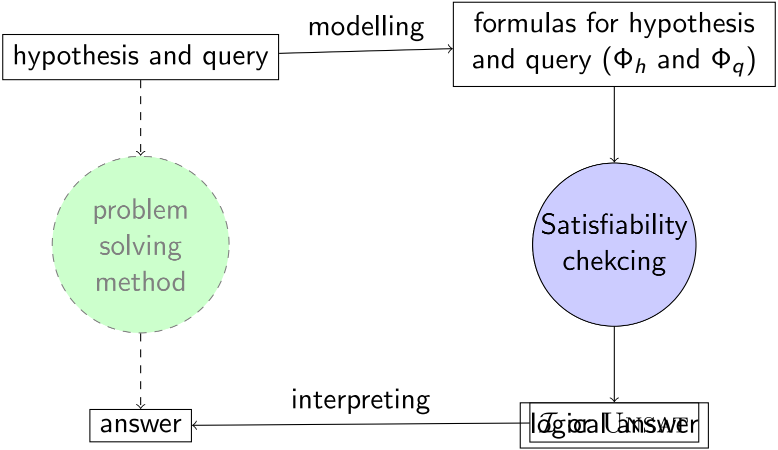
Logic based problem solving



Formalizing/modelling informal statements

- Natural langauge is one of the most common way to specify knowledge.
- We need a way to represent the knowledge expressed in common langauge in terms of propositional logical formulas.

Formalizing natural language sentences

To formalize text which is composed of complex sentences:

1. provide a set propositional variables corresponding to the simplest sentences of the text;
2. compose the propositional variables in formula using the logical connectives in accordance with the natural language connectives;

Conjunctions

Conjunction in english

but, yet, although, though, even though, moreover, furthermore, however, and whereas are all connectives that express some conjunctive information. Although these expressions have different connotations, they are all truthfunctionally equivalent to one another.

"and"

$$\left.\begin{array}{r} \text{it is raining, and I am happy} \\ \text{it is raining, but I am happy} \\ \text{although it is raining, I am happy} \\ \text{it is raining, yet I am happy} \end{array}\right\} \rightarrow \mathit{rain} \wedge \mathit{happy}$$

Other ways to express conjunctive statements

- Bill is a former player who coaches basketball
- Pelé is a Brasilian soccher player
- John and mary are students

Warning! there are cases in which "and" does not convey conjunctive information

- Jay and Kay are friends

This usually mean that Jay and key are friends eachother. However there are cases in which this interpretation is not unique

- Jay and Kay are married

Can be that Jay and Kay are married eachother, or that they are merried with some other person. The context can help to disambiguate.

Negation

"not"

$$\left.\begin{array}{r}\text{it is not raining}\\ \text{it is not true that it is raining}\\ \text{it is false that it is raining}\end{array}\right\} \rightarrow \neg raining$$

When the sentence which is negated is not atomic the usage of the first formulation might lead to some confusion. For instance are these two statements equivalent?

"not" and conjunction

$$\left.\begin{array}{r}\text{This car is not red and fast}\\ \text{It is false that this care is red and fast}\end{array}\right\} \Rightarrow \neg(red \wedge fast)$$

Using "it is true that . . . " and "it is false that . . . " will generate less confusion.

Disjunction

The standard English expression for disjunction is ‘or’, a variant of which is ‘either ... or ...’. ‘or’ has two senses – an inclusive sense and an exclusive sense.

“or’ (inclusive and exclusive)’

Jones will win or Smith will win (possibly both) $\Rightarrow J \vee S$

Jones will win or Smith will win (but not both) $\Rightarrow (J \vee S) \wedge \neg(J \wedge S)$

$\Rightarrow J \equiv \neg S$ (alternative)

“neither ... nor ...”

'Neither ... nor...’ is the negation of ‘either...or...’

neither Jones will win nor Smith will win $\Rightarrow \neg(J \vee S)$

$\Rightarrow \neg J \wedge \neg S$ (alternative)

Conditional

"if ... then ..."

if it is sunny then I wear a hat
if it is sunny I wear a hat
I wear a hat if it is sunny
in case of sun I wear a hat
I wear a hat in case of sun

$\Rightarrow S \rightarrow H$

"only if "

only if it is sunny I wear a hat
I wear a hat only if it is sunny
only in case of sun I wear a hat
I wear a hat only in case of sun
I don't wear a hat unless it is sunny

$\Rightarrow \neg S \rightarrow \neg H$

"if and only if"

is the conjunction of "if ... then ..." and "... only if ..."

Conditional

Unless

'Unless' is very similar to 'only if', in the sense that it has a built-in negation. The difference is that, whereas 'only if' scardi incorporates two negations, 'unless' incorporates only one.

I will graduate only if I pass the DB exam
I will not graduate unless I pass DB exam
unless I pass the DB exam, I will not graduate

$\Rightarrow \neg P \rightarrow \neg G$

I will pass DB exam only if I study
I will not pass the DB exam unless I study
unless I study, I will not pass DB exam

$\Rightarrow \neg S \rightarrow \neg P$

Conditional

'Otherwise'/'else'

'otherwise' is a three-place connective expressing conditional knowledge.

if it is sunny, then I'll play tennis, otherwise, I'll play racquetball

if it is sunny, I'll play tennis, otherwise, I'll play racquetball

I'll play tennis if it is sunny, otherwise, I'll play racquetball

$$\Rightarrow S \rightarrow T \wedge \neg S \rightarrow R$$

Paraphrasing Complex Statements

1. Identify the simple (atomic) statements, and associate to them a propositional variable;
2. Identify all the connectives
3. Identify the scope of the connectives (the scope of a connective is the (complex or simple statements on which it is applied)
4. apply the translation of the connectives in logical formulas

Formalizing English Sentences

Example

To ormalize in propositional logic the following english statements:

- if Sonia is happy and paints a picture then Renzo isn't happy
- if Sonia is happy, then she paints a picture
- When Sonia paints a picture is happy

We proveed as follows:

1. find the basic propositions] and associate to it a propositional variable:
    - Sonia is happy; $\longrightarrow s$;
    - Sonia paints a picture: $\longrightarrow p$;
    - Renzo is happy: $\longrightarrow r$;
2. replace them in the sentences
    - if s and p then not r
    - if s, then p
    - When p s
3. Translate the connectives in logical connectives
    - $s \wedge p \rightarrow r$
    - $s \rightarrow p$
    - $p \rightarrow s$

Checking informal arguments

- An informal argument is a test that contains a set of sentences that are considered as hypothesis (assumed to be true) and a sentence that is the *conclusion* that is supposed to be a consequence of the hypothesis.
- To check if an informal argument is correct one has to formalize the hypothesis in a set of formulas H, and the conclusion in a formiula ϕ,
- The argument is considered valid if ϕ is a logical consequence of H. I.e., if

$$H \models \phi$$

- This can be checked via sat by verifying that $H \cup \{\neg\phi\}$ is not satisfiable.
- If $H \cup \neg\phi$ is satisfiable, then Truth table returns an interpretation $\mathcal{I}$ that satisfies H and do not satisfy ϕ, which is a counter-example of the argument.

Checking informal arguments

Example

- *"If you play and you study you'll pass the exams, while if you play and don't study you won't pass. Thus, if you play, either you study and you'll pass the exams, or you don't study and you won't pass."*
  1. $p \wedge s \rightarrow e$
  2. $p \wedge \neg s \rightarrow \neg e$
  3. $p \rightarrow (s \wedge e) \vee (\neg s \wedge \neg e)$
- We need to prove that $1. \wedge 2. \models 3.$
  Use truth tables[a]

[a] To check $A_1, \ldots, A_n \models A$ via truth table, you have to build a unique truth table for $A_1, \ldots, A_n$ and A and check that every line in in which all A_i's are true A is also true

Solving puzzles

The Three Door Problem

John is in a room with a killing monster, he the room has three colored doors. Behind one of the doors is a path to freedom. Behind the other two doors, however, is an evil fire-breathing dragon. Opening a door to the dragon means almost certain death. On each door there is an inscription:

Freedom is behind this door	Freedom is not behind this door	Freedom is behind the red door

Given the fact that AT LEAST ONE of the three statements on the doors is true and At LEAST ONE of them is false, which door would lead the boys to safety?

Solving puzzles

Language

- r: *"freedom is behind the red door"*
- b: *"freedom is behind the blue door"*
- g: *"freedom is behind the green door"*

Axioms

1. *"behind one of the door is a path to freedom, behind the other two doors is an evil dragon"*
   $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
2. *"at least one of the three statements is true"*
   $r \vee \neg b$
3. *"at least one of the three statements is false"*
   $\neg r \vee b$

The 3 doors: Solution (2)

Axioms

1. $(r \wedge \neg b \wedge \neg g) \vee (\neg r \wedge b \wedge \neg g) \vee (\neg r \wedge \neg b \wedge g)$
2. $r \vee \neg b$
3. $\neg r \vee b$

Solution

r	b	g	2	3	$2 \wedge 3$
T	F	F	T	F	F
F	T	F	F	T	F
F	F	T	T	T	T

Freedom is behind the **green door!**

Graph Coloring Problem

Problem

Provide a propositional language and a set of axioms that formalize the graph coloring problem of a graph with at most n nodes, with connection degree $\leq m$, and with less then $k+1$ colors.

- node degree: number of adjacent nodes
- connection degree of a graph: max among all the degree of its nodes
- Graph coloring problem: given a non-oriented graph, associate a color to each of its nodes in such a way that no pair of adjacent nodes have the same color.

Graph Coloring: Propositional Formalization

Language

- For each $1 \leq i \leq n$ and $1 \leq c \leq k$, color_{ic} is a proposition, which intuitively means that *"the i-th node has the c color"*
- For each $1 \leq i \neq j \leq n$, edge_{ij} is a proposition, which intuitively means that *"the i-th node is connected with the j-th node"*.

Axioms

- for each $1 \leq i < j \leq n$, $\text{edge}_{ij} \leftrightarrow \text{color}_{ji}$
  "each node has at least one color"
- for each $1 \leq i \leq n$, $\bigvee_{c=1}^{k} \text{color}_{ic}$
  "each node has at least one color"
- for each $1 \leq i \leq n$ and $1 \leq c, c' \leq k$, $\text{color}_{ic} \rightarrow \neg\text{color}_{ic'}$
  "every node has at most 1 color"
- for each $1 \leq i, j \leq n$ and $1 \leq c \leq k$,
  $\text{edge}_{ij} \rightarrow \neg(\text{color}_{ic} \wedge \text{color}_{jc})$

Sudoku Example

Problem

Sudoku is a placement puzzle. The aim of the puzzle is to enter a numeral from 1 through 9 in each cell of a grid, most frequently a 9×9 grid made up of 3×3 subgrids (called "regions"), starting with various numerals given in some cells (the "givens"). Each row, column and region must contain only one instance of each numeral. Its grid layout is like the one shown in the following schema

		9				7		
	4		5		9		1	
3				1				2
	1			6			7	
		2	7		1	8		
	5			4			3	
7				3				4
	8		2		4		6	
		6				5		

Provide a formalization in propositional logic of the sudoku problem, so that any truth assignment to the propositional variables that satisfy the axioms is a solution for the puzzle.

Sudoku Example: Solution

Language

For $1 \leq n, r, c \leq 9$,define the proposition

$$in(n, r, c)$$

which means that the number n has been inserted in the cross between row r and column c.

Sudoku Example: Solution

Axioms

1. *"A raw contains all numbers from 1 to 9 "*

$$\bigwedge_{r=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{c=1}^{9} in(n,r,c)$$

2. *"A column contains all numbers from 1 to 9"*

$$\bigwedge_{c=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{r=1}^{9} in(n,r,c)$$

3. *"A block contains all numbers from 1 to 9"*

$$\bigwedge_{rb=0}^{2}\bigwedge_{cb=0}^{2}\bigwedge_{n=1}^{9}\bigvee_{r=1}^{3}\bigvee_{c=1}^{3} in(n, rb\cdot 3+r, rc\cdot 3+c)$$

4. *"A cell cannot contain two numbers"*

$$\bigwedge_{r=1}^{9}\bigwedge_{c=1}^{9}\bigwedge_{n=1}^{9}\bigwedge_{n'=n+1}^{9} in(n,r,c)\rightarrow \neg in(n',r,c)$$

notice that $in(n,r,c)\rightarrow\neg in(n',r,c)$ when $n' < n$ is not necessary because it is equivalent to $in(n',r,c)\rightarrow in(n,r,c)$ which is included in the formula.

Modelling constraints

Let $X = \{x_1, x_2, \ldots, x_n\}$ be an ordered list of propositional variables. Write formulas with the following meaning:

1. Two consecutive variables cannot take the same value:

$$\bigwedge_{i=1}^{n-1} (x_i \equiv \neg x_{i+1})$$

2. $k > 1$ consecutive variables cannot take the same value:

$$\bigwedge_{i=1}^{n=k} (x_i \wedge x_{i+1} \ldots x_{i+k-2}) \rightarrow \neg x_{k-1}$$

$$\bigwedge_{i=1}^{n=k} (\neg x_i \wedge \neg x_{i+1} \ldots \neg x_{i+k-2}) \rightarrow x_{k-1}$$

Modelling constraints

1. No zero occurs after a one:

$$\bigwedge_{i=1}^{n-1} x_i \rightarrow x_{i+1}$$

2. No zero occurs after $k > 1$ consecutive ones:

$$\bigwedge_{i=1}^{n-k} (x_i \wedge x_{i+1} \wedge \cdots \wedge x_{i+k-1}) \rightarrow x_{i+k}$$

Cardinality constraints

At least k

Given a set of boolean variables $X = \{x_1, x_2, \ldots, x_n\}$, the constraint " at least k propositional variables in X are true" is formalized by

$$\bigvee_{\substack{I \subseteq [n] \\ |I| = k}} \bigwedge_{i \in I} x_i \qquad \text{where } [n] = \{1, 2, 3, \ldots n\}$$

Example (at least 2 among $X = \{a, b, c, d\}$)

$$(a \wedge b) \vee (a \wedge c) \vee (a \wedge d) \vee (b \wedge c) \vee (b \wedge d) \vee (c \wedge d)$$

. . . in CNF

$$\bigwedge_{\substack{I \subseteq [n] \\ |I| = n-k+1}} \bigvee x_i$$

Example (at least 2 among $X = \{a, b, c, d\}$ in CNF)

Cardinality constraints

At most k (= not at least $k+1$)

The constraint " at most k propositional variables in X are true" can be rephrased as "it is not the case that at least $k+1$ variables are true" and can be formalized as the negation of "at least $k+1$":

$$\neg\left(\bigvee_{\substack{I\subseteq[n]\\|I|=k+1}}\bigwedge_{i\in I} x_i\right) \quad \text{which is equivalent to} \quad \bigwedge_{\substack{I\subseteq[n]\\|I|=k+1}}\bigvee_{i\in I}\neg x_i$$

Example (at most 2 among $X = \{a, b, c, d\}$)

$$(\neg a \vee \neg b \vee \neg c) \wedge (\neg a \vee \neg b \vee \neg d) \wedge$$
$$(\neg a \vee \neg c \vee \neg d) \wedge (\neg b \vee \neg c \vee \neg d)$$

Cardinality constraints

Exactly k

The constraint " exactly k propositional variables in X are true" can be rephrased as the conjunction of "at least k" and "at most k".

$$\bigwedge_{\substack{I\subseteq[n]\\|I|=n-k+1}} \bigvee x_i \wedge \bigwedge_{\substack{I\subseteq[n]\\|I|=k+1}} \bigvee_{i\in I} \neg x_i$$

Example (Exactly k among $X = \{a, b, c, d\}$)

$$(a \vee b \vee c) \wedge (a \vee b \vee d) \wedge (b \vee c \vee d) \wedge (\neg a \vee \neg b \vee \neg c) \wedge (\neg a \vee \neg b \vee \neg d) \wedge (\neg a \vee \neg c \vee \neg d) \wedge (\neg b \vee \neg c \vee \neg d)$$

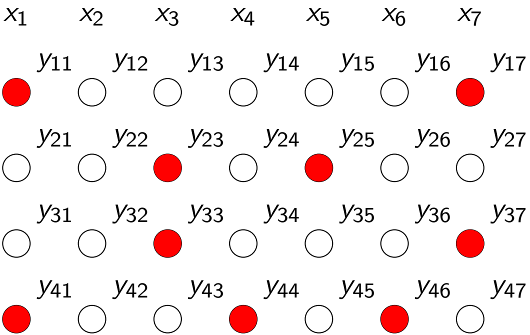
Complexity $= (n - k + 1)\binom{n}{k-1} + (k + 1)\binom{n}{k+1}$

Cardinality Constraints. An alternative for exactlty k

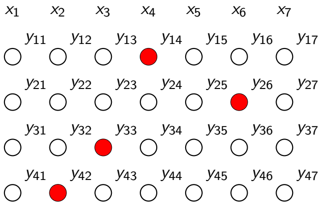
Task: select a set I of indices with $|I| = k$ such that $i \in I$ implies x_i is true

- Use auxiliary variables
- for every $1 \leq i \leq k$ and for every $1 \leq j \leq n$ add the variable

y_{ij} $\qquad$ x_j is the i-th element of I



Cardinality Constraints. An alternative for exactlty k



1. x_j implies that column j is active

$$x_j \equiv \bigvee_{i=1}^{k} y_{ij}$$

2. exactly 1 y_{ij} for every row i:

$$\bigwedge_{i=1}^{k} \left(\bigwedge_{j<j'=1}^{n} \neg(y_{ij} \wedge y_{ij'}) \wedge \bigvee_{j=1}^{n} y_{ij} \right)$$

3. at most 1 y_{ij} for every column j:

$$\bigwedge_{j=1}^{n} \bigwedge_{i<i'=1}^{k} \neg(y_{ij} \wedge y_{i'j})$$

bibliography

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