Knowledge Representation and Learning

Propositional Logic - Syntax and Semantics

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March 6, 2023

Propositional logic - Intuition

- Propositional logic is the logic of propositions
- a proposition can be true or false in the state of the world.
- the same proposition can be expressed in different ways. E.g.
  - "B. Obama is drinking a bier"
  - " The U.S.A. president is drinking a bier", and
  - "B. Obama si sta facendo una birra"

  express the same proposition.
- The language of propositional logic allows us to express propositions.

Propositions

Example (Propositions)

1. Today is Monday.
2. The derivative of sin x is cos x.
3. Every even number has at least two factors.

Example (Not Propositions)

4. The dog of my girlfriend
5. When is the pretest?
6. Do your homework!

Propositional logic language

Definition (Propositional alphabet)

Logical symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\equiv$

Non logical symbols A set $\mathcal{P}$ of symbols called propositional variables

Separator symbols "(" and ")"

,

Definition (Well formed formulas (or simply formulas))

- every $P \in \mathcal{P}$ is an atomic formula
- every atomic formula is a formula
- if A and B are formulas then $\neg A$, $A \wedge B$, $A \vee B$ $A \rightarrow B$, e $A \equiv B$ are formulas

Formulas cont'd

Example ((non) formulas)

Formulas	Non formulas
$P \to Q$	PQ
$P \to (Q \to R)$	$(P \to \wedge((Q \to R)$
$P \wedge Q \to R$	$P \wedge Q \to \neg R \neg$

Reading formulas

Problem

How do we read the formula $P \wedge Q \rightarrow R$?
The formula $P \wedge Q \rightarrow R$ can be read in two ways:

$$(P \wedge Q) \rightarrow R \qquad\qquad P \wedge (Q \rightarrow R)$$

Symbol priority

$\neg$ has higher priority, then $\wedge$, $\vee$, $\rightarrow$ and $\equiv$. Parenthesis can be used around formulas to stress or change the priority.

Symbol	Priority
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\equiv$	5

Furthermore binary connectives are right associative, e.g.:

$$a \rightarrow b \rightarrow c \text{ reads as } a \rightarrow (b \rightarrow c)$$

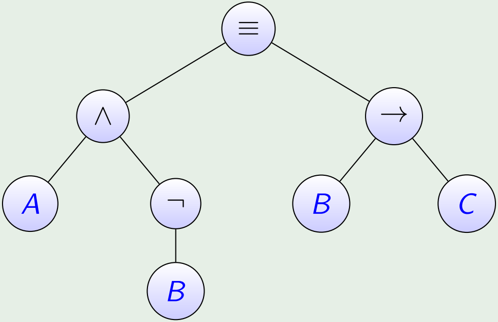
$$a \wedge b \wedge c \text{ reads as } a \wedge (b \wedge c)$$

Formulas as trees

Tree form of a formula

A formula can be seen as a tree. The leaves of the tree associated to a formula are propositional variables, while intermediate (non-leaf) nodes are associated to connectives.

Example (Tree of the formula $(A \wedge \neg B) \equiv (B \rightarrow C)$)

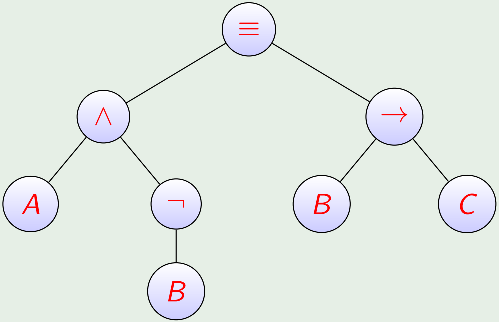


Subformulas

intuition

The subformulas of a formulas ϕ are the formulas associated to the subtrees of the tree associated to the formula ϕ.

Example



- A
- B
- C
- $\neg B$
- $A \wedge \neg B$
- $B \rightarrow C$
- $(A \wedge \neg B) \equiv (B \rightarrow C)$

Subformulas

Definition

(Proper) Subformula

- A is a subformula of itself
- A and B are subformulas of $A \wedge B$, $A \vee B$ $A \rightarrow B$, e $A \equiv B$
- A is a subformula of $\neg A$
- if A is a subformula of B and B is a subformula of C, then A is a subformula of C.
- A is a proper subformula of B if A is a subformula of B and A is different from B.

Remark

The subformulas of a formula represented as a tree correspond to all the different subtrees of the tree associated to the formula, one for each node.

Formalization in Propositional Logic

Exercise 1

Transform the following sentences in propositional logic formulas:

1. John will make supper only if Mary is working late.
2. Mary will make supper if John is working late.
3. John will not make supper unless he is very hungry.
4. John works late if and only if Mary does not.
5. Not both John and Mary will make supper.

Answer of exercise 1

1. $JS \rightarrow MW$
2. $JW \rightarrow MS$
3. $\neg JH \rightarrow \neg JS$
4. $JW \leftrightarrow \neg MW$
5. $\neg(JS \wedge MS)$

Exercise 2

Which of the following sentences express a proposition?

1. The sum of the numbers 3 and 5 equals 8.
2. Jane reacted violently to Jack's accusations.
3. Every even natural number is the sum of two prime numbers.
4. Could you please pass me the salt?
5. Ready, steady, go.
6. May fortune come your way.

Answer of exercise 2

- 1–3 are declarative
- 3–6 are not declarative, as they cannot be clearly spoken to be true or false.

Exercise 3

For each of the following compound propositions, construct the parse tree. What is the main connective in each case?

1. $(\neg p) \wedge q$
2. $\neg(p \wedge q)$
3. $(q_1 \wedge q_2) \wedge q_3$
4. $q_1 \wedge (q_2 \wedge q_3)$
5. $\neg((p \vee q) \wedge r)$
6. $((\neg p) \wedge (\neg q)) \vee (\neg r)$

Exercise 4

Parse each of the following formulas, add the parenthesis in the correct position according to the connectives' priorities

1. $p \wedge q \wedge \neg r$
2. $p \vee q \vee \neg r$
3. $q \wedge \neg p \vee q$
4. $p \wedge q \vee \neg r \wedge p$
5. $p \rightarrow q \wedge r \rightarrow s \vee t$

Answer of exercise 4

1. $p \wedge (q \wedge (\neg r))$
2. $p \vee (q \vee (\neg r)$
3. $(q \wedge (\neg p)) \vee q$
4. $((p \wedge q) \vee ((\neg r) \wedge p))$
5. $p \rightarrow ((q \wedge r) \rightarrow (s \vee (\neg t)))`,$

Exercise 5

Which of the following words are well formed?

1. $(P \wedge Q) \vee R$
2. $\vee(\wedge PQ)R$
3. $(P \wedge Q)$
4. $P\&Q$
5. $((\neg P)) \vee Q$
6. $(((P \wedge Q) \vee R))$

Answer of exercise 5

1. $(P \wedge Q) \vee R$: is a well formed formula (wff).
2. $\vee(\wedge PQ)R$: Is not a wff since the $\vee$ operator needs two arguments;
3. $(P \wedge Q)$: is a wff;
4. $P\&Q$: is not a wff, since & is not part of the alphabet;
5. $((\neg P)) \vee Q$: it is a wff, though parantesis are redundant and can be simplified in $\neg P \vee Q$;
6. $(((P \wedge Q) \vee R))$: it is a wff, but it can be simplified by removing parentesis in $P \wedge Q \vee R$.

Interpretation of Propositional Logic

Definition (Interpretation)

A Propositional interpretation is a function $\mathcal{I} : \mathcal{P} \rightarrow \{\text{True}, \text{False}\}$

Remark

If $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$, then there are $2^{|\mathcal{P}|}$ different interpretations, i.e. all the different subsets of $\mathcal{P}$. If $|\mathcal{P}|$ is finite then there is a finite number of interpretations.

Remark

A propositional interpretation can be thought as a subset S of $\mathcal{P}$, and $\mathcal{I}$ is the characteristic function of S, i.e., $A \in S$ iff $\mathcal{I}(A) = True$.

Interpretation of Propositional Logic

Example

	Functional form			Set theoretic form
	p	q	r	
$\mathcal{I}_1$	True	True	True	$\{p, q, r\}$
$\mathcal{I}_2$	True	True	False	$\{p, q\}$
$\mathcal{I}_3$	True	False	True	$\{p, r\}$
$\mathcal{I}_4$	True	False	False	$\{p\}$
$\mathcal{I}_5$	False	True	True	$\{q, r\}$
$\mathcal{I}_6$	False	True	False	$\{q\}$
$\mathcal{I}_7$	False	False	True	$\{r\}$
$\mathcal{I}_8$	False	False	False	$\{\}$

Satisfiability of a propositional formula

Definition ($\mathcal{I}$ satisfies a formula, $\mathcal{I} \models A$)

A formula A is true in/satisfied by an interpretation $\mathcal{I}$, in symbols $\mathcal{I} \models A$, according to the following inductive definition:

- If $P \in \mathcal{P}$, $\mathcal{I} \models P$ if $\mathcal{I}(P) = \text{True}$.
- $\mathcal{I} \models \neg A$ if not $\mathcal{I} \models A$ (also written $\mathcal{I} \not\models A$)
- $\mathcal{I} \models A \wedge B$ if, $\mathcal{I} \models A$ and $\mathcal{I} \models B$
- $\mathcal{I} \models A \vee B$ if, $\mathcal{I} \models A$ or $\mathcal{I} \models B$
- $\mathcal{I} \models A \rightarrow B$ if, when $\mathcal{I} \models A$ then $\mathcal{I} \models B$
- $\mathcal{I} \models A \equiv B$ if, $\mathcal{I} \models A$ iff $\mathcal{I} \models B$

Functional Notation

Sometimes it is useful to consider an interpretation $\mathcal{I}$ as a function from formulas to truth values. In this case we use the notation $\mathcal{I}(A) = \text{true}$ or $\mathcal{I}(A) = \text{false}$, as an alternative to $\mathcal{I} \models A$ and $\mathcal{I} \not\models A$, respectively

Satisfiability of a propositional formula

Proposition

For every pair of interpretations $\mathcal{I}$ and $\mathcal{I}'$, if $\mathcal{I}(p) = \mathcal{I}'(p)$ for all the propositional variables p of a formula A, then $\mathcal{I} \models A$ iff $\mathcal{I}' \models A$

- In other words if $\mathcal{I}$ and $\mathcal{I}'$ differs only on the propositional letters that do not appears in A, then the truth values of A w.r.t., $\mathcal{I}$ and $\mathcal{I}'$ are the same.
- This means that to check if $\mathcal{I} \models A$ it is enough to consider the truth assignments to the propositional variables appearing in A.

Material implication

- The semantic of the perator $\rightarrow$ is called material implication;
- it does not assert causality, i.e, that the antecedent causes the consequent;
- the interpretation of $\rightarrow$ as material implication implies that

$$\mathcal{I} \models a \rightarrow b \text{ if and only if } \mathcal{I} \models \neg a \vee b$$

- Counter intuitive

$$\text{the moon is made of cheese} \rightarrow \text{the earth is flat}$$

is always true, since the antecedent is false.

The low of excluded middle

- Notice that the formula $p \vee \neg p$ is valid, This is related to the fact that we have only *two truth values*;
- There are logics (e.g, constructive logics or intuitionistic logic) which does not assume the law of excluded middle.
- This results in a "3-valued" logic in which one allows for a third possibility, namely,"other". In this system proving that a statement is "not true" is not the same as proving that it is "false", so that indirect proofs,
- there is also the entire field of many valued logics that admits a partially ordered set of truth values $(T, \prec)$ where T can contain more than two truth values.
- e.g., in fuzzy logic there are infinitely many truth values in the real interval $[0, 1]$.

Valid, Satisfiable, and Unsatisfiable formulas

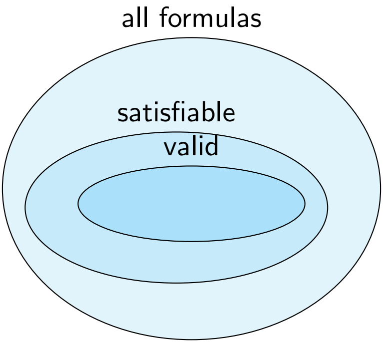
Definition

A formula A is

Valid if for all interpretations $\mathcal{I}$, $\mathcal{I} \models A$

Satisfiable if there is an interpretations $\mathcal{I}$ s.t., $\mathcal{I} \models A$

Unsatisfiable if tere is no interpretations $\mathcal{I}$, $\mathcal{I} \models A$



Valid, Satisfiable, and Unsatisfiable formulas

Example

$$\text{Satisfiable}\left\{\begin{array}{c} A \rightarrow A \\ A \vee \neg A \\ \neg\neg A \equiv A \\ \neg(A \wedge \neg A) \\ A \wedge B \rightarrow A \\ A \rightarrow A \vee B \\ p \vee q \\ p \rightarrow q \\ \neg(p \vee q) \rightarrow r \end{array}\right.$$

$$\text{Uatisfiable}\left\{\begin{array}{c} A \wedge \neg A \\ \neg(A \rightarrow A) \\ A \equiv \neg A \\ \neg(A \equiv A) \end{array}\right.$$

Valid: $A \rightarrow A$, $A \vee \neg A$, $\neg\neg A \equiv A$, $\neg(A \wedge \neg A)$, $A \wedge B \rightarrow A$, $A \rightarrow A \vee B$

Non Valid: $p \vee q$, $p \rightarrow q$, $\neg(p \vee q) \rightarrow r$, $A \wedge \neg A$, $\neg(A \rightarrow A)$, $A \equiv \neg A$, $\neg(A \equiv A)$

Prove that the blue formulas are valid, that the magenta formulas are satisfiable but not valid, and that the red formulas are unsatisfiable.

Models of a formula

Definitions (Models of ϕ)

For every formula ϕ the set $Models(\phi)$, the models of ϕ is the set $\{\mathcal{I} \mid \mathcal{I} \models \phi\}$, i.e., the set of truth assignments to the propositional variables $\mathcal{P}$ of ϕ that satisfy ϕ;

- if A is satisfiable $\text{models}(A) \neq \emptyset$;
- if A is valid $\text{models}(A) = 2^{\text{prop}(A)}$, i.e., the set of all interpretations of $\text{prop}(A)$;
- if A is unsatisfiable then $\text{models}(A) = \emptyset$.
- $\text{models}(\neg A) = 2^{\text{prop}(A)} \setminus \text{models}(A)$;
- $\text{models}(A \wedge B) = \text{models}_{\text{prop}(A \wedge B)}(A) \cap \text{models}_{\text{prop}(A \wedge B)}(B)$;
- $\text{models}(A \vee B) = \text{models}_{\text{prop}(A \vee B)}(A) \cup \text{models}_{\text{prop}(A \vee B)}(B)$;
- $\text{models}(A \rightarrow B) =$ $\text{models}_{\text{prop}(A \rightarrow B)}(\neg A) \cup \text{models}_{\text{prop}(A \rightarrow B)}(B)$

Valid, Satisfiable, and Unsatisfiable sets of formulas

Definition

A set of formulas Γ is

Valid if for all interpretations $\mathcal{I}$, $\mathcal{I} \models A$ for all formulas $A \in \Gamma$

Satisfiable if there is an interpretations $\mathcal{I}$, $\mathcal{I} \models A$ for all $A \in \Gamma$

Unsatisfiable if for no interpretations $\mathcal{I}$,, s.t. $\mathcal{I} \models A$ for all $A \in \Gamma$

Proposition

For any finite set of formulas Γ, (i.e., $\Gamma = \{A_1, \dots, A_n\}$ for some $n \geq 1$), Γ is valid (resp. satisfiable and unsatisfiable) if and only if $A_1 \wedge \cdots \wedge A_n$ is valid (resp, satisfiable and unsatisfiable).

Logical consequence

Definition (Logical consequence)

A formula A is a logical consequence of a set of formulas Γ, in symbols

$$\Gamma \models A$$

Iff for any interpretation $\mathcal{I}$ that satisfies all the formulas in Γ, $\mathcal{I}$ satisfies A,

Example (Logical consequence)

- $p \models p \vee q$
- $q \vee p \models p \vee q$
- $p \vee q, p \rightarrow r, q \rightarrow r \models r$
- $p \rightarrow q, p \models q$
- $p, \neg p \models q$

Proving Logical consequence in a direct manner

Example

Proof of $p \models p \lor q$ Suppose that $\mathcal{I} \models p$, then by definition $\mathcal{I} \models p \lor q$.

Proof of $q \lor p \models p \lor q$ Suppose that $\mathcal{I} \models q \lor p$, then either $\mathcal{I} \models q$ or $\mathcal{I} \models p$. In both cases we have that $\mathcal{I} \models p \lor q$.

Proof of $p \lor q, p \to r, q \to r \models r$ Suppose that $\mathcal{I} \models p \lor q$ and $\mathcal{I} \models p \to r$ and $\mathcal{I} \models q \to r$. Then either $\mathcal{I} \models p$ or $\mathcal{I} \models q$. In the first case, since $\mathcal{I} \models p \to r$, then $\mathcal{I} \models r$, In the second case, since $\mathcal{I} \models q \to r$, then $\mathcal{I} \models r$.

Proof of $p, \neg p \models q$ Suppose that $\mathcal{I} \models \neg p$, then not $\mathcal{I} \models p$, which implies that there is no $\mathcal{I}$ such that $\mathcal{I} \models p$ and $\mathcal{I} \models \neg p$. This implies that all the interpretations that satisfy p and $\neg p$ (actually none) satisfy also q.

Proof of $(p \land q) \lor (\neg p \land \neg q) \models p \equiv q)$ Left as an exercise

Proof of $(p \to q) \models \neg p \lor q$ Left as an exercise

Chesking Validity and (un)satisfiability of a formula

Truth Table

Checking (un)satisfiability and validity of a formula A can be done by enumerating all the interpretations (truth assignments) of the propositional variables of A, and for each interpretation $\mathcal{I}$ compute $\mathcal{I}(A)$.

Example (Truth table $p \rightarrow (q \vee \neg r)$)

p	q	r	p	$\rightarrow$	(q	$\vee$	$\neg$	r)
True	True	True	True	True	True	True	False	True
True	True	False	True	True	True	True	True	False
True	False	True	True	False	False	False	False	True
True	False	False	True	True	False	True	True	False
False	True	True	False	True	True	True	False	True
False	True	False	False	True	True	True	True	False
False	False	True	False	True	False	False	False	True
False	False	False	False	True	False	True	True	False

Valid, Satisfiable, and Unsatisfiable formulas

Proposition

if A is	*then ¬A is*
Valid	*Unsatisfiable*
Satisfiable	*not Valid*
not Valid	*Satisfiable*
Unsatisfiable	*Valid*

Truth Tables: Example

Compute the truth table of $(F \vee G) \wedge \neg(F \wedge G)$.

F	G	$F \vee G$	$F \wedge G$	$\neg(F \wedge G)$	$(F \vee G) \wedge \neg(F \wedge G)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Intuitively, what does this formula represent?

Truth Tables: Example (2)

Use the truth tables method to determine whether $(p \to q) \vee (p \to \neg q)$ is valid.

p	q	$p \to q$	$\neg q$	$p \to \neg q$	$(p \to q) \vee (p \to \neg q)$
T	T	T	F	F	**T**
T	F	F	T	T	**T**
F	T	T	F	T	**T**
F	F	T	T	T	**T**

The formula is valid since it is satisfied by every interpretation.

Truth Tables: Example (3)

Use the truth tables method to determine whether $(\neg p \vee q) \wedge (q \rightarrow \neg r \wedge \neg p) \wedge (p \vee r)$ (denoted with Φ) is satisfiable.

p	q	r	$\neg p \vee q$	$\neg r \wedge \neg p$	$q \rightarrow \neg r \wedge \neg p$	$(p \vee r)$	Φ
T	T	T	T	F	F	T	F
T	T	F	T	F	F	T	F
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	F	F
F	F	T	T	F	T	T	**T**
F	F	F	T	T	T	F	F

There exists an interpretation satisfying Φ, thus Φ is satisfiable.

Truth Tables: Exercises

Compute the truth tables for the following propositional formulas:

- $(p \rightarrow p) \rightarrow p$
- $p \rightarrow (p \rightarrow p)$
- $p \vee q \rightarrow p \wedge q$
- $p \vee (q \wedge r) \rightarrow (p \wedge r) \vee q$
- $p \rightarrow (q \rightarrow p)$
- $(p \wedge \neg q) \vee \neg(p \leftrightarrow q)$

Truth Tables: Exercises

Use the truth table method to verify whether the following formulas are valid, satisfiable or unsatisfiable:

- $(p \rightarrow q) \wedge \neg q \rightarrow \neg p$
- $(p \rightarrow q) \rightarrow (p \rightarrow \neg q)$
- $(p \vee q \rightarrow r) \vee p \vee q$
- $(p \vee q) \wedge (p \rightarrow r \wedge q) \wedge (q \rightarrow \neg r \wedge p)$
- $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $(p \vee q) \wedge (\neg q \wedge \neg p)$
- $(\neg p \rightarrow q) \vee ((p \wedge \neg r) \leftrightarrow q)$
- $(p \rightarrow q) \wedge (p \rightarrow \neg q)$
- $(p \rightarrow (q \vee r)) \vee (r \rightarrow \neg p)$

Proving Logical consequence using the truth tables

Use the truth tables method to determine whether $p \wedge \neg q \rightarrow p \wedge q$ is a logical consequence of $\neg p$.

p	q	$\neg p$	$p \wedge \neg q$	$p \wedge q$	$p \wedge \neg q \rightarrow p \wedge q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	**T**	F	F	**T**
F	F	**T**	F	F	**T**

Truth Tables: Exercises

Use the truth table method to verify whether the following logical consequences and equivalences are correct:

- $(p \to q) \models \neg p \to \neg q$
- $(p \to q) \wedge \neg q \models \neg p$
- $p \to q \wedge r \models (p \to q) \to r$
- $p \vee (\neg q \wedge r) \models q \vee \neg r \to p$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $(p \vee q) \wedge (\neg p \to \neg q) \equiv q$
- $(p \wedge q) \vee r \equiv (p \to \neg q) \to r$
- $(p \vee q) \wedge (\neg p \to \neg q) \equiv p$
- $((p \to q) \to q) \to q \equiv p \to q$

Logical Equivalence

Definition

Logical Equivalence Two formulas Φ and Ψ are **logically equivalent** (denoted with $\Phi \equiv \Psi$) if for each interpretation $\mathcal{I}$, $\mathcal{I}(\Phi) = \mathcal{I}(\Psi)$.

Truth Tables: Example (5)

Use the truth tables method to determine whether $p \to (q \wedge \neg q)$ and $\neg p$ are logically equivalent.

p	q	$q \wedge \neg q$	$p \to (q \wedge \neg q)$	$\neg p$
T	T	F	**F**	**F**
T	F	F	**F**	**F**
F	T	F	**T**	**T**
F	F	F	**T**	**T**

Properties of propositional logical consequence

Proposition

If Γ *and* Σ *are two sets of propositional formulas and* A *and* B *two formulas, then the following properties hold:*

- Reflexivity: $\{A\} \models A$
- Monotonicity: *If* $\Gamma \models A$ *then* $\Gamma \cup \Sigma \models A$
- Cut: *If* $\Gamma \models A$ *and* $\Sigma \cup \{A\} \models B$ *then* $\Gamma \cup \Sigma \models B$
- Deduction theorem: *If* $\Gamma, A \models B$ *then* $\Gamma \models A \rightarrow B$
- Refutation principle: $\Gamma \models A$ *iff* $\Gamma \cup \{\neg A\}$ *is unsatisfiable*
- Compactness: *If* $\Gamma \models A$, *then there is a finite subset* $\Gamma_0 \subseteq \Gamma$, *such that* $\Gamma_0 \models A$

Reflexivity $\{A\} \models A$.
PROOF: For all $\mathcal{I}$ if $\mathcal{I} \models A$, then $\mathcal{I} \models A$.

Monotonicity If $\Gamma \models A$ then $\Gamma \cup \Sigma \models A$
PROOF: For all $\mathcal{I}$ if $\mathcal{I} \models \Gamma \cup \Sigma$, then $\mathcal{I} \models \Gamma$, by hypothesis ($\Gamma \models A$) we can infer that $\mathcal{I} \models A$, and therefore that $\Gamma \cup \Sigma \models A$

Cut If $\Gamma \models A$ and $\Sigma \cup \{A\} \models B$ then $\Gamma \cup \Sigma \models B$.
PROOF: For all $\mathcal{I}$, if $\mathcal{I} \models \Gamma \cup \Sigma$, then $\mathcal{I} \models \Gamma$ and $\mathcal{I} \models \Sigma$. The hypothesis $\Gamma \models A$ implies that $\mathcal{I} \models A$. Since $\mathcal{I} \models \Sigma$, then $\mathcal{I} \models \Sigma \cup \{A\}$. The hypothesis $\Sigma \cup \{A\} \models B$, implies that $\mathcal{I} \models B$. We can therefore conclude that $\Gamma \cup \Sigma \models B$.

Deduction theorem If $\Gamma, A \models B$ then $\Gamma \models A \rightarrow B$

PROOF: Suppose that $\mathcal{I} \models \Gamma$. If $\mathcal{I} \not\models A$, then $\mathcal{I} \models A \rightarrow B$. If instead $\mathcal{I} \models A$, then by the hypothesis $\Gamma, A \models B$, implies that $\mathcal{I} \models B$, which implies that $\mathcal{I} \models B$. We can therefore conclude that $\mathcal{I} \models A \rightarrow B$.

Refutation principle $\Gamma \models A$ iff $\Gamma \cup \{\neg A\}$ is unsatisfiable

PROOF:

($\Longrightarrow$) Suppose by contradiction that $\Gamma \cup \{\neg A\}$ is satisfiable. This implies that there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Gamma$ and $\mathcal{I} \models \neg A$, i.e., $\mathcal{I} \not\models A$. This contradicts that fact that for all interpretations that satisfies Γ, they satisfy A

($\Longleftarrow$) Let $\mathcal{I} \models \Gamma$, then by the fact that $\Gamma \cup \{\neg A\}$ is unsatisfiable, we have that $\mathcal{I} \not\models \neg A$, and therefore $\mathcal{I} \models A$. We can conclude that $\Gamma \models A$.

Proof of compactness theorem

Definition (Finitely satisfiable set)

A set of formulas Γ is finitely satisfiable if all the finite subsets of Γ are satisfiable.

Theorem (Compactness)

Γ is satisfiable if and only if Γ is finitely satisfiable.

This formulation of the compactness theorem is equivalent to the formulation given in the previous slide. We can indeed easily prove the original compactness theorem by combining the refutation principle, monotonicity and the the new formulaiton of the compactness theorem.

- $\Gamma \models A$ if and only if $\Gamma \cup \{\neg A\}$ is not satisfiable (by the refutation principle)
- $\Gamma \cup \{\neg A\}$ is not satisfiable if and only if there is a finite subset Γ_0 of $\Gamma \cup \{\neg A\}$ which is not satisfiable (new formulation of the compactness theorem).
- This implies that $\Gamma_0 \cup \{\neg A\}$ is not satisfiable, by monotonicity, and therefore by refutation principle that $\Gamma_0 \models A$.

Proof of compactness theorem

Lemma

If Γ is finitely satisfiable then either $\Gamma \cup \{\phi\}$ or $\Gamma \cup \{\neg\phi\}$ is finitely satisfiable.

Proof.

- Suppose the conclusion of the lemma does not hold: Both $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are not finitely satisfiable.
- Hence, there are two finite subsets Γ_1 and Γ_2 of Γ such that both $\Gamma_1 \cup \{\phi\}$ and $\Gamma_2 \cup \{\neg\phi\}$ are not satisfiable.
- Let us show that $\Gamma_1 \cup \Gamma_2$ does not have models
- If $\mathcal{I}$ is a model of Γ_1, than it cannot be a model of ϕ, therefore it is a model of $\neg\phi$. But since $\Gamma_2 \cup \{\neg\phi\}$ is not satisfiable, then $\mathcal{I}$ cannot be a model of Γ_2.
- This implies that every model of Γ_1 is not a model for Γ_2 and therefore, $\Gamma_1 \cup \Gamma_2$ is not satisfiable.
- Since $\Gamma_1 \cup \Gamma_2$ is finite and it is a subset of Γ, then Γ cannot be finitely satisfiable.

□

Proof of compactness theorem

Proof of the main theorem.

- let enumerate all the formulas $\phi_1, \phi_2, \phi_3, \ldots$
- we define the sequence $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$

$$\Sigma_0 = \Gamma \qquad \Sigma_n = \begin{cases} \Sigma \cup \{\phi_n\} & \text{if } \Sigma_{n-1} \cup \{\phi_n\} \text{ is fin. sat.} \\ \Sigma \cup \{\neg\phi_n\} & \text{if } \Sigma_{n-1} \cup \{\neg\phi_n\} \text{ is fin. sat.} \end{cases}$$

- By induction, using previous lemma, Σ_i is finitely satisfiable;
- Let $\Sigma = \bigcup_{n \geq 0} \Sigma_i$
- By construction Σ is finitely satisfiable. Furthermore
  1. For every formula ϕ either $\phi \in \Sigma$ or $\neg\phi \in \Sigma$ but not both.
  2. For every $p \in \mathcal{P}$, $p \in \Sigma$ or $\neg p \in \Sigma$ but not both.

□

Proof of compactness theorem (cont'd)

Proof of the main theorem.

- By construction Σ is finitely satisfiable. Furthermore
  1. For every formula ϕ either $\phi \in \Sigma$ or $\neg\phi \in \Sigma$ but not both.
  2. For every $p \in \mathcal{P}$, $p \in \Sigma$ or $\neg p \in \Sigma$ but not both.
- We define the interpretation $\mathcal{I}(p) = \begin{cases} \text{True} & \text{if } p \in \Sigma \\ \text{False} & \text{if } \neg p \in \Sigma \end{cases}$
- $\mathcal{I} \models \phi$ for all $\phi \in \Sigma$. Consider the finite set Σ_i that contains ϕ and either p or $\neg p$ for all p in ϕ. Since it is finite, and Σ is finitely satisfiable, there is an interpretation $\mathcal{I}'$ that satisfies Σ_i, and therefore $\mathcal{I}' \models \phi$.
- $\mathcal{I}'$ and $\mathcal{I}$ agree on the interpretations of all the p's of ϕ and therefore $\mathcal{I} \models \phi$.
- Hence, $\mathcal{I} \models \Sigma$. Since $\Gamma \subset \Sigma$, then $\mathcal{I} \models \Gamma$

□

Propositional theory

Definition (Propositional theory)

A theory is a set of formulas closed under the logical consequence relation. I.e. T is a theory iff $T \models A$ implies that $A \in T$

Example (Of theory)

- T_1 is the set of valid formulas $\{A | A \text{ is valid}\}$
- T_2 is the set of formulas which are true in the interpretation $\mathcal{I} = \{P, Q, R\}$
- T_3 is the set of formulas which are true in the set of interpretations $\{I_1, I_2, I_3\}$
- T_4 is the set of all formulas

Show that T_1, T_2, T_3 and T_4 are theories

Propositional theory (2)

Example (Of non theory)

- N_1 is the set $\{A, A \rightarrow B, C\}$
- N_2 is the set $\{A, A \rightarrow B, B, C\}$
- N_3 is the set of all formulas containing P

Show that N_1, N_2 and N_3 are not theories

Axiomatization

Remark

A propositional theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas. which are infinite) (e.g.,$A \rightarrow A$ for all formulas A)

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all $A \in T$, $\Omega \models A$.

Definition

Finitely axiomatizable theory A theory T is finitely axiomatizable if it has a finite set of axioms.

Propositional theory (cont'd)

Definition (Logical closure)

For any set Γ, $cl(\Gamma) = \{A | \Gamma \models A\}$

Proposition (Logical closure)

For any set Γ*, the logical closure of* Γ*,* $cl(\Gamma)$ *is a theory*

Proposition

Γ *is a set of axioms for* $cl(\Gamma)$*.*

Axioms and theory - intuition

Compact representation of knowldge

The axiomatization of a theory is a compact way to represent a set of interpretations, and thus to represent a set of possible (acceptable) world states. In other words is a way to represent all the knowledge we have of the real world.

minimality

The axioms of a theory constitute the basic knowledge, and all the *generable knolwledge* is obtained by logical consequence. An important feature of a set of axioms, is that they are minimal, i.e., no axioms can be derived from the others.

Axioms and theory - intuition

Example

Suppose that we want to write a theory about the possible status of a traffic-light.

$$Red \rightarrow \neg Orange \qquad (1)$$

$$Red \rightarrow \neg Green \qquad (2)$$

$$Orange \rightarrow \neg Green \qquad (3)$$

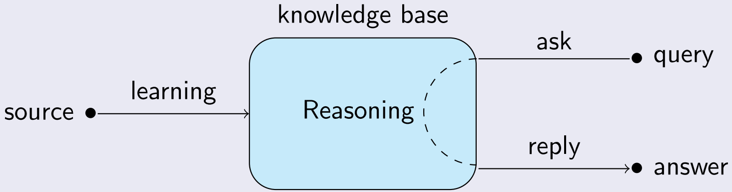
$$Red \lor Orange \lor Green \qquad (4)$$

The axioms above constitute the basic knowledge about three propositions *Red*, *Orange* and *Green*, that are mutual exclusive, and such that at leas one is true. The formula $\neg Red \rightarrow Orange \lor Green$, corresponding to the proposition "if the traffic light is not red then it must be orange or green", which is also valid, should not be added to the set of axioms for the traffic light since it is entailed by (1)–(4)

Logic based systems

A logic-based system for representing and reasoning about knowledge is composed by a Knowledge base and a Reasoning system. A knowledge base consists of a finite collection of formulas in a logical language. The main task of the knowledge base is to answer queries which are submitted to it by means of a Reasoning system

Logic based system for knowledge representation



Learn: this action incorporates the new knowledge encoded in an axiom (formula). This allows to build/extend/update a *KB*.

Ask: allows to query what is known, i.e., whether a formula ϕ is a logical consequences of the axioms contained in the KB

Propositional theory (cont'd)

Proposition

Given a set of interpretations S, the set of formulas A which are satisfied by all the interpretations in S is a theory. i.e.

$$T_S = \{A | \mathcal{I} \models A \textit{ for all } \mathcal{I} \in S\}$$

is a theory.

Knowledge representation problem

Given a set of interpretations S which correspond to admissible situations find a set of axioms Ω for T_S.