Static games in normal form
Choice of strategies

• Dominating strategies

• MiniMax Theorem (von Newmann)

Saddle points existence not guaranteed
Nash Equilibrium

A set of strategies constitutes a Nash equilibrium if no single player is interested in changing his strategy unless one of the other players changes his own.

That is:

Keeping the choices of other players fixed, Nobody is interested in changing his own.
Example with No saddle point but there exists 1 Nash equilibria

Set of strategies \((a, \beta)\) s.t.:
Knowing that \(G1\) plays \(a\) then for \(G2\) has not choice (convenience) but to play \(\beta\)
Knowing that \(G2\) plays \(\beta\) then for \(G1\) has not choice (convenience) but to play \(a\)
Example with No saddle point but there exist 2 Nash equilibria

Set of strategies \((a, \beta)\) s.t.:
Knowing that G1 plays \(a\) then for G2 has not choice (convenience) but to play \(\beta\)
Knowing that G2 plays \(\beta\) then for G1 has not choice (convenience) but to play \(a\)
CAR RACE

\[ \begin{array}{c|cc}
A & H & T \\
\hline
H & (1,1) & (-1,0) \\
T & (-1,0) & (1,1) \\
\end{array} \]

MATCHING PENNY

\[ \begin{array}{c|cc}
A & H & T \\
\hline
H & (1,1) & (-1,1) \\
T & (-1,1) & (1,1) \\
\end{array} \]

Chickren game

\[ \begin{array}{c|cc}
A & H & T \\
\hline
H & (1,1) & (-1,1) \\
T & (-1,1) & (1,1) \\
\end{array} \]

Dom Strael

Saddle point

2 Nash eq.

PURE STRATEGIES

I mixed strategies

Expected

\[ E(A) = \frac{1}{2} \]

\[ \theta \in [a, b] \rightarrow a = \frac{1}{2}, b = \frac{1}{2} \]
Nash equilibria for static games

Existence of Nash equilibrium Kakutani fixed point theorem for multivalued maps. Consequence of the classical Brouwer fixed point theorem.

In a zero-sum game, if a Nash equilibrium exists, then all Nash equilibria yield the same payoff \( V \) (value of the game) (von Neumann)
Nash Equilibrium Existence Theorem

(1950)

In a finite game there exists at least one Nash equilibrium (eventually mixed strategies)

\[
P1 = \begin{bmatrix}
z1 & 3 & 0 
z2 & -1 & 1 
\end{bmatrix}
P2 = \begin{bmatrix}
y1 & y2 
z1 & 3 & 0 
z2 & -1 & 1 
\end{bmatrix}
\]

\( (u_i^N, u_{-i}^N) \) Nash equilibrium

\[ J_i(u_i^N, u_{-i}^N) > J_i(u_i, u_{-i}^N) \text{ for all } u_i \in U_i \]
**Nash Equilibrium**

- There might be other combination of strategies that increase the payoff of some players without reducing the payoffs of the others. Or, more, that increase the payoff of all players: *Prisoners’ dilemma*. 
**Prisoners’ Dilemma**

- If **only one** confesses, and puts the blame on the other one, then he is set free and the other will be sentenced to 6 years of jail;
- If **both** confess, they will be sentenced to 5 years.
- If **neither one** confesses, they will be sentenced to 1 year.

$$
\begin{array}{c|cc}
   & C & NC \\
\hline
C & (-5, -5) & (0, -6) \quad -5 \\
NC & (-6, 0) & (-1, -1) \quad -6 \\
\end{array}
$$

Min A
**Prisoners’ Dilemma A**

\[
\begin{array}{cc}
  C & NC \\
  C & (-5, -5) & (0, -6) \\
  NC & (-6, 0) & (-1, -1) \\
\end{array}
\]

\[\text{Min A} = \begin{cases} 
-5 \\
-6
\end{cases}\]
Prisoners’ Dilemma $B$

\[
\begin{pmatrix}
\text{C} & \text{NC} \\
\text{C} & \begin{pmatrix}
(-5, -5) \\
(-6, 0)
\end{pmatrix} & \begin{pmatrix}
(0, -6) \\
(-1, -1)
\end{pmatrix}
\end{pmatrix}
\]
Prisoners’ Dilemma

Max Min of B

Max Min of A

Cooperative solution

NASH eq. (-5, -5)
Prisoners’ Dilemma
Nash equilibrium

\[
\begin{pmatrix}
\text{A} & \text{NA} \\
\text{A} & (-5, -5) & (0, -6) \\
\text{NA} & (-6, 0) & (-1, -1)
\end{pmatrix}
\]
Nash Equilibrium

• Existence and uniqueness is not guaranteed
  → There might exist more than one NE

• It gives solution when there might be uncertainty
• Each player does what is better for him (noncooperative)
• It might not be the better solution for everybody.
• Someone might increase his payoff moving far from the equilibrium.

Nash Equilibrium might not be Pareto Optimum.
Nash equilibrium
Noncooperative simultaneous game

• Symmetric Information structure

Max $J_1(u_1,u_2)$
$u_1 \in U^1$

Max $J_2(u_1,u_2)$
$u_2 \in U^2$

$u_1^{BR} = u_1(u_2)$

$u_2^{BR} = u_2(u_1)$

$(u_1^N, u_2^N)$
**Stackelberg game**

Noncooperative sequential game

- Asymmetric information structure

1. LEADER: declares his action \( u_L \)
2. FOLLOWER: computes his best response \( u_F(u_L) \) (to any Leader’s strategy \( u_L \))
3. LEADER: computes his optimal Stackelberg strategy \( u_L^S \)
4. FOLLOWER: adjust his strategy to obtain the Stackelberg strategy \( u_F^S \)

\[
\begin{align*}
\text{Max } J_F(u_L, u_F) & \quad u_F^{BR}=u_F(u_L) \\
 & \quad \text{Max } J_L(u_L, u_F(u_L)) \\
& \quad u_L \in U_L \\
& \quad (u_L^S, u_F^S)
\end{align*}
\]
Coordination game

Cooperative simultaneous game

• Symmetric information structure

Max \( J_1(u_1, u_2) + J_2(u_1, u_2) \)

\( u_1, u_2 \in U^1 \times U^2 \)

A: 7 + \( \frac{1}{2} \)  
F: 10 + \( \frac{1}{2} \)

Bargaining — Rationally/Fair

Nash Bargaining Solution

\( 18 - (7 + 10) = \frac{1}{2} \)
Example Cournot duopoly static game with infinite strategy sets

\[ J_1 = (\alpha - \beta (Q_1 + Q_2)) Q_1 - K_1 Q_1^2 \]

\[ J_2 = (\alpha - \beta (Q_1 + Q_2)) Q_2 - K_2 Q_2^2 \]

\[ \text{NASH: } (Q_1^N, Q_2^N) = \left( \frac{\alpha}{2K_1 + 3\beta}, \frac{\alpha}{2K_2 + 3\beta} \right) \]

\[ \text{Symm. case } \alpha = \beta = 1, K_i = 0 \Rightarrow (Q_1^N, Q_2^N) = (1/3, 1/3). J_1^N = J_2^N = 1/9 \]

\[ \text{STACKELBERG: } (Q_{LS}, Q_{FS}) = \left( \frac{\alpha \left( 1 - \frac{\beta}{2(K_F + \beta)} \right)}{2(K_L + \beta) - \beta^2 / (K_F + \beta)}, \frac{\alpha - \beta Q_L^S}{2(K_F + \beta)} \right) \]

\[ \text{Symm. case } \alpha = \beta = 1, K_i = 0 \Rightarrow (Q_1^S, Q_2^S) = (1/2, 1/4). J_1^S = 1/8, J_2^S = 1/16 \]

\[ \text{COOPERATIVE: } \text{Symm. case } \alpha = \beta = 1, K_i = 0 \Rightarrow J_C = 2/9 = J_1^N + J_2^N \]

\[ \text{IN GENERAL } (\alpha, \beta \in \mathbb{R}, K_i = 0) \Rightarrow J_C > J_1^N + J_2^N \]
\[ \max J_1(Q_1, Q_2) \]  
**Quadratic**  
\[ d - \beta (Q_1 + Q_2) - \beta Q - 2k Q = 0 \]
\[ \frac{\partial J_1}{\partial Q_1} = d - \beta (Q_1 + Q_2) - \beta Q - 2k Q = 0 \]
\[ \frac{\partial J_1}{\partial Q_2} = d - \beta (Q_1 + Q_2) - \beta Q_2 - 2k Q_2 = 0 \]

\[ \beta Q_1 - 2k Q + \beta Q_2 + 2k Q_2 = 0 \]
\[ (\beta + 2k) Q_2 = (\beta + 2k) Q_1 \]

Stackelberg  
\[ \frac{\partial J_F}{\partial Q_F} = d - \beta (Q_L + Q_F) - \beta Q_F - 2k Q_F = 0 \]
\[ Q_F^{BR} (Q_L) = \frac{d - \beta Q_L}{2 \beta + 2k} \]
\[ J_l = (d - \beta (Q_l + \frac{d - \beta Q_l}{2 \beta + 2 K_F})) Q_l - K_l Q_l^2 \]

\[
\max_{Q_l} \quad \frac{\partial J_l}{\partial Q_l} = 0
\]

\[ Q^* \]

\[ Q_l \]