Introduction to

differential games

PhD Program in Mathematical Sciences

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Course contents
(12 hours)

• Recall of basic concepts of game theory, best response strategies, dominating strategies, Nash equilibrium
• Dynamic games: formalization of a differential game
• Simultaneous Noncooperative differential games (Nash equilibrium)
• Hierarchic differential games (Stackelberg equilibrium)
References

Exam

1. The lecturer will suggest a set of recent scientific publications on differential games
2. Each student will choose a paper among the suggested ones to read, comprehend and present in class
Introduction to game theory

Buratto Alessandra
Game theory

Quantitative methods
for strategic interactions
among entities
Our logical thread

<table>
<thead>
<tr>
<th></th>
<th>One player</th>
<th>Many players</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static</td>
<td>Mathematical programming</td>
<td>(Static) game theory</td>
</tr>
<tr>
<td>Dynamic</td>
<td>Optimal control theory</td>
<td>Dynamic (and/or differential) game theory</td>
</tr>
</tbody>
</table>
From Mathematical programming to Game Theory

One decision maker

\[
\max_u J(u), \ u \in U, \quad U \text{ set of actions}
\]

Two decision makers P1, P2

\[
\begin{align*}
\max_{u_1} J(u_1, u_2), \quad & u_1 \in U_1 \\
\max_{u_2} J(u_1, u_2), \quad & u_2 \in U_2
\end{align*}
\]

Game Theory
Game

Basic elements:

- **Players** with clear preferences, represented by a **Payoff** function.
- Each **Action** leads to an associated Consequence

Axioms:

- Players are rational:
  
  They are aware of their alternatives, forms expectations about any unknowns, have clear preferences, and choose their action deliberately after some process of optimization.

- And think strategically.
  
  When designing his strategy for playing the game, each player takes into account any knowledge or expectation he may have regarding his opponents' behaviour.
Rational Behavior

A  Set of Actions from which the decision-maker makes a choice.
C  Set of possible Consequences of these actions.

J: A --> C

Consequence function that associates a Consequence with each Action.

Preference relation (a complete transitive reflexive binary relation) on C.
Static games (One-shot games)

• Each player makes one choice and this completely determines the payoffs.
• Zero-Sum (Noncooperative) matrix games $\leftrightarrow$ NonZero-sum bimatrix
• Normal (strategic) form: all possible sequences of decisions of each player are set out against each other (no dynamic)
  • Matrix structure
Normal form

G2

\[
\begin{bmatrix}
\alpha & \beta \\
(3, -2) & (2, 0) \\
(0, 2) & (1, 1) \\
\end{bmatrix}
\]

Non-zero sum game

Existence questions
Pure and mixed strategies

Extensive form for G1

Single-act games
Multi-act games
Choice of strategies

WHAT IS OPTIMAL?
**Best response strategies**

<table>
<thead>
<tr>
<th></th>
<th>P2</th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td></td>
<td>(1,-1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>b</td>
<td></td>
<td>(2,-2)</td>
<td>(0,-3)</td>
</tr>
<tr>
<td>c</td>
<td></td>
<td>(1,-1)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>

$u_i^b$, best reply (response) by player $i$ to a profile of strategies for all other players $u_{-i}$ if

$$J^i(u_i^b, u_{-i}) \geq J^i(u_i, u_{-i})$$

for all $u_i \in U^i$.
### Strictly Dominating strategies

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>(1,0)</td>
<td>(0,0)</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>(2,-2)</td>
<td>(1,0)</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>(1,-1)</td>
<td>(0,-1)</td>
<td></td>
</tr>
</tbody>
</table>

**ui^d** of player I

\[ J^i(u_i^d, u_{-i}) > J^i(u_i, u_{-i}) \] for all \( u_i \in U^i \), for all \( u_{-i} \in U^1 \times U^2 \times \cdots \times U^{i-1} \times U^{i+1} \times \cdots \times U^N \)
Dominating strategies

• Eliminating some rows and/or columns which are known from the beginning to have no influence on the equilibrium solution

• Looking for Saddle points

• best reply to any feasible profile of the $N - 1$ rivals:
Example: Zero Sum Marketing game

<table>
<thead>
<tr>
<th>Market 1</th>
<th>Market 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FIRM A</strong></td>
<td>4 units of capital</td>
</tr>
</tbody>
</table>

**Payoffs of A**

<table>
<thead>
<tr>
<th>STRATEGIES of B</th>
<th>2, 0</th>
<th>1, 1</th>
<th>0, 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4, 0</strong></td>
<td>1+0=1</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>3, 1</strong></td>
<td></td>
<td>1+1=2</td>
<td></td>
</tr>
<tr>
<td><strong>2, 2</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>1, 3</strong></td>
<td></td>
<td>-1+1=0</td>
<td></td>
</tr>
<tr>
<td><strong>0, 4</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example: Zero Sum Marketing game -2-

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 , 0</td>
<td>1 , 1</td>
<td>0 , 2</td>
<td></td>
</tr>
<tr>
<td>4 , 0</td>
<td>(1,-1)</td>
<td>(0,0)</td>
<td>(0,0)</td>
<td></td>
</tr>
<tr>
<td>3 , 1</td>
<td>(2,-2)</td>
<td>(1,-1)</td>
<td>(0,0)</td>
<td></td>
</tr>
<tr>
<td>2 , 2</td>
<td>(1,-1)</td>
<td>(2,-2)</td>
<td>(1,-1)</td>
<td></td>
</tr>
<tr>
<td>1 , 3</td>
<td>(0,0)</td>
<td>(1,-1)</td>
<td>(2,-2)</td>
<td></td>
</tr>
<tr>
<td>0 , 4</td>
<td>(0,0)</td>
<td>(0,0)</td>
<td>(1,-1)</td>
<td></td>
</tr>
</tbody>
</table>
### Dominating strategies Player A

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4, 0</td>
<td>2, 0</td>
<td>1, 1</td>
<td>0, 2</td>
</tr>
<tr>
<td>3, 1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2, 2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1, 3</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>0, 4</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
MaxiMin rule
(von Neumann)

- non-probabilistic decision-making rule
- decisions are ranked on the basis of their worst-case outcomes
- the optimal decision is one with the least worst outcome.

“In the worst of cases...”
MaxiMin rule

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$s_1^B$</th>
<th>$s_2^B$</th>
<th>$s_3^B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1^A$</td>
<td>(7,-7)</td>
<td>(5,-5)</td>
<td>(4,-4)</td>
<td></td>
</tr>
<tr>
<td>$s_2^A$</td>
<td>(2,-2)</td>
<td>(6,-6)</td>
<td>(3,-3)</td>
<td></td>
</tr>
<tr>
<td>$s_3^A$</td>
<td>(8,-8)</td>
<td>(0,0)</td>
<td>(1,-1)</td>
<td></td>
</tr>
</tbody>
</table>
MaxiMin rule

**Saddle point**

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$S_1^B$</th>
<th>$S_2^B$</th>
<th>$S_3^B$</th>
<th>MIN of A</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1^A$</td>
<td>(7,-7)</td>
<td>(5,-5)</td>
<td>(4,-4)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$S_2^A$</td>
<td>(2,-2)</td>
<td>(6,-6)</td>
<td>(3,-3)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$S_3^A$</td>
<td>(8,-8)</td>
<td>(0,0)</td>
<td>(1,-1)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

MIN of B  
-8  -6  -4

MAX MIN of A: 4
MAX MIN of B: 0
Saddle points may not exist (♯)
Static games in normal form

Choice of strategies

- Dominating strategies
- MiniMax Theorem (von Newmann)

Saddle points existence not guaranteed
Nash Equilibrium

A set of strategies constitutes a Nash equilibrium if no single player is interested in changing his strategy unless one of the other players changes his own.

That is:

Keeping the choices of other players fixed,
Nobody is interested in changing his own.
**Example with No saddle point but there exists 1 Nash equilibria**

Set of strategies \((a, \beta)\) s.t.:
Knowing that G1 playes \(a\) then for G2 has not choise (convenience) but to play \(\beta\)
Knowing that G2 playes \(\beta\) then for G1 has not choise (convenience) but to play \(a\)
Example with No saddle point but there exist 2 Nash equilibria

Set of strategies \((a, \beta)\) s.t.:
Knowing that G1 plays \(a\) then for G2 has not choice (convenience) but to play \(\beta\)
Knowing that G2 playes \(\beta\) then for G1 has not choice (convenience) but to play \(a\)

\[
\begin{pmatrix}
& \alpha & \beta \\
\alpha & (-3, -2) & (2, 0) \\
b & (0, 2) & (1, 1)
\end{pmatrix}
\]