Time consistency and Stackelberg games

Alessandra Buratto
The time consistency issue

(Dockner et al. p.98)

Notation

- Weak time consistency (WTC) ≡ Time consistency (TC)
- Strong time consistency (STC) ≡ Subgame-perfect (SP)
(Weak) time consistency

Definition ((Weak) time consistency)

A MNE in $\Gamma(0, x_0)$ is time consistent if it is a MNE in any subgame $\Gamma(t, x)$ that starts in $x^*(t)$

- Any OLNE is (weakly) time consistent
- Any MNE is (weakly) time consistent
Strong time consistency

Definition ((Strong) time consistency)

A MNE in $\Gamma(0, x_0)$ is subgame perfect (strongly time consistent) if it is a MNE in any subgame $\Gamma(t, x)$, $\forall x \in X$ (either on the optimal equilibrium trajectory OR not). Any $\Gamma(t, x)$ is identical to $\Gamma(0, x_0)$ except for the initial point.

Markov Perfect Nash Equilibrium

Theorems

- Any OLNE is NOT subgame perfect (in general)
- Any MNE is subgame perfect
- A MNE with $T = +\infty$ is subgame perfect if $\phi^*$ is independent of $x_0$
OLNE NOT subgame perfect: Example

N players

\[ J^i(u^i()) = - \int_0^T (u^i(t))^2 \, dt - x(T)^2 \]
\[ \dot{x}(t) = \sum_{j=1}^{N} u^j(t) \]
\[ x(0) = 0 \]
\[ u(t) \in \mathcal{R} \]

\[ J^i(u^i()) \leq 0 \text{ for any feasible control} \Rightarrow \text{Optimal value } J^i(u^i()) = 0 \]

Optimal control \( u^i(t) \equiv 0 \Rightarrow \text{Optimal path } x^i(t) \equiv 0 \)

\( x(t) \equiv 0 \), \( \Rightarrow \) eq. trajectory \( u^i(t) = \Phi^i(x(t), t) = x(t) \)

\( u^i^*(t) \) is **Time consistent**: (strategies credible along the eq. trajectory)

Let all players \( j \neq i \) use \( \Phi^j(x, t) = x \), then player \( i \) has to face

\[ \begin{cases} 
\dot{x}(t) = u^i + (N - 1)x \\
x(0) = 0 
\end{cases} \Rightarrow u^{i^*}(t) = 0 \Rightarrow x^*(t) = 0. \]
OLNE NOT subgame perfect: Example

Strategies not credible along any trajectory

\( \Phi^i \) not credible as optimal behaviour OFF the equilibrium path

If there exists some time \( t \) such that \( x(t) \neq 0 \), then:

- All players sticking to \( \Phi^i \) would have to choose non-zero controls \( \Phi^i(x(t), t) = x(t) \neq 0 \) state is driven away from 0
- Each player prefers to choose \( u^i(t) = 0 \) to avoid the cost associated with a non-zero control value and to reduce the speed at which the system diverges from 0.

Although the strategies \( \Phi^i(x, t) = x \) are credible along the equilibrium trajectory \( x^*(t) \), they are not credible as specifications of optimal behaviour out of the equilibrium path.
MNE are subgame perfect: Example

HJB ...

\[ \Phi^i(x, t) = \frac{x}{(2N - 1)(t - T) - 1} \]

\[ V(x, t) = \frac{x^2}{(2N - 1)(t - T) - 1} \]

\[ \limsup_{t \to +\infty} e^{-rt} V(x_f(t), t) \leq 0 \text{ for any } x_f \text{ feasible trajectory.} \]

Markov perfect Nash equilibrium
Stackelberg equilibrium

Sequential, asymmetric information, hierarchical

Leader (L), Follower (F)

a) L: declares his strategy $u^L$

b) F: computes his best response (rational choice) $u^F = u^F(u^L)$

c) L: 

$$\max_{u^L \in \mathcal{U}^L} J^L(u^L, u^F(u^L))$$

backward induction.

Time consistency
Open-Loop Stackelberg Equilibrium (OLSE)

System dynamics

\[
\begin{aligned}
\dot{x}_i(t) &= f_i(x_i(t), u^L(t), u^F(t), t) \\
x_i(0) &= x_0 \\
x_i(T) &\in R, u^L(t) \in \mathcal{U}^L, u^F(t) \in \mathcal{U}^F
\end{aligned}
\]

a) L: declares his control path \( u^L(t) \)
b) F: computes his best response

\[
\max_{u^F \in \mathcal{U}^F} J^F = \int_0^T e^{-rF} t v^F(x(t), u^L, u^F(t), t) \, dt
\]

\[
H^F_C(x, u^F, \lambda, t) = v^F(x, u^L, u^F, t) + \sum_{i=1}^n \lambda_i(t) f_i(x, u^L, u^F, t)
\]

concavity, \( \mathcal{U}^F \) open, stationary points.
\[
\frac{\partial H^F}{\partial u^F} = \frac{\partial v^F(x, u^L, u^F, t)}{\partial u^F} + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i(x, u^L, u^F, t)}{\partial u^F} = 0
\]

\[
\dot{\lambda}_i(t) = -\frac{\partial H^F}{\partial x_i} = -\frac{\partial v^F(x, u^L, u^F, t)}{\partial x_i} - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i(x, u^L, u^F, t)}{\partial x_i}
\]

\[
\lambda_i(T) = 0
\]

\[
\exists\ u^F(t) = g(x(t), \lambda(t), u^L(t), t) \text{ best response of } F \text{ to the actions of the leader}
\]

The co-state equation becomes

\[
\dot{\lambda}_i(t) = -\frac{\partial v^F(x(t), u^L(t), g(x(t), \lambda(t), u^L(t), t))}{\partial x_i} + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i(x(t), u^L(t), g(x(t), \lambda(t), u^L(t), t))}{\partial x_i}
\]

\[
\lambda_i(T) = 0
\]

Time consistency
1. What do we know about $\lambda_i(0)$?

2. Do they depend on the leader’s announced time path $u^*(t)$ or not?

The answer depends on the structure of the problem.

$\lambda_i(0)$ is **controlled** by $u^l$

$\lambda_i(0)$ is **not controlled** by $u^l$
Example 5.1 $\lambda_i(0)$ Controlled by $L$

$$J^F = \int_0^T \left( u^F - \frac{(u^F)^2}{2} - \frac{x^2}{2} \right) dt$$

$$\dot{v}^F(x, u^F) = u^F - \frac{(u^F)^2}{2} - \frac{x^2}{2}$$

$$\begin{cases} 
\dot{x}(t) = u^F(t) + u^L(t) \\
x(0) = x_0 
\end{cases}$$

$$H^F(x, u^F, \lambda, t) = u^F - \frac{(u^F)^2}{2} - \frac{x^2}{2} + \lambda(u^F + u^L)$$

$$u^*(t) = 1 + \lambda(t)$$

$$\begin{cases} 
\dot{\lambda}(t) = -\frac{\partial H^F}{\partial x} = x(t), \quad \lambda(T) = 0 \\
\dot{x}(t) = (1 + \lambda(t)) + u^L(t) \quad x(0) = x_0 
\end{cases}$$

The Follower’s control variable $u^F(t)$ at time $t$ depends also on the future values of $u^L(t)$, i.e. on $u^L(s)$, $s > t$. 
Example 5.2 $\lambda_i(0)$ NOT Controlled by L

\[ V^F(x, u^F) = u^F - \frac{(u^F)^2}{2} - x \]

\[
\begin{align*}
\dot{x}(t) &= u^F(t) + u^L(t) \\
x(0) &= x_0
\end{align*}
\]

\[ H^F(x, u^F, \lambda, t) = u^F - \frac{(u^F)^2}{2} - x + \lambda(u^F + u^L) \]

\[ u^*(t) = 1 + \lambda(t) \]

\[
\begin{align*}
\dot{\lambda}(t) &= -\frac{\partial H^F}{\partial x} = 1 \\
\lambda(T) &= 0
\end{align*}
\]

\[ \lambda(t) = t - T \]

State redundant

The Leader has no influence on the follower’s best response.
Controllable co-state

**Definition**

The initial value $\lambda(0)$ of the Follower’s co-state function is called

- **Controllable** if $\lambda(0)$ depends on $u^L(t)$ (Ex 5.1)
- **Uncontrollable** if $\lambda(0)$ does not depend on $u^L(t)$ (Ex 5.2)
The Leader’s problem

L knows the best response of the Follower

\[
\max_{u^L} J^L = \int_0^T e^{-r^L t} v^L(x(t), u^L(t), u^{FBR}(t), t)
\]

\[ u^{FBR}(t) = g(x(t), \lambda(t), u^L(t), t), t) \]

The co-state function of F becomes a state function for L →

additive co-state function \( \pi \) associated with \( \lambda \)

\[ x(0) = x_0 \text{ fixed} \]

\( \lambda(0) \) is fixed iff it is uncontrollable

\[
H_C^L(x, \lambda, u^L, \psi, \pi, t) = v^L(x, u^L, g(x(t), \lambda(t), u^L(t), t), t)
\]

\[ + \sum_{i=1}^n \psi_i(t) f_i(x, u^L, g(x(t), \lambda(t), u^L(t), t), t), t) + \]

\[ + \sum_{i=1}^n \pi_i k_i(x, \lambda, u^L, t) \]
\[ \frac{\partial H^L(x(t), \lambda(t), u^L(t), \psi(t), \pi(t), t)}{\partial u^L} = 0 \]

\[ \dot{\psi}(t) = r^L \pi_i(t) - \frac{\partial H^L(x(t), \lambda(t), u^L(t), \psi(t), \pi(t), t)}{\partial x_i} \]

\[ \dot{\pi}(t) = r^L \pi(t) - \frac{\partial H^L(x(t), \lambda(t), u^L(t), \psi_i(t), \pi(t), t)}{\partial \lambda_i} \]

\[ \psi_i(T) = 0 \text{ because } x(T) \in \mathcal{R} \]

\[ \pi_i(0) = ? \]

- If \( \lambda(0) \) is controllable \( \Rightarrow \) \( \lambda(0) \) treated as a state function of L associated co-state \( \pi_i(0) = 0 \)
- If \( \lambda(0) \) is non-controllable \((\lambda(t) = t - T) \Rightarrow \) no need to consider it as a state function of L
Non consistent Stackelberg equilibrium

\[ J^L = \int_0^T u^L(t) - \frac{1}{2}[(u^L(t))^2 + (x(t))^2] \, dt \]
\[ \dot{x}(t) = 1 + \lambda(t) + u^L(t) \]
\[ \dot{\lambda}(t) = x(t) \]
\[ x(0) = 0, \quad x(T) \in \mathcal{R} \]
\[ \lambda(T) = 0, \quad \lambda(0) \text{ controllable} \]

\[ H^L(x, \lambda, u^L, \psi, \pi) = u^L - \frac{1}{2}(u^L + x^2) + \psi(1 + \lambda + u^L) + \pi x \]

\[
\begin{cases}
1 - u^L(t) + \psi(t) = 0 \\
\dot{\psi}(t) = x(t) - \pi(t) \\
\dot{\pi}(t) = -\psi(t) \\
\psi(T) = 0 \\
\pi(T) = 0
\end{cases}
\]
\[ z = (x, \lambda, \psi, \pi) \]
\[ B = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ \dot{z} = Bz + k \]

\[ \exists! \; SOL \]

At a given time \( t_1 > 0 \), we have \( \pi(t_1) \neq 0 \)
If L can replan his strategy at the time \( t_1 \), he will choose a new solution such that \( \pi(t_1) = 0 \) (because his co-state fct at \( t_1 \) is free) and therefore he will deviate.
The Leader has no longer an incentive to keep his promises.
Consistent Stackelberg equilibrium

(Example 5.2 (continued))

\[ \lambda(t) = t - T \quad \lambda(0) = -T \]

\[ 1 + \lambda(t) = 1 + t - T \]

\[ J^L = \int_0^T u^L(t) - \frac{1}{2}[(u^L(t))^2 + (x(t))^2] \, dt \]

\[ \dot{x}(t) = 1 + t - T + u^L(t) \]

\[ x(0) = 0, \quad x(T) \in \mathcal{R} \]

\[ H^L(x, \lambda, u^L, \psi, \pi) = u^L - \frac{1}{2}(u^L + x^2) + \psi(1 + t - T + u^L) \]

\[ 1 - u^L(t) + \psi(t) = 0 \quad \Rightarrow \quad u^L(t) = 1 + \psi(t) \]

\[ \begin{cases} 
\dot{x}(t) = \psi(t) + 2 + t - T, & x(0) = 0 \\
\dot{\psi}(t) = x(t), & \psi(T) = 0 
\end{cases} \]