Introduction to differential games
PhD Program in Mathematical Sciences

Buratto Alessandra
buratto@math.unipd.it
328 7058243
Course contents
(12 hours)

- Recall of basic concepts of game theory, best response strategies, dominating strategies, Nash equilibrium
- Dynamic games: formalization of a differential game
- Simultaneous Noncooperative differential games (Nash equilibrium)
- Hierarchic differential games (Stackelberg equilibrium)
References

Exam

1. The lecturer will suggest a set of recent scientific publications on differential games
2. Each student will choose a paper among the suggested ones to read, comprehend and present in class
Introduction to game theory

Buratto Alessandra
Game theory

Quantitative methods for strategic interactions among entities
## Motivations

<table>
<thead>
<tr>
<th>MILITARY</th>
<th>Gulf war,…</th>
</tr>
</thead>
<tbody>
<tr>
<td>ECONOMICS - MARKETING</td>
<td>Advertising, Promotion, Price, …</td>
</tr>
<tr>
<td>ECONOMICS – FINANCE</td>
<td>Portfolio Management</td>
</tr>
<tr>
<td>POLITICS</td>
<td>Voting systems,…</td>
</tr>
<tr>
<td>SPORT</td>
<td>Attack / Defense Strategies</td>
</tr>
<tr>
<td>SOCIOLOGY</td>
<td>Migration, …</td>
</tr>
<tr>
<td>MEDICINE- BIOLOGY</td>
<td>Neurons, Bacterial evolution</td>
</tr>
<tr>
<td>PSICOLOGY</td>
<td>Prisoners’ dilemma, …</td>
</tr>
<tr>
<td>ENVIRONMENT</td>
<td>Pollution, Kyoto cartel, …</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>... LOGIC – PHILOSOPHY– RELIGION ...</td>
<td></td>
</tr>
<tr>
<td>Year</td>
<td>Author(s)</td>
</tr>
<tr>
<td>------</td>
<td>-----------------------------------</td>
</tr>
<tr>
<td>1928</td>
<td>von Neumann</td>
</tr>
<tr>
<td>1940</td>
<td>von Neumann, Turing, Zu</td>
</tr>
<tr>
<td>1944</td>
<td>von Neumann, Morgenstern</td>
</tr>
<tr>
<td>1950</td>
<td>Nash</td>
</tr>
<tr>
<td>1951</td>
<td>Isaacs</td>
</tr>
<tr>
<td>1953</td>
<td>Nash, Gillies, Shapley</td>
</tr>
<tr>
<td>1957</td>
<td>Bellmann</td>
</tr>
<tr>
<td>1962</td>
<td>Pontryagin</td>
</tr>
</tbody>
</table>

A little bit of history
## Nobel prizes in Economics

<table>
<thead>
<tr>
<th>Year</th>
<th>Winners</th>
<th>Categories</th>
</tr>
</thead>
<tbody>
<tr>
<td>1994</td>
<td>John F. Nash Jr.</td>
<td>Perfect equilibrium</td>
</tr>
<tr>
<td></td>
<td>John Harsanyi</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Reinhard Selten</td>
<td></td>
</tr>
<tr>
<td>2005</td>
<td>Y. Robert J. Aumann</td>
<td>Cooperation &amp; conflict</td>
</tr>
<tr>
<td></td>
<td>Thomas C. Schelling</td>
<td></td>
</tr>
<tr>
<td>2007</td>
<td>Roger Myerson</td>
<td>Mechanism design</td>
</tr>
<tr>
<td></td>
<td>Leonid Hurwicz</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Eric Maskin</td>
<td></td>
</tr>
<tr>
<td>2012</td>
<td>Lloyd Shapley</td>
<td>Market design &amp; stable allocations</td>
</tr>
<tr>
<td></td>
<td>Alvin Roth</td>
<td></td>
</tr>
<tr>
<td>2014</td>
<td>Jean Tirole</td>
<td>Market regulations</td>
</tr>
</tbody>
</table>

Non cooperative: Sargent Sims 2011
Our logical thread

<table>
<thead>
<tr>
<th></th>
<th>One player</th>
<th>Many players</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Static</strong></td>
<td>Mathematical programming</td>
<td><em>(Static)</em> game theory</td>
</tr>
<tr>
<td><strong>Dynamic</strong></td>
<td>Optimal control theory</td>
<td>Dynamic (and/or differential) game theory</td>
</tr>
</tbody>
</table>
From Mathematical programming to Game Theory

One decision maker

\[ \max_u J(u), \quad u \in U, \quad \text{U set of actions} \]

Two decision makers P1, P2

\[ \begin{align*}
    \max_{u_1} J(u_1, u_2), & \quad u_1 \in U_1 \\
    \max_{u_2} J(u_1, u_2), & \quad u_2 \in U_2
\end{align*} \]

Game Theory
Game

Basic elements:
• **Players** with clear preferences, represented by a **Payoff** function.
• Each **Action** leads to an associated **Consequence**

Axioms:
• Players are rational:
  They are aware of their alternatives, forms expectations about any unknowns, have clear preferences, and choose their action deliberately after some process of optimization.

• And think strategically.
  When designing his strategy for playing the game, each player takes into account any knowledge or expectation he may have regarding his opponents' behaviour.
Rational Behavior

A  Set of **Actions** from which the decision-maker makes a choice.

C  Set of possible **Consequences** of these actions.

J: A --> C

Consequence function that associates a Consequence with each Action.

Preference relation (a complete transitive reflexive binary relation) on C.
Static games (One-shot games)

• Each player makes one choice and this completely determines the payoffs.
• Zero-Sum (Noncooperative) matrix games $\leftrightarrow$ NonZero-sum bimatrix
• Normal (strategic) form: all possible sequences of decisions of each player are set out against each other (no dynamic)
  • Matrix structure
Normal form

\[
\begin{pmatrix}
\alpha & \beta \\
\alpha & \beta
\end{pmatrix}
\]

\[
\begin{pmatrix}
-3 & -2 \\
0 & 2
\end{pmatrix}
\]

Existence questions
Pure and mixed strategies

Extensive form for G1

Single-act games
Multi-act games
Choice of strategies

WHAT IS OPTIMAL?
**Best response strategies**

<table>
<thead>
<tr>
<th></th>
<th>P2</th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>a</td>
<td>(1,-1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>(2,-2)</td>
<td>(0,-3)</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>(1,-1)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>

$u_i^b$ best reply (response) by player 1 to a profile of strategies for all other players $u_{-i}$ if

$$J^i(u_i^b , u_{-i}) \geq J^i(u_i , u_{-i})$$ for all $u_i \in U^i$
**Strictly Dominating strategies**

<table>
<thead>
<tr>
<th></th>
<th>P2</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>(1,0)</td>
<td>(0,0)</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>(2,-2)</td>
<td>(1,0)</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>(1,-1)</td>
<td>(0,-1)</td>
<td></td>
</tr>
</tbody>
</table>

\( u_i^d \) of player I

\[ J^i(u_i^d, u_{-i}) > J^i(u_i, u_{-i}) \] for all \( u_i \in U^i \),

for all \( u_{-i} \in U^1 \times U^2 \times \ldots \times U^{i-1} \times U^{i+1} \times \ldots \times U^N \)
Dominating strategies

• Eliminating some rows and/or columns which are known from the beginning to have no influence on the equilibrium solution

• Looking for Saddle points

• best reply to any feasible profile of the $N - 1$ rivals:
Example: Zero Sum Marketing game

<table>
<thead>
<tr>
<th>Market</th>
<th>STRATEGIES of B</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>2, 0</td>
</tr>
<tr>
<td>M2</td>
<td>1, 1</td>
</tr>
<tr>
<td></td>
<td>0, 2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STRATEGIES of A</th>
<th>Payoffs of A</th>
</tr>
</thead>
<tbody>
<tr>
<td>4, 0</td>
<td>1+0=1</td>
</tr>
<tr>
<td>3, 1</td>
<td></td>
</tr>
<tr>
<td>2, 2</td>
<td>1+1=2</td>
</tr>
<tr>
<td>1, 3</td>
<td>-1+1=0</td>
</tr>
<tr>
<td>0, 4</td>
<td></td>
</tr>
</tbody>
</table>

FIRM A 4 units of capital
FIRM B 2 units of capital
Example: Zero Sum Marketing game -2-

```
<table>
<thead>
<tr>
<th></th>
<th></th>
<th>TV</th>
<th>VE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>B</strong></td>
<td><strong>A</strong></td>
<td>2, 0</td>
<td>1, 1</td>
</tr>
<tr>
<td>4, 0</td>
<td>(1,-1)</td>
<td>(0,0)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>3, 1</td>
<td>(2,-2)</td>
<td>(1,-1)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>2, 2</td>
<td>(1,-1)</td>
<td>(2,-2)</td>
<td>(1,-1)</td>
</tr>
<tr>
<td>1, 3</td>
<td>(0,0)</td>
<td>(1,-1)</td>
<td>(2,-2)</td>
</tr>
<tr>
<td>0, 4</td>
<td>(0,0)</td>
<td>(0,0)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>
```
### Dominating strategies Player A

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>2, 0</th>
<th>1, 1</th>
<th>0, 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4, 0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3, 1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2, 2</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1, 3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0, 4</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
MaxiMin rule
(von Neumann)

- non-probabilistic decision-making rule
- decisions are ranked on the basis of their worst-case outcomes
- the optimal decision is one with the least worst outcome.

“In the worst of cases...”
MaxiMin rule

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$s_1^B$</th>
<th>$s_2^B$</th>
<th>$s_3^B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1^A$</td>
<td></td>
<td>(7,-7)</td>
<td>(5,-5)</td>
<td>(4,-4)</td>
</tr>
<tr>
<td>$s_2^A$</td>
<td></td>
<td>(2,-2)</td>
<td>(6,-6)</td>
<td>(3,-3)</td>
</tr>
<tr>
<td>$s_3^A$</td>
<td></td>
<td>(8,-8)</td>
<td>(0,0)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>
MaxiMin rule

Saddle point

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$S_1^B$</th>
<th>$S_2^B$</th>
<th>$S_3^B$</th>
<th>MIN of A</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1^A$</td>
<td>(7,-7)</td>
<td>(5,-5)</td>
<td>(4,-4)</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$S_2^A$</td>
<td>(2,-2)</td>
<td>(6,-6)</td>
<td>(3,-3)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$S_3^A$</td>
<td>(8,-8)</td>
<td>(0,0)</td>
<td>(1,-1)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>MIN of B</td>
<td>-8</td>
<td>-6</td>
<td>-4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

MAX MIN of A

MAX MIN of B
Saddle points may not exist (∅)
Static games in normal form

Choice of strategies

• Dominating strategies

• MiniMax Theorem (von Newmann)

• Saddle points existence not guaranteed
Nash Equilibrium

A set of strategies constitutes a Nash equilibrium if no single player is interested in changing his strategy unless one of the other players changes his own.

That is:

Keeping the choices of other players fixed, nobody is interested in changing his own.
Example with No saddle point but there exists 1 Nash equilibria

Set of strategies \((a, \beta)\) s.t.:
Knowing that G1 playes \(a\) then for G2 has not choise (convenience) but to play \(\beta\)
Knowing that G2 playes \(\beta\) then for G1 has not choise (convenience) but to play \(a\)
Example with No saddle point but there exist 2 Nash equilibria

Set of strategies \((a, \beta)\) s.t.:
Knowing that G1 plays a then for G2 has not choice (convenience) but to play \(\beta\)
Knowing that G2 plays \(\beta\) then for G1 has not choice (convenience) but to play a
Nash equilibria for static games

Existence of Nash equilibrium Kakutani fixed point theorem for multivalued maps. Consequence of the classical Brouwer fixed point theorem.

In a zero-sum game, if a Nash equilibrium exists, then all Nash equilibria yield the same payoff $V$ (value of the game) (von Neumann)
Nash Equilibrium Existence Theorem (1950)

In a finite game there exists at least one Nash equilibrium (eventually mixed strategies)

\[
\begin{pmatrix}
  y1 & y2 \\
  z1 & 3 & 0 \\
  z2 & -1 & 1 \\
\end{pmatrix}
\]

\((u_i^N, u_{-i}^N)\) Nash equilibrium

\[J^i (u_i^N, u_{-i}^N) > J^i(u_i, u_{-i}^N)\] for all \(u_i \in U_i\)
Nash Equilibrium

- There might be other combination of strategies that increase the payoff of some players without reducing the payoffs of the others. Or, more, that increase the payoff of all players: **Prisoners’ dilemma**.
**Prisoners’ Dilemma**

- If **only one** confesses, and puts the blame on the other one, then he is set free and the other will be sentenced to 6 years of jail;
- If **both** confess, they will be sentenced to 5 years.
- If **neither one** confesses, they will be sentenced to 1 year.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>NC</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(-5, -5)</td>
<td>(0, -6)</td>
</tr>
<tr>
<td>NC</td>
<td>(-6, 0)</td>
<td>(-1, -1)</td>
</tr>
</tbody>
</table>

Min A: 
-如果没有一个囚犯认罪，则他们将被判处1年。
Prisoners’ Dilemma A

\begin{array}{ccc}
\text{C} & \text{NC} \\
\text{C} & (-5, -5) & (0, -6) \\
\text{NC} & (-6, 0) & (-1, -1) \\
\end{array}

Min A

-5

-6

Prisoners’ Dilemma B

\[
\begin{array}{c|cc}
 & C & NC \\ 
C & (-5, \ -5) & (0, \ -6) \\ 
NC & (-6, \ 0) & (-1, \ -1) \\ 
\end{array}
\]
**Prisoners’ Dilemma**

Max Min of B

Max Min of A

Cooperative solution
Prisoners’ Dilemma
Nash equilibrium

\[
\begin{array}{c|cc}
 & A & NA \\
\hline
A & (-5, -5) & (0, -6) \\
NA & (-6, 0) & (-1, -1)
\end{array}
\]
Nash Equilibrium

• Existence and uniqueness is not guaranteed
  → There might exist more than one NE

• It gives solution when there might be uncertainty
• Each player does what is better for him (noncooperative)
• It might not be the better solution for everybody.
• Someone might increase his payoff moving far from the equilibrium.

Nash Equilibrium might not be Pareto Optimum.
Nash equilibrium
Noncooperative simultaneous game

• Symmetric Information structure

\[ \text{Max } J_1(u_1,u_2) \quad \Rightarrow \quad u_1^{BR} = u_1(u_2) \quad \text{if } u_1 \in U^1 \]

\[ \text{Max } J_2(u_1,u_2) \quad \Rightarrow \quad u_2^{BR} = u_2(u_1) \quad \text{if } u_2 \in U^2 \]

\( (u_1^N, u_2^N) \)
Stackelberg game
Noncooperative sequential game

• Asymmetric information structure

1. LEADER: declares his action $u_L$
2. FOLLOWER: computes his best response $u_F(u_L)$ (to any Leader’s strategy $u_L$)
3. LEADER: computes his optimal Stackelberg strategy $u_L^S$
4. FOLLOWER: adjust his strategy to obtain the Stackelberg strategy $u_F^S$

$$\text{Max } J_F(u_L, u_F) \quad \Rightarrow \quad u_F^{BR} = u_F(u_L) \quad \Rightarrow \quad \text{Max } J_L(u_L, u_F(u_L))$$

$$u_F \in U^F, \quad (u_L^S, u_F^S) \quad \quad u_L \in U^L$$
Coordination game

Cooperative simultaneous game

- Symmetric information structure

\[
\begin{align*}
\text{Max } & J_1(u_1,u_2) + J_2(u_1,u_2) \\
& u_1,u_2 \in U^1 \times U^2
\end{align*}
\]
Example Cournot duopoly static game with infinite strategy sets

\[ J_1 = (\alpha - \beta (Q_1 + Q_2)) Q_1 - K_1 Q_1^2 \]
\[ J_2 = (\alpha - \beta (Q_1 + Q_2)) Q_2 - K_2 Q_2^2 \]

**NASH:** \((Q_1^N, Q_2^N) = \left( \frac{\alpha}{2K_1 + 3\beta}, \frac{\alpha}{2K_2 + 3\beta} \right)\)

Symm. case \(\alpha = \beta = 1, K_i = 0 \Rightarrow (Q_1^N, Q_2^N) = (1/3, 1/3). J_1^N = J_2^N = 1/9\)

**STACKELBERG:** \((Q_{LS}, Q_{FS}) = \left( \frac{\alpha \left(1 - \frac{\beta}{2(K_F + \beta)}\right)}{2(K_L + \beta) - \beta^2/(K_F + \beta)}, \frac{\alpha - \beta Q_L^S}{2(K_F + \beta)} \right)\)

Symm. case \(\alpha = \beta = 1, K_i = 0 \Rightarrow (Q_1^S, Q_2^S) = (1/2, 1/4). J_1^S = 1/8, J_2^S = 1/16\)

**COOPERATIVE:** Symm. case \(\alpha = \beta = 1, K_i = 0 \Rightarrow J^C = 2/9 = J_1^N + J_2^N\)

IN GENERAL \((\alpha, \beta \in \mathbb{R}, K_i = 0) \Rightarrow J^C > J_1^N + J_2^N\)

\[ J^C > J_1^N + J_2^N \]