Absorption (I)

We now consider the excitation from the atomic state $|a\rangle$ to the atomic state $|b\rangle$ due to the absorption of one photon. Thus we suppose that the initial state is

$$|I\rangle = |a\rangle |n_{ks}\rangle ,$$

while the final state is

$$|F\rangle = |b\rangle |n_{ks}-1\rangle ,$$

where $E_a < E_b$. From the Golden rule one finds

$$W_{absorp}^{ab,ks} = \frac{2\pi}{\hbar} \left(\frac{e}{m}\right)^2 \left(\frac{\hbar}{2\varepsilon_0 \omega_k V}\right) n_{ks} |\varepsilon_{ks} \cdot \langle b|\hat{p}|a\rangle|^2 \delta(E_a + \hbar\omega_k - E_b) ,$$

because

$$\langle F|\hat{p}\hat{a}_{k's'}|I\rangle = \sqrt{n_{ks}} \langle b|\hat{p}|a\rangle \delta_{k',k} \delta_{s',s} , \quad \langle F|\hat{p}\hat{a}_{k's'}^+|I\rangle = 0 .$$
We can follow the procedure of the previous Unit to get

\[ W_{ab, ks}^{\text{absorp}} = \frac{\pi \omega_b^2}{V \varepsilon_0 \omega_k} n_{ks} |\varepsilon_{ks} \cdot \langle b| e \ r | a \rangle|^2 \delta(\hbar \omega_{ba} - \hbar \omega_k). \tag{5} \]

Again the delta function can be eliminated by integrating over the final photon states but here one must choose the functional dependence of \( n_{ks} \). We simply set

\[ n_{ks} = n(\omega_k), \tag{6} \]

and after integration over \( k \) and \( s \), from Eq. (5) we get

\[ W_{ab}^{\text{absorp}} = \frac{\omega_{ba}^3}{3 \pi \varepsilon_0 \hbar c^3} |\langle b| d | a \rangle|^2 n(\omega_{ba}) = W_{ba}^{\text{spont}} n(\omega_{ba}). \tag{7} \]
Absorption (III)

For a thermal distribution of photons, with $\rho(\omega)$ the energy density per unit of angular frequency specified by the thermal-equilibrium Planck formula

$$\rho(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3} n(\omega) \text{,} \quad n(\omega) = \frac{1}{e^{\hbar \omega/(k_B T)} - 1},$$

where $k_B$ is the Boltzmann constant and $T$ the absolute temperature, we can also write

$$W_{ab}^{\text{absorp}} = \left| \langle b | d | a \rangle \right|^2 \frac{1}{3 \epsilon_0 \hbar^2} \rho(\omega_{ba}) = W_{ba}^{\text{spont}} \frac{\pi^2 c^3 \hbar}{\hbar \omega_{ba}^3} \rho(\omega_{ba}).$$

(8)
Finally, we consider the stimulated emission of a photon from the atomic state $|b\rangle$ to the atomic state $|a\rangle$. Thus we suppose that the initial state is

$$|I\rangle = |b\rangle|n_{ks}\rangle,$$  \hspace{1cm} (10)

while the final state is

$$|F\rangle = |a\rangle|n_{ks} + 1\rangle,$$  \hspace{1cm} (11)

where $E_b > E_a$. From the Golden rule one finds

$$W_{stimul}^{ba,ks} = \frac{2\pi}{\hbar} \left(\frac{e}{m}\right)^2 \left(\frac{\hbar}{2\varepsilon_0 \omega_k V}\right) (n_{ks} + 1) |\epsilon_{ks} \cdot \langle a|\hat{p}|b\rangle|^2 \delta(E_b - E_a - \hbar\omega_k).$$  \hspace{1cm} (12)

Note that with respect to Eq. (3) in Eq. (12) there is the factor $n_{ks} + 1$ instead of $n_{ks}$.
It is straightforward to follow the previous procedure obtaining

\[
W_{ba}^{\text{stimul}} = \frac{\omega_{ba}^3}{3\pi\epsilon_0\hbar c^3} |\langle a|d|b \rangle|^2 (n(\omega_{ba}) + 1) = W_{ab}^{\text{absorp}} + W_{ba}^{\text{spont}},
\]

which shows that the probability \(W_{ba}^{\text{stimul}}\) of stimulated emission reduces to the spontaneous one \(W_{ba}^{\text{spont}}\) when \(n(\omega_{ba}) = 0\). It is then useful to introduce

\[
\tilde{W}_{ba}^{\text{stimul}} = W_{ba}^{\text{stimul}} - W_{ba}^{\text{spont}}
\]

which is the effective stimulated emission, i.e. the stimulated emission without the contribution due to the spontaneous emission. Clearly, for a very large number of photons \((n(\omega_{ba}) \gg 1)\) one gets \(\tilde{W}_{ba}^{\text{stimul}} \simeq W_{ba}^{\text{stimul}}\). Moreover, for a thermal distribution of photons, with the energy density per unit of angular frequency \(\rho(\omega)\), we can also write

\[
\tilde{W}_{ba}^{\text{stimul}} = W_{ab}^{\text{absorp}} = W_{ba}^{\text{spont}} \frac{\pi^2 c^3}{\hbar \omega_{ba}^3} \rho(\omega_{ba}).
\]
Stimulated emission (III)

For a thermal distribution of photons, with the energy density per unit of angular frequency \( \rho(\omega) \), we can also write

\[
\tilde{W}_{ba}^{\text{stimul}} = W_{ab}^{\text{absorp}} = W_{ba}^{\text{spont}} \frac{\pi^2 c^3}{\hbar \omega_{ba}^3} \rho(\omega_{ba}).
\] 

(16)

Remarkably, the probability of stimulated emission is different from zero only if the emitted photon is in the same single-particle state \( |k_s\rangle \) of the stimulating ones, apart when the stimulating light is the vacuum \( |0\rangle \).

In the stimulated emission the emitted photon is said to be “coherent” with the stimulating ones, having the same frequency and the same direction.