Lesson 3 - Matter-Radiation Interaction
Unit 3.2 Fermi golden rule and spontaneous emission

Luca Salasnich

Dipartimento di Fisica e Astronomia “Galileo Galilei”, Università di Padova

Structure of Matter - MSc in Physics
We have seen that the total Hamiltonian of the matter-radiation system in the dipole approximation is given by

$$\hat{H} = \hat{H}_0 + \hat{H}_D,$$  \hspace{1cm} (1)

where

$$\hat{H}_0 = \hat{H}_{\text{matt}} + \hat{H}_{\text{rad}}$$  \hspace{1cm} (2)

is the unperturbed Hamiltonian and $\hat{H}_D$ is the dipole Hamiltonian, which couples matter and radiation.

**Fermi golden rule:** Given the initial $|I\rangle$ and final $|F\rangle$ eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ under the presence to the perturbing Hamiltonian $\hat{H}_D$, the probability per unit time of the transition from $|I\rangle$ to $|F\rangle$ is given by

$$W_{IF} = \frac{2\pi}{\hbar} |\langle F|\hat{H}_D|I\rangle|^2 \delta(E_I - E_F),$$  \hspace{1cm} (3)

with the constraint of energy conservation.
Let us now apply the Fermi golden rule to the very interesting case of the hydrogen atom in the state $|b\rangle$ and the radiation field in the vacuum state $|0\rangle$. We are thus supposing that the initial state is

$$ |I\rangle = |b\rangle|0\rangle . $$

(4)

Notice that, because we are considering the hydrogen atom, one has

$$ \hat{H}_{\text{mat}} |b\rangle = E_b |b\rangle , $$

(5)

where

$$ E_b = -\frac{13.6 \text{ eV}}{n_b^2} $$

(6)

is the well-known quantization formula of the nonrelativistic hydrogen atom with quantum number $n_b = 1, 2, 3, \ldots$. In addition we suppose that the final state is

$$ |F\rangle = |a\rangle|k_s\rangle , $$

(7)

i.e. the final atomic state is $|a\rangle$ and the final photon state is

$$ |k_s\rangle = |1_{k_s}\rangle = \hat{a}^+_s |0\rangle . $$
From Eq. (3) one finds

$$W_{ba,ks}^{spont} = \frac{2\pi}{\hbar} \left( \frac{e^2}{m} \right)^2 \left( \frac{\hbar}{2\varepsilon_0 \omega_k V} \right) |\varepsilon_{ks} \cdot \langle a | \hat{p} | b \rangle|^2 \delta(E_b - E_a - \hbar \omega_k) , \quad (8)$$

because

$$\hat{a}_{k's'} |I\rangle = \hat{a}_{k's'} |b\rangle |0\rangle = |b\rangle \hat{a}_{k's'} |0\rangle = 0 , \quad (9)$$

while

$$\hat{a}_{k's'}^+ |I\rangle = \hat{a}_{k's'}^+ |b\rangle |0\rangle = |b\rangle \hat{a}_{k's'}^+ |0\rangle = |b\rangle |k's'\rangle , \quad (10)$$

and consequently

$$\langle F | \hat{p} \hat{a}_{k's'} | I \rangle = 0 , \quad \langle F | \hat{p} \hat{a}_{k's'}^+ | I \rangle = \langle a | \hat{p} | b \rangle \delta_{k',k} \delta_{s',s} . \quad (11)$$
On the basis of Heisenberg equation of motion of the linear momentum operator $\hat{p}$ of the electron

$$\frac{\hat{p}}{m} = \frac{dr}{dt} = \frac{1}{i\hbar} [r, \hat{H}_{matt}] , \tag{12}$$

we get

$$\langle a|\hat{p}|b\rangle = \langle a|m\frac{1}{i\hbar}[r, \hat{H}_{matt}]|b\rangle = \frac{m}{i\hbar} \langle a|r\hat{H}_{matt} - \hat{H}_{matt}r|b\rangle$$

$$= \frac{m}{i\hbar} (E_b - E_a) \langle b|r|a\rangle = -im\omega_{ba} \langle a|r|b\rangle , \tag{13}$$

where $\omega_{ba} = (E_b - E_a)/\hbar$, and consequently

$$W_{ba,ks}^{spont} = \frac{\pi \omega_{ba}^2}{\sqrt{e_0 \omega_k}} |\epsilon_{ks} \cdot \langle a|e|r|b\rangle|^2 \delta(\hbar\omega_{ba} - \hbar\omega_k) . \tag{14}$$
Spontaneous emission (IV)

The delta function is eliminated by integrating over the final photon states

\[ W_{ba}^{spont} = \sum_k \sum_s W_{ba,ks}^{spont} = V \int \frac{d^3k}{(2\pi)^3} \sum_{s=1,2} W_{ba,ks}^{spont} \]

\[ = \frac{V}{8\pi^3} \int dk k^2 \int d\Omega \sum_{s=1,2} W_{ba,ks}^{spont} , \tag{15} \]

where \( d\Omega \) is the differential solid angle.

Because \( \varepsilon_{k1}, \varepsilon_{k2} \) and \( n = k/k \) form a orthonormal system of vectors, setting \( r_{ab} = \langle a|r|b \rangle \) one finds

\[ |r_{ab}|^2 = |\varepsilon_{k1} \cdot r_{ab}|^2 + |\varepsilon_{k2} \cdot r_{ab}|^2 + |n \cdot r_{ab}|^2 = \sum_{s=1,2} |\varepsilon_{ks} \cdot r_{ab}|^2 + |r_{ab}|^2 \cos^2 (\theta) , \tag{16} \]

where \( \theta \) is the angle between \( r_{ba} \) and \( n \).
Spontaneous emission (V)

It follows immediately

\[ \sum_{s=1,2} |\mathbf{e}_{ks} \cdot \mathbf{r}_{ab}|^2 = |\mathbf{r}_{ab}|^2 (1 - \cos^2 (\theta)) = |\mathbf{r}_{ab}|^2 \sin^2 (\theta) = |\langle a | \mathbf{r} | b \rangle|^2 \sin^2 (\theta). \] \hspace{1cm} (17)

In addition, in spherical coordinates one can choose \( d \Omega = \sin (\theta) d \theta d \phi \), with \( \theta \in [0, \pi] \) the zenith angle of colatitude and \( \phi \in [0, 2\pi] \) the azimuth angle of longitude, and then

\[ \int d \Omega \sin^2 (\theta) = \int_0^{2\pi} d \phi \int_0^\pi d \theta \sin^3 (\theta) = \frac{8\pi}{3}. \] \hspace{1cm} (18)

In this way from Eq. (15) we finally obtain

\[ W_{ba}^{\text{spont}} = \frac{\omega_{ba}^3}{3\pi \epsilon_0 \hbar c^3} |\langle a | d | b \rangle|^2, \] \hspace{1cm} (19)

where the \( \mathbf{d} = -e \mathbf{r} \) is the classical electric dipole momentum of the hydrogen atom, i.e. the dipole of the electron-proton system where \( \mathbf{r} \) is the position of the electron of charge \(-e < 0\) with respect to the proton of charge \( e > 0 \), and \( \langle a | d | b \rangle = -\langle a | e \mathbf{r} | b \rangle \) is the so-called dipole transition element.